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Reconstruction of Almost Discrete Spaces

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The *deck* of a topological space X is the set $\mathcal{D}(X) = \{[X - \{x\}] : x \in X\}$, where $[Z]$ denotes the homeomorphism class of Z. A space X is called (topologically) *reconstructible* if whenever $\mathcal{D}(X) = \mathcal{D}(Y)$, then X is homeomorphic to Y. In this paper, we prove the property that whether a almost discrete spaces have isolated point or not is reconstructible. We also prove that almost discrete spaces with isolated points and almost discrete spaces without isolated points but having at least one finite point open set, n −open set topological spaces are reconstructible.
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1. Introduction

A vertex-deleted subgraph or card $G - v$ of a graph G is obtained by deleting the vertex v and all edges incident with v. The collection of all cards of G is called the deck of G and it is denoted by $\mathcal{D}(G)$. A graph H is a reconstruction of G if H has the same deck as G. A graph is said to be reconstructible if it is isomorphic to all its reconstructions. A parameter p defined on graphs is reconstructible if, for any graph G, it takes the same value on every reconstruction of G. The graph reconstruction conjecture, posed by Kelly and Ulam [13] in 1941, asserts that every graph G on n (≥ 3) vertices is reconstructible. More precisely, if G and H are finite graphs with at least three vertices such that $\mathcal{D}(H) = \mathcal{D}(G)$, then G and H are isomorphic. For a reconstructible graph G, Harary and Plantholt [8] defined the reconstruction number of a graph G, denoted by $rn(G)$, to be the minimum number of cards which can only belong to the deck of G and not to the deck of any other graph H, $H \ncong G$, these cards thus uniquely identifying G.

In 2016, Pitz and Suabedissen [12] have introduced the concept of reconstruction in topological spaces as follows. For a topological space X, the one-point deleted subspace $X - \{x\}$ is called a card of X and it is denoted by X_x. The set $\mathcal{D}(X) = \{[X_x] : x \in X\}$ of subspaces of X is called the deck of X, where$[X_x]$ denotes the homeomorphism class of the card X_x. Given topological spaces X and Z, we say that Z is a reconstruction of X if their decks agree. A topological space X is said to be reconstructible if the only reconstructions of it are the spaces homeomorphic to X. Formally, a space X is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Z)$ implies $X \cong Z$ and a property P of topological spaces is reconstructible if $\mathcal{D}(X) = \mathcal{D}(Z)$ implies "X has P if and only if Z has P". A space X is weakly reconstructible if it is reconstructible from the collection of all the cards of X, that is, it is reconstructible from the collection $\{X_x : x \in X\}$.

Gartside et al. [6, 7, 12] have proved that the space of real numbers, the space of rational numbers, the space of irrational numbers, every compact Hausdorff space that has a card with a maximal finite compactification, and every Hausdorff continuum X with weight $\omega(X) < |X|$ are reconstructible. In their papers, they also proved certain properties of a space, namely all hereditary separation axioms and all cardinal invariants are reconstructible. Manvel et al. [11] have done similar work in 1991 itself and they have reconstructed all finite sequences from their subsequences. Recently, Jini and Monikandan [1] have reconstructed most of the finite topological spaces.

On the other side, Pitz and Suabedissen [12] have shown that the Cantor set is not reconstructible. They have also proved some properties of a space are not reconstructible, which include connectedness, compactness, lindelofness, countable compactness and pseudo compactness.

By the order of a topological space (X, τ) , we mean the number of elements in the set (that is, $|X|$). By the size of the topological space, we mean the number of open sets in the space (that is, $|\tau|$). Terms not defined here are taken as in [5].

In this paper, we show the property that whether almost discrete spaces having isolated points or not, almost discrete spaces with isolated points, almost discrete spaces without isolated points but containing a finite point open set and n -open set topological spaces are reconstructible.

Almost Discrete Space

A space X is almost discrete if every open set is closed in X. That is, X has only clopen sets. In this chapter, we prove that the almost discrete property and almost discrete spaces with a finite open set are reconstructible as below.

Lemma 1. A space X is almost discrete if and only if every card of X is almost discrete. Thus, the almost discrete property is reconstructible.

Proof. Necessity is obvious. For sufficiency, assume that all the cards of X are almost discrete but X is not. Then X contains an open set that is not closed. Among these open sets, choose one, say U such that $|U|$ is minimum. Two cases arise depending on the cardinality of U.

Case 1. $|U| \geq 2$.

Now, there exists a point $x \in U$ such that $X - U \subseteq X_x$ and hence $X - U$ is not open in X_x as in (i) below. Thus, X_x is a card containing the open set $U - \{x\}$ and the non-open set $X - U$, where $X - U = X - (U - \{x\})$. That is, $U - \{x\}$ is open but not closed in X_x (since $|U| \geq 2$, the existence of the set $U - \{x\}$ is guaranteed), contradicting our assumption.

