Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

Lattice Points on the Cone

$$y^2 + 5x^2 = 86z^2$$

Dr. R. Sathiyapriya[1], Dr. N. Thiruniraiselvi[2], Dr. M. A. Gopalan[3]

[1]Associate Professor, Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Samayapuram, India.
[2]Assistant Professor, Department of Mathematics, M.A.M. College of Engineering and Technology, Affiliated to Anna University (Chennai), India.
[3]Professor, Department of Mathematics, Shrimati Indira Gandhi College, Affiliated to Bharathidasan University, India.
Email: charukanishk@gmail.com

Our aim is to find different solutions in integers to the second-degree Diophantine equation with 3 variables $y^2 + 5x^2 = 86z^2$. Through different methods of solutions in integers are found. A construction of formula to get sequence of solutions from the given solutions are found.

Keywords: Ternary Quadratic, Homogeneous Quadratic, Integer Solutions.

1. Introduction

The subject of second degree Diophantine equations has numerous variety of interesting problems. The problem on cone are rich in variety and attached many mathematicians. For extensive review problems, one may refer [1-16]. This article focuses on various choices of solutions in integers to the second degree Diophantine equation with three variables $y^2 + 5x^2 = 86z^2$. Different sets of solutions in integers are found by applying the method of factorization and also, through employing the linear transformations.Given an integer solution, formulae for getting sequence of solutions based on the given solution are illustrated

2. Method of analysis

The cone under consideration is

$$y^2 + 5x^2 = 86z^2 \quad (1)$$

Different methods for finding different non-zero solutions in integers to (1) are given below.

Set 1

Assume $$z = a^2 + 5b^2 \quad (2)$$

Express 86 on the Right hand side of (1) in the form of multiplication of complex

Conjugates as $86 = (9 + i\sqrt{5})(9 - i\sqrt{5})$ (3)

Substituting (2) and (3) in (1) and factorizating

Consider

$$y + i\sqrt{5}\,x = (9 + i\sqrt{5})\,(a + i\sqrt{5}\,b)^2$$

Equate real and imaginary parts, we get

$$\begin{aligned} x &= a^2 - 5b^2 + 18ab, \\ y &= 9(a^2 - 5b^2) - 10ab \end{aligned} \quad (4)$$

Thus, (2) and (4) give the solutions in integers to (1).

Set 2

(1) Can be written as $y^2 + 5x^2 = 86z^2 * 1$ (5)

Consider the integer 1 on the Right-hand side of (5) as

$$1 = \frac{(2 + i\sqrt{5})(2 - i\sqrt{5})}{9} \quad (6)$$

Putting (2), (3) & (6) in (5) and in view of factorization, consider

$$y + i\sqrt{5}\,x = (9 + i\sqrt{5})\frac{(2 + i\sqrt{5})}{3}(a + i\sqrt{5}\,b)^2$$

Equating the real and imaginary parts in the above equation and taking

a by 3 A & b by 3 B, the following solutions in integers to (1) are as follows

$$x = 33(A^2 - 5B^2) + 78AB,$$

$$y = 39(A^2 - 5B^2) - 330AB,$$

$$z = 9(A^2 + 5B^2)$$

Note 1

It is to be noted that, in addition to (6), one has

$$1=\frac{(2+i\,3\sqrt{5})(2-i\,3\sqrt{5})}{49},$$

$$1=\frac{(1+i\,4\sqrt{5})(1-i\,4\sqrt{5})}{81},$$

$$1=\frac{(5r^2-s^2+i\sqrt{5}\,2rs)(5r^2-s^2-i\sqrt{5}\,2rs)}{(5r^2+s^2)^2}$$

Repeating the same methos as above, we getthree sets of solutions in integer to (1).

