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Optimal Control Strategies for the SEAIR Model: Dynamics of COVID-19 in Metropolitan Cities of Tamil Nadu, India

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Abstract: This study investigates the optimal control strategies for the dynamics of COVID-19 transmission in metropolitan cities of Tamil Nadu, India, by using the SEAIR (Susceptible-Exposed-Asymptomatic-Infected-Recovered) model. Two control interventions were introduced to target the susceptible and asymptomatic populations, which aims to minimize both disease transmission and its intervention costs. The SEAIR model as an optimal control problem was formulated and solved using Pontryagin's Maximum Principle and numerical optimization techniques. Results highlight the effectiveness of dual control strategies in reducing infection rates while balancing intervention costs. Comparison of SEAIR model dynamics with and without optimal control are visulaized. Optimal control strategies and its cost analysis for various control scenarios has studied numerically by using NumPy and SciPy libraries. The dynamics of the SEAIR model with optimal control is analysed using a Python-based simulation to generate phase portraits. Logistic growth and Gaussian models were used to fit the curve into the dynamics of COVID-19 across three waves in Tamil Nadu and its metropolitan cities. This comprehensive approach provides a framework for policymakers to make a proper decisions on disease management.

1. Introduction

Mathematical modeling is an essential tool to understand the transmission dynamics of infectious diseases, like COVID-19, , which remains a significant public health challenge around the world. The SEAIR model is an extension of the classical SEIR framework that takes into account asymptomatic carriers who contribute to the silent spread of the disease. In this study, an optimal control measures in the SEAIR model [1] is analyzed which is to minimize the infection rates and associated costs.

The theory of optimal control provides a mathematical framework that determines intervention strategies, weighing disease mitigation against economic and societal costs. Theoretical foundations for problems in optimal control were first established by Pontryagin et al. [2] and later further developed by Fleming and Rishel [3], who applied these techniques to dynamic systems.

In the last two decades, optimal control has been highly applied in epidemiological models toward solving several public health issues. For example, Hethcote [4] and Zhou et al. [5] analyzed vaccination and isolation strategies and the corresponding disease dynamics. In fact,

Maurer et al. [6] and do Ros'ario de Pinho et al. [7] highlighted that combined intervention strategies are cost-effective in SEIR and SEAIR models. The Wang et al. [8] and Giordano et al. [9] studies highlight the fact that real-time data integration along with parameter estimation during an outbreak optimizes the control measures.

Specifically, in the COVID-19 period, NPIs became a focus of much discussion. Kucharski et al. [10] were able to emphasize the usefulness of social distancing measures while Li et al. [11] demonstrated the vital role of testing and isolation in controlling transmission. Research carried out by Prem et al. [12] and Arenas et al. [13] explore further how patterns of mobility and social interactions influence the COVID-19 epidemic.

Optimal control strategies in the management of epidemic outbreaks typically focus on minimizing a cost functional involving both infections and costs in relation to intervention. For instance, Silva and Batista [14] utilized Python-based simulations to construct low-cost control strategies, while Zhou et al. [15] used reinforcement learning to optimize control. In more recent work, McKinlay et al. [16] and Larremore et al. [17] have recently integrated vaccination dynamics and testing efficacy into optimal control models, showing that such approaches can be very effective at controlling epidemic effects.

Pandey et al. [18] have incorporated vaccination dynamics and waning immunity in SEIR models to increase the prediction accuracy for nonuniform vaccination. Sharma et al. [19] proposed an extension of the SEAIR model with respect to multiple variants and stressed on transmissibility and immune escape. Lee et al. [20] emphasized the asymptomatic carriers to achieve a long-term prediction of the epidemic, along with the importance of testing.

Akhtar et al. [21] designed a stochastic SEIR model with real-time data for dynamic outbreak forecasting. Bhardwaj et al. [22] also optimized cost-effective NPIs targeting the susceptible and exposed populations. Martinez et al. [23] studied multi-wave epidemics by relating wave periodicity to intervention time. Nguyen et al. [24] modeled mobility patterns of lockdown and reopening effects and Gupta and Verma proposed resource-constrained SEIR models for guiding interventions in low-resource settings [25].

Some recent works highlighted the importance of compartmental models, for example, SEIR and SEAIR in epidemic management. For example, Hethcote [4] shows that SEIR models have been successfully applied in making predictions for different scenarios about the dynamics of diseases. Zhou et al. [5] suggested asymptomatic carriers for better accuracy of prediction in the model. Optimal control theory as applied in epidemiological models also attracts a great deal of interest these days. Fleming and Rishel [3] and Pontryagin et al. [2] provided the necessary theoretical basis for optimal control applications in dynamic systems.