(i) If $X - U$ were open in X_x, then either $X - U$ or $(X - U) \cup \{x\}$ would be open in X. Therefore $(X - U) \cup \{x\}$ would be open in X and so $((X - U) \cup \{x\}) \cap U = \{x\}$ would be open in X. Since U is of minimum order (≥ 2) among all open sets that are not closed in X, it follows that

$\{x\}$ would be a closed set in X. Therefore $\{x\}$ would be a clopen set in X and so X_x would be open in X. Hence $X_x \cap ((X - U) \cup \{x\} = X - U$ would be open in X, giving a contradiction.

Case 2. $|U| = 1$.

Let $U = \{s\}$, where $s \in X$. Then $\{s\}$ is open in X but $X - U = X_s$ is not open in X.

Consider a card X_y, where $y \in X_s$. We proceed by two subcases.

Case 2.1. The set $X - \{s, y\}$ is not open in X.

Now $\{s\}$ is open but not closed in X_y. That is, $\{s\}$ is open in X_y but $X_y - \{s\} = X - \{s, y\}$ is not open in X_y as otherwise either X_s or $X - \{s, y\}$ is open in X and thus X_y is not an almost discrete card, a contradiction.

Case 2.2. The set $X - \{s, y\}$ is open in X.

Now $X_y = (X - \{s, y\}) \cup \{s\}$ is open in X but $X_y$is not closed in X (as otherwise, $\{y\}$ is open in X and hence $X_s = (X - \{s, y\}) \cup \{y\}$, is open in X, a contradiction). Therefore $\{y\}$ is not open in X. Let $V = X - \{s, y\}$. Then $|V| \neq \varphi$. Clearly, $V \cup \{s\}$ is open in X and $V \cup \{y\}$ is not open in X, since $V \cup \{y\} = X_s$. Now consider the card $X_t, t \in V$. Then $\{s\}$ is open but not closed in $X_t$as in (ii) below. Therefore X_t is not an almost discrete card of X, a contradiction.

(ii) If $\{s\}$ is closed in X_t, then $X - \{s, t\}$ is open in X_t. Therefore X_s or $X - \{s, t\}$ is open in X. Since X_s is not open in X, $X - \{s, t\} = (V - \{t\}) \cup \{y\}$ is open in X. Therefore $V \cup \{y\}$, where $V \cup \{y\} = ((V - \{t\}) \cup \{y\}) \cup V$, is open in X, giving a contradiction.

Lemma 2. Let X be an almost discrete space with $|\mathcal{D}(X)| \geq 2$. Then the property that whether X has an isolated point or not is reconstructible.

Proof. If $\mathcal{D}(X)$ has a card (say X_z) containing two isolated points (say x, y), then $\{x\}, \{y\}$ or $\{z\}$ must be an isolated point of the given space X. So, we assume that every card has at most one isolated point and we proceed by three cases as below.

Case 1. Every card of X has exactly one isolated point.

Assume, to the contrary, that X has no isolated point. Since every card in $\mathcal{D}(X)$ has an isolated point, every isolated point in the card must be contained in a $2-$point open set of X. Also an isolated point in a card X_z and an isolated point in a card non-homeomorphic to X_z would not be in the same $2-$point open set of X. If there exists a point not contained in any $2-$point open set of X, then the card corresponding to that point has no isolated point, a contradiction to our assumption in Case 1. Otherwise, all the cards of X are homeomorphic and $|\mathcal{D}(X)| = 1$, again a contradiction.

Case 2. At least two cards have no isolated point.

Now X has no isolated point (as otherwise, every card X_x, where x is a non-isolated point, would contain at least one isolated point, a contradiction).

Case 3. Exactly one card has no isolated point.

First we assume that $|\mathcal{D}(X)| = 2$. If the card with no isolated point does not contain $2-$point open sets, then X has an isolated point. Otherwise, since the other card in $\mathcal{D}(X)$ has an isolated point, X has at least one $2-$point open set. Now the two cards X_x and X_y, where $\{x, y\}$ is the $2-$point open set, are homeomorphic. Therefore, the card without isolated point must contain a $2-$point open set, which is a contradiction. Suppose the card with no isolated point contains $2-$point open sets. Then we shall prove that X has no isolated point. Suppose, if possible, X has an isolated point, say x. Then clearly every card other than X_x contains the isolated point corresponding to x. Now, by our assumption, we have a card without isolated point; let it be the same X_x. Since X is almost discrete and $\{x\}$ is open in X, X_x is also open in X. Therefore, every open set of X_xs also open in X. Thus τ_x contains at least one $2-$point open set and it does not contain the point x; let it be $\{a, b\}$. Hence the card X_a must contain at least the two points x and b as isolated points, which is a contradiction.