Set 3

Rewrite (1) as

$$86z^2-y^2=5x^2 \tag{7}$$

Assume

$$x=86a^2-b^2 \tag{8}$$

Consider the 5 on the Right-hand side of (7) as

$$5=(\sqrt{86}+9)(\sqrt{86}-9) \tag{9}$$

Using (8) & (9) in (7) and applying factorization, consider

$$\sqrt{86}\,z+y=(\sqrt{86}+9)(\sqrt{86}\,a+b)^2 \tag{10}$$

Equating the rational and irrational parts in (10), one has

$$\begin{aligned} y&=9(86a^2+b^2)+172ab,\\ z&=(86a^2+b^2)+18ab \end{aligned} \tag{11}$$

Thus, (8) & (11) gives solutions in integers to (1).

Set 4

Taking (1) as

$$86z^2-5x^2=y^2*1 \tag{12}$$

Assume

$$y=86a^2-5b^2 \tag{13}$$

1 of (12) can be written as

$$1=\frac{(\sqrt{86}+\sqrt{5})(\sqrt{86}-\sqrt{5})}{81} \qquad (14)$$

Substituting (13) & (14) in (12) and in view of factorization, consider

$$\sqrt{86}\,z+\sqrt{5}\,x=\frac{(\sqrt{86}+\sqrt{5})}{9}\,(\sqrt{86}\,a+\sqrt{5}\,b)^2 \qquad (15)$$

Equating the coefficients of corresponding terms in (15) and replacing a, b

By $3A, 3B$ respectively, the following solutions in integers to (1) are given below

$$x=86A^2+5B^2+172AB,$$
$$y=9(86A^2-5B^2),$$
$$z=86A^2+5B^2+10AB$$

Set 5

Choosing

$$z=X+5T,\ x=X+86T,\ y=9Y \qquad (16)$$

in (1) leads to

$$X^2=430T^2+Y^2 \qquad (17)$$

which is satisfied by

$$T=2rs,\ Y=430r^2-s^2,\ X=430r^2+s^2 \qquad (18)$$

From (16), thefollowing solutions to (1) are as follows

$$x=430r^2+s^2+172rs,\ y=9(430r^2-s^2),\ z=430r^2+s^2+10rs$$

In addition to (18), there are other solutions to (17) that are illustrated below:

Express (17) as the simultaneous equations as follows in Table 1:

Table 1-Simultaneous Equations

System	I	II	III	IV	V	VI	VII
$X+Y$	$430T$	$215T$	$86T$	$43T$	$215\,T^2$	$43\,T^2$	$5\,T^2$
$X-Y$	T	$2T$	$5T$	$10T$	2	10	86

Solving each of the above system of equations, the values of X, Y, T are found.

From (16), thefollowing solutions in integers are got. For the sake of simplicity and brevity, the respective solutions to (1) are represented below:

Solutions from System I

$$x = 603\ k, y = 3861\ k, z = 441\ k$$

Solutions from System II

$$x = 389\ k, y = 1917\,k, z = 227\ k$$

Solutions from System III

$$x = 263\ k, y = 729\ k, z = 101\ k$$

Solutions from System IV

$$x = 225\ k, y = 297\ k, z = 63\ k$$

Solutions from System V

$$x = 430\,k^2 + 172\,k + 1, y = 9(430\,k^2 - 1), z = 430\,k^2 + 10\,k + 1$$

Solutions from System VI

$$x = 86\,k^2 + 172\,k + 5, y = 9(86\,k^2 - 5), z = 86\,k^2 + 10\,k + 5$$

Solutions from System VII

$$x = 10\,k^2 + 172\,k + 43, y = 9(10\,k^2 - 43), z = 10\,k^2 + 10\,k + 43$$

Note 2

Apart from (16), we can choose z,x,y as follows

$$z = X - 5T, x = X - 86T, y = 9Y$$

Leading to distinct sets of solutions in integers to (1).