During the COVID-19 pandemic, a large number of studies came out that focused on non-pharmaceutical interventions and vaccination strategies, to understand the dynamics of the COVID-19 disease. For example, Kucharski et al. [10] conducted a comprehensive analysis that quantified the impact of social distancing on COVID-19 transmission to show that the interventions were effective in reducing the rates of infections, which emphasized the significance of behavioral changes in public health responses.

In a related work, Li et al. [11] underlined the importance of testing and isolation in breaking the chain of virus spread. They illustrated that if cases are detected early, isolation of confirmed cases indeed limited the spread substantially, requiring effective testing

infrastructure and public adherence to isolation procedures. This study was particularly important for outbreak settings where the rate of change in an outbreak can be extremely fast, so detection is very crucial for timely containment measures.

Maurer et al. [6] and do Ros'ario de Pinho et al. [7] specifically focused on the Indian scenario to study the cost-effectiveness of combined intervention strategies with the help of SEIR (Susceptible – Exposed – Infectious – Recovered) and SEAIR (Susceptible – Exposed – Asymptomatic – Infectious – Recovered) models, to understand how multiple interventions could be used in a resource-constrained country like India to optimize resource allocation while improving health outcomes.

The SEAIR model [1] is governed by the following system of nonlinear differential equations:

$$\begin{cases} \frac{dS}{dt} &= \Lambda - \beta S(I + \kappa A) - dS \\ \frac{dE}{dt} &= \beta S(I + \kappa A) - (\epsilon + d)E \\ \frac{dA}{dt} &= \delta\epsilon E - (\gamma' + d)A \\ \frac{dI}{dt} &= (1-\delta)\epsilon E - (\gamma + \alpha + d)I \\ \frac{dR}{dt} &= \gamma I + \gamma' A - dR \end{cases} \quad (1)$$

where, S, E, A, I, and R represent the populations of susceptible, exposed, asymptomatic, infected, and recovered individuals, respectively. The parameters λ is the recruitment rate, β is the transmission rate, $\kappa$is the relative infectiousness of asymptomatic individuals compared to symptomatic individuals, d is the natural death rate, ϵ is the rate at which exposed individuals become infectious, δ is the proportion of exposed individuals who become asymptomatic, γ and γ' is the recovery rate of symptomatic and asymptomatic infected individuals respectively, and α is the disease-induced death rate of symptomatic infected individuals.

The SEAIR model developed for this study consists of two major control variables: the first one being the application of measures of social distancing and vaccination for the control of the susceptible population and the second being the measure of reducing the asymptomatic transmission through testing and isolation. The optimal control problem is framed based on these two controls and solved by the Pontryagin's Maximum Principle along with the numerical optimization technique.

The structure of this paper is as follows: Section 2 presents the formulation of the optimal control problem on the SEAIR model given in Eq.(1) with the description of state equations, control variables, and the objective functional, and the existence of optimal controls is established theoretically by using Pontryagin's Maximum Principle. In Section 3, the numerical solutions of the optimal control problem, the comparison of SEAIR model dynamics with and without optimal control, Optimal Control Strategy for the SEAIR Model, and the cost analysis for various control scenarios are described numerically. Furthermore, the logistic growth model and the Gaussian curve fitting have been employed to analyze the dynamics of three waves of COVID-19 transmission in Tamil Nadu, India, and the five metropolitan cities in Tamil Nadu: Chennai, Coimbatore, Madurai, Trichy, and Salem. This paper ends with a conclusion of this work in section 4.

2. Optimal Control Problem

The formulation of an optimal control problem for the SEAIR model (1) aims to minimize the spread of disease while balancing the costs of interventions. The major goals are to reduce the number of individuals susceptible (S) in the population, by limiting contact with infected people, through maneuvers such as vaccination, individual protection measures and social distancing; and, to reduce the asymptomatic infected population (A), with strategies like isolation and quarantine, testing, and medical treatment to prevent the apparently healthy from being able to transmit the infection and to reduce the severity of their infections.

Existence of an Optimal Control Pair

Consider two Lebsegure measurable control functions $u_1(t)$ and $u_2(t)$. The control variable $u_1(t)$ is defined to reduce the susceptible populations (S) and the control variable $u_2(t)$ is defined to reduce the asymptomatic (A) populations over a finite time interval $[0, T]$. The control problem is to minimize the cost objective functional:

$$J(u_1, u_2) = \int_0^T [w_1 S(t) + w_2 A(t) + \frac{1}{2} C_1 u_1^2(t) + \frac{1}{2} C_2 u_2^2(t)]\, dt, \quad (2)$$

subject to the following state equations:

$$\frac{dS}{dt} = \Lambda - \beta S(I + \kappa A) - dS - u_1 S, \quad (3)$$

$$\frac{dE}{dt} = \beta S(I + \kappa A) - (\epsilon + d)E, \quad (4)$$

$$\frac{dA}{dt} = \delta \epsilon E - (\gamma' + d)A - u_2 A, \quad (5)$$

$$\frac{dI}{dt} = (1 - \delta)\epsilon E - (\gamma + \alpha + d)I, \quad (6)$$

$$\frac{dR}{dt} = \gamma I + \gamma' A - dR, \quad (7)$$

with control constraints:

$$0 \le u_1(t), u_2(t) \le u_{\max}, \quad \forall t \in [0, T]. \quad (8)$$

Here, w_1, w_2 are weights for minimizing $S(t)$ and $A(t)$, and C_1, C_2 represent the cost coefficients associated with the controls.