Now, consider the case that $|\mathcal{D}(X)| \geq 3$. Let $X_c = \{\varphi, \{a\}, \ldots, X_c\}$ and $X_d = \{\varphi, \{b\}, \ldots, X_d\}$ be two non-homeomorphic cards with isolated points. Suppose X has no isolated point. Then $\{a, c\}$ and $\{b,d\}$ are open in X. Two more subcases arise here as below.

(i). $\{a, c\}$ and $\{b, d\}$ are equal in τ_X.

(ii). $\{a, c\}$ and $\{b, d\}$ are disjoint.

Suppose that (i) holds. Then as $c \neq d$, we have $c = b$ and $a = d$. Since X has no isolated point, $\{a, c\} \subseteq U$, for every U with $a \in U$ or $c \in U$. But in this case X_a and X_c are homeomorphic by the mapping $f: X_a \longrightarrow X_c$ defined by $f(W - \{a\}) = W - \{c\}$ for all $W \in \tau_X$. That is, $X_a = X_d$ and X_c are homeomorphic cards of X, which is a contradiction.

Suppose (ii) holds. Now a, b, c, d are pairwise distinct points of X. Since $\{a, c\}$ and $\{b, d\}$ are in τ_X, it follows that $\{a, b, c, d\}$ and hence the complement $X - \{a, b, c, d\}$ are in τ_X. Now we consider the card, say X_e, with no isolated point. Clearly $e \in V$ for some $V \in \tau_X$ and $|V| \geq 3$, since X and X_e have no isolated point. Suppose any point of X, say t, together with the open set $\{a, c\}$ or $\{b, d\}$ form an open set of order three. Then $\{a, c, t\}$ or $\{b, d, t\}$, say $\{a, c, t\} \in \tau_X$. Therefore $(X - \{a, c\}) \cap \{a, c, t\} = \{t\} \in \tau_X$, which is a contradiction to our assumption. Hence there exists no open set W of order 3 containing $\{a, c\}$ in X. If there exists a 3-point open set A containing the point e, then A does not contain the points a, b, c and d. Therefore no 2-point open set of X contained in any 3-point open set of X. If $V - \{e\} \in \tau_X$, then $(X - (V - \{e\})) \cap V = \{e\} \in \tau_X$, giving a contradiction. Otherwise, that is, $V - \{e\}$ is not open in X. From the above arguments, we conclude $\tau_X = \{\varphi, \{a, c\}, \{b, d\}, \{a, b, c, d\}, V, \ldots, X\}$.

If τ_X contains no other two point open sets, then the cards X_a, X_b, X_c and X_d are homeomorphic. Suppose X has some other 2-point open set. Then they are mutually disjoint, since X has no isolated point. Since no 2-point open set contained in 3-point open sets, all the cards X_z, where z is in a 2-point open set, are homeomorphic. Therefore, exactly one card in $\mathcal{D}(X)$ has an isolated point, giving a contradiction and completing the proof.

A topological space X is an n -open set topological space if there exist pairwise disjoint open sets $U_1, U_2, \ldots, U_m$, where $|U_i| = n \geq 2$, for $i = 1$ to m such that $\bigcup_{i=1}^{m} U_i = X$ and X has no non-empty open set of order fewer than n.

Lemma 3. A space (X, τ) is an n -open set topology if and only if every card X_x of X has exactly one $(n-1)$ -point open set, say U, and the union of all disjoint n-point open sets equal $X_x - U$

and $|\mathcal{D}(X)| = 1$.

Proof. Necessity: Since $\bigcup_{i=1}^{m} U_i = X$ and $|U_i| = n \geq 2$, for all $i = 1$ to m, every card must contain a (n−1) -point open set. Suppose, to the contrary, that X_x has two (n−1) -point open sets, say U_1, U_2. If U_1 or U_2 is open in X , then we have an (n − 1) -point open set in X, which is a contradiction. If both U_1 and U_2 are not open in X, then $U_1 \cup \{x\}$ and $U_2 \cup \{x\}$ are open in X and hence {x} is open in X, again a contradiction. Since $\bigcup_{i=1}^{m} U_i = X$ every $x \in X$ is contained in exactly one U_i and so every card X_x must contain a unique open set of order $n-1$ and all other n -point open sets not containing a point x of X are open in X_x. Hence the union of n -point open sets of X_x is equal to $X_x - (U_i - \{x\})$ and $U_i - \{x\}$ is not contained in any other U_j, $j \neq i$. Now a mapping f defined from a card X_x into a card X_y by

$$f(U) = \begin{cases} U & \text{if } U \in \tau_{X_x};\ y \notin U \\ (U - \{y\}) \cup \{x\} & \text{if } U \in \tau_{X_x};\ y \in U \end{cases}$$

is clearly a homeomorphism and hence $|\mathcal{D}(X)| = 1$.