Set 6

Express (1) in the form of ratios as

$$\frac{y + 9z}{z + x} = \frac{5(z - x)}{y - 9z} = \frac{P}{Q}, Q \neq 0 \qquad (19)$$

which isa double equations

$$P\,x - Q\,y + (P - 9\,Q)\,z = 0$$
$$5\,Q\,x + P\,y - (5\,q + 9\,P)\,z = 0 \qquad (20)$$

Solving (20) by the method of cross-multiplication, we get

$$x = 5\,Q^2 - P^2 + 18\,P\,Q,$$
$$y = -45\,Q^2 + 9\,P^2 + 10\,P\,Q,$$
$$z = 5\,Q^2 + P^2$$

which gives the required solutions in integers to (1).

Note 3

(1) May also be represented as other forms of ratios as given below.

(i) $$\frac{y + 9\,z}{5\,(z + x)} = \frac{(z - x)}{y - 9\,z} = \frac{P}{Q}, Q \neq 0$$

(ii) $$\frac{y + 9\,z}{z - x} = \frac{5\,(z + x)}{y - 9\,z} = \frac{P}{Q}, Q \neq 0$$

(iii) $$\frac{y + 9\,z}{5\,(z - x)} = \frac{(z + x)}{y - 9\,z} = \frac{P}{Q}, Q \neq 0$$

Repeating the above procedure, three more sets of solutions in integers to (1) are got.

Generation of solutions

Let $(x_0\,, y_0\,, z_0)$ be any given solution to (1). The process of getting sequence of solutions from the given solution is illustrated below:

Illustration 1

Let $$x_1 = h - 6\,x_0\,, y_1 = h - 6\,y_0\,, z_1 = 6\,z_0 \qquad (21)$$

be the second solution to (1). Putting (21) in (1) and simplifying,

we have $$h = 10\,x_0 + 2\,y_0 \qquad (22)$$

Using (22) in (21), we get

$$x_1 = 4\,x_0 + 2\,y_0\,, y_1 = 10\,x_0 - 4\,y_0$$

Following the same method as above leads to the general solution to (1) given by

$$x_n = \frac{(10\alpha^n + 2\beta^n)x_0 + 2(\alpha^n - \beta^n)y_0}{12},$$

$$y_n = \frac{10(\alpha^n - \beta^n)x_0 + (2\alpha^n + 10\beta^n)y_0}{12},$$

$$z_n = 6^n z_0 \quad , n = 1,2,3,...$$

where $\alpha = 6, \beta = -6$

Illustration 2

Let $x_1 = 5\ x_0, y_1 = 9h + 5\ y_0, z_1 = h - 5\ z_0$ (23)

be the second solution to (1). Substituting (23) in (1) and simplifying,

we have $h = 18y_0 + 172z_0$ (24)

Using (24) in (23), we get $y_1 = 167y_0 + 1548z_0, z_1 = 18y_0 + 167z_0$

Following the same method as above leads to general solutions (1) given by

$$y_n = \frac{\sqrt{86}(\alpha^n + \beta^n)y_0 + 86(\alpha^n - \beta^n)z_0}{2\sqrt{86}},$$

$$z_n = \frac{(\alpha^n - \beta^n)y_0 + \sqrt{86}(\alpha^n + \beta^n)z_0}{2\sqrt{86}},$$

$$x_n = 5^n x_0 \quad , n = 1,2,3,...$$

where

$$\alpha = 167 + 18\sqrt{86}, \beta = 167 - 18\sqrt{86}$$

Illustration 3

Let

$$x_1 = 4h + 6\ x_0, y_1 = 6\ y_0, z_1 = h - 6\ z_0 \qquad (25)$$

be the second solution to (1). Substituting (25) in (1) and simplifying,

we have

$$h = 40x_0 + 172z_0 \qquad (26)$$

Using (26) in (25), we get

$$x_1 = 166\,x_0 + 688\,z_0\,,\,z_1 = 40\,x_0 + 166\,z_0$$

Following the same method as above leads to general solutions (1) given by

$$x_n = \frac{\sqrt{6898}\,(\alpha^n + \beta^n)\,x_0 + 344\,(\alpha^n - \beta^n)\,z_0}{2\sqrt{6898}}\ ,$$