Theorem 1 *There exists an optimal control pair* (u_1^*, u_2^*) *and corresponding state variables* S^*, E^*, A^*, I^*, R^* *to the model ((1)) that minimize the cost functional* $J(u_1, u_2)$ *over all admissible control pairs in*

$$\Gamma = \{(u_1, u_2) \in L^\infty([0, T]) \times L^\infty[0, T] \ : \ (u_1(t), u_2(t)) \in [0, u_{max}] \times [0, u_{max}] \ \forall \ t \in [0, T]\}.$$

Proof. The proof follows the standard existence results in optimal control theory, utilizing conditions from the theorem of Fleming and Rishel [3]. We verify the following requirements:

1. Nonemptiness of the admissible set: The set of admissible controls (u_1, u_2) is nonempty since the constraints $0 \le u_1, u_2 \le u_{\max}$ allow for feasible control functions.

2. Boundedness of solutions: Define $N(t) = S(t) + E(t) + A(t) + I(t) + R(t)$ as the total population at time t. From the state equations, $\frac{dN}{dt} \le \Lambda - dN$, ensuring $N(t) \le \max\left(N(0), \frac{\Lambda}{d}\right)$. Thus, each state variable S, E, A, I, R is nonnegative and bounded. Furthermore, the existence and boundedness of solutions follow from [1].

3. Convexity of the integrand in the controls: The cost functional integrand

$$L(S, A, u_1, u_2) = w_1 S + w_2 A + \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2 \quad (9)$$

is convex in the control variables u_1 and u_2 because it includes quadratic terms for the controls.

4. Compactness of the admissible control set: The admissible control set Γ is closed, convex, and bounded in $L^{\infty}([0, T])$, ensuring compactness.

5. Continuity and Lipschitz property of the state equations: The state equations are linear in the controls u_1 and u_2, and the nonlinear terms involving S, E, A, I, R are continuously differentiable. To prove that the state equations satisfy the Lipschitz condition, we need to show that there exists a constant $C > 0$ such that for any two state vectors $X_1 = (S_1, E_1, A_1, I_1, R_1)$ and $X_2 = (S_2, E_2, A_2, I_2, R_2)$, the following inequality holds:

$$\| \dot{X}_1 - \dot{X}_2 \| \leq L \| X_1 - X_2 \|,$$

where $\|\cdot\|$ denotes the Euclidean norm.

Let $\Delta X = X_1 - X_2 = (S_1 - S_2, E_1 - E_2, A_1 - A_2, I_1 - I_2, R_1 - R_2)$. We compute the differences in the derivatives:

$$\dot{\Delta}S = \dot{S}_1 - \dot{S}_2 = (\mu N - \beta S_1(I_1 + \kappa A_1) - dS_1 - u_1 S_1) - (\mu N - \beta S_2(I_2 + \kappa A_2) - dS_2 - u_2 S_2), \quad (10)$$

$$\dot{\Delta}E = \dot{E}_1 - \dot{E}_2 = (\beta S_1(I_1 + \kappa A_1) - \epsilon E_1 - dE_1) - (\beta S_2(I_2 + \kappa A_2) - \epsilon E_2 - dE_2), \quad (11)$$

$$\dot{\Delta}A = \dot{A}_1 - \dot{A}_2 = (\delta\epsilon E_1 - \gamma' A_1 - dA_1 - u_2 A_1) - (\delta\epsilon E_2 - \gamma' A_2 - dA_2 - u_2 A_2), \quad (12)$$

$$\dot{\Delta}I = \dot{I}_1 - \dot{I}_2 = ((1 - \delta)\epsilon E_1 - \gamma I_1 - dI_1) - ((1 - \delta)\epsilon E_2 - \gamma I_2 - dI_2), \quad (13)$$

$$\dot{\Delta}R = \dot{R}_1 - \dot{R}_2 = (\gamma I_1 - dR_1) - (\gamma I_2 - dR_2). \quad (14)$$

These equations (10) to (14) can be rewirtten as a simplifed form:

$$\dot{\Delta}X = F(X_1) - F(X_2),$$

where $F(X)$ represents the vector of state equations.

The equation (10) can be reformed as:

$$\dot{\Delta