Sufficiency: Suppose X has an open set of order less than n. Then there exists a card containing an open set of order fewer than n − 1, giving a contradiction. Therefore the order of an open set in X must be at least n. By hypothesis, every card has a unique (n − 1) -point open set. Since n is the minimum order of an open set in X, any (n − 1) -point open set in the cards is not open in X. Therefore the (n − 1) -point open set along with the deleted point forms an n-point open set in X. Now the n -point open sets of X are pairwise disjoint and $\bigcup_{i=1}^{m} U_i = X$, since $X_x = \bigcup_{j=1}^{m} U_j - (U_i - \{x\})$, where $i \neq j$.

By Lemma 3, it is clear that whether the unknown topology is an n -open set topology or not can be reconstructed from the given deck. Suppose that the unknown topology on X is an n -open set topology. Then any card X_x contains an $(n - 1)$ -point open set, say U, that is, not open in X. Consequently, $U \cup \{x\}$ must be open in X. Also all the n -point open sets are open in X. Hence $\tau_X = \{V : V \in \tau_{X_x}, V \cap U = \varphi\} \cup \{V \cup \{x\} : V \in \tau_{X_x}, V \cap U \neq \varphi\}$, which is clearly the required n -open set topology on X. This is concluded in the next theorem.

Lemma 4. Every n -open set topological space is reconstructible.

Lemma 5. Let X be a space endowed with a topology other than the 2 -open set topology and let $|\mathcal{D}(X)| = 1$. Then X has an isolated point if and only if the card has an isolated point.

Proof. If X has an isolated point (say x), then x must be an isolated point of every card except X_x. Since $|\mathcal{D}(X)| = 1$, the card X_x also contains an isolated point. Conversely, the only topology on X yielding isolated points to all the cards but not to X is the 2 -open set topology. Therefore, by hypothesis, X has an isolated point.

Every n -open set topology on X is an almost discrete topology. For, any open set U is a union of n-point open sets. Also the union of the remaining n -point open sets clearly equals U^c, since $\bigcup_{i=1}^{m} U_i = X$. Therefore U is closed.

Theorem 6. Any almost discrete space with an isolated point is reconstructible.

Proof. By Lemmas 1, 2 and 5, we can assume that X is almost discrete with an isolated point, say x. Now the card X_x is open in X and $X \cong X_x \oplus \{x\}$.

Lemma 7. Let X be an almost discrete space without isolated point. Then X has an open set of order n, where $2 \leq n < \infty$ and n is minimum if and only if there exists a card containing a unique $(n-1)$ -point open set and no card contains an open set of order fewer than $n-1$.

Proof. Necessity: If X has an open set of order n, then at least n cards have an open set of order $n-1$. Therefore at least one card in $\mathcal{D}(X)$, say X_y, must contain an open set of order $n-1$. Suppose, to the contrary, that X_y contains two distinct $(n-1)$ -point open sets, say U_1 and U_2. If U_1 or U_2 is open in X, then we have an $(n-1)$ -point open set in X, a contradiction. Otherwise, U1 $\cup$ {y} and $U_2 \cup \{y\}$ are open in X and so {y} is open in X, again a contradiction. Hence the $(n-1)$ -point open set is unique in X_y. If any card would contain an open set of order fewer than $n-1$, then the space X would contain an open set of order fewer than n, a contradiction.

Sufficiency: If X would contain an $(n-1)$ -point open set, then totally $n-1$ cards would contain an $(n-2)$ −point open set and hence $\mathcal{D}(X)$ would contain a card having an $(n-2)$ −point open set, giving a contradiction and completing the proof.

Theorem 8. Any almost discrete space X without isolated point is reconstructible if X has a finite open set.

Proof. Let the minimum order of an open set in X be n. Then, by Lemma 7, there exists a card, say X_x, containing a unique $(n-1)$-point open set, say U, such that no card in $\mathcal{D}(X)$contains an open set of order fewer than $n-1$. Now $\tau_{X_x} = \{\varphi, U, U_1, U_2, \dots, X_x\}$. Since U is not open in X, $U \cup \{x\}$ must be open in X. Also all n −point open sets of X_x are open in X. Thus, $\tau_X = \{V : V \in \tau_{X_x}, V \cap U = \varphi\} \cup \{V \cup \{x\} : V \in \tau_{X_x}, V \cap U \neq \tau(X_x)\ \varphi\}$ is the desired topology on X.

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