$$z_n = \frac{20\,(\alpha^n - \beta^n)\,x_0 + \sqrt{6898}\,(\alpha^n + \beta^n)\,z_0}{2\sqrt{6898}}\ ,$$

$$y_n = 6^n\,y_0 \quad ,n = 1,2,3,...$$

where

$$\alpha = 166 + 2\sqrt{6898}\,,\beta = 166 - 2\sqrt{6898}$$

3. Conclusion

In this article we have attempted to find various method through which we get solutions to integers to the second-degree equations with three unknowns. The researchers who are interested may find more types of methods to extend this problem as Diophantine equations has infinitely may solutions.

References

1. M.A.Gopalan, A.Vijayashankar. Integral points on the homogeneous cone .IJIRSET, 2(1), 682-685, (2013).
2. M.A.Gopalan,S.Vidhyalakshmi,V.Geetha: Lattice points on the homogeneous cone . IJESRT, 2(2), 775-779, (2013).
3. M.A.Gopalan, S.Vidhyalakshmi, E.Premalatha: On the Ternary quadratic Diophantine equation .Diophantus.J.Math1 (1), 51-57, (2012).
4. M.A.Gopalan, S.Vidhyalakshmi, A.Kavitha: Integral points on the homogeneous cone .Diophantus.J.Math1 (2), 127-136, (2012).
5. M.A.Gopalan, G.Sangeetha, Observations on .Antarctica J.Math., 9(4),359-362,(2012).
6. M.A.Gopalan, Manju Somanath, Sangeetha: Observations on the Ternary Quadratic Diophantine Equation .Bessel J.Math., 2(2),101-105,(2012).
7. M.A.Gopalan, S.Vidhyalakshmi,E.Premalatha: On the Ternary quadratic equation .Diophantus.J.Math1(2),69-76,(2012).
8. M.A.Gopalan,S.Vidhyalakshmi , E.Premalatha: On the homogeneous quadratic equation with three unknowns . Bulletin of Mathematics and Statistics Research.1 (1), 38-41,(2013).
9. K.Meena,M.A.Gopalan,S.Vidhyalakshmi,N.Thiruniraiselvi: Observations on the quadratic equation . International Journal of Applied Research .1(2), 51-53, (2015).
10. R.Anbuselvi, S.A. Shanmugavadivu: On homogeneous Ternaryquadratic Diophantine equation . IJERA.7 (11), 22-25, (2017).

11. S. Vidhyalakshmi, T. Mahalakshmi: A Study on the Homogeneous Cone .International Research Journal of Engineering and Technology (IRJET).6, Issue 3, 5000-5007, (2019)
12. J.Shanthi,T.Mahalakshmi ,S.Vidhyalakshmi ,M.A.Gopalan : A search on Integer solutions to t6he Homogeneous Quadratic Equation Three Unknowns .IJRPR , 4, 3610-3619,(2023).
13. J. Shanthi,M. Parkavi: On finding Integer solutions to the HomogeneousTernary Quadratic Diophantine equation .InternationalJournal of Research Publication andReviews. 4, 700-708, (2023).
14. T. Mahalakshmi, E. Shalini: On Finding Integer Solutions to the Homogeneous Ternary Quadratic Diophantine Equation .International Journal of Research Publication and Reviews. 4, 452-462(2023).
15. S.Vidhyalakshmi, T. Mahalakshmi, M.A. Gopalan, S. Shanthia: Observations OnThe Homogeneous Ternary Quadratic DiophantineEquation with Three Unknowns . Journal of Information and Computational Science .10(3), 822-831, (2020).
16. M.A. Gopalan, S. Vidhyalakshmi, J. Shanthi, V. Anbuvalli: On Finding theInteger Solutions of Ternary Quadratic Diophantine Equation . International Journal of Precious Engineering Research and Applications (IJPERA).7, 34-38, (2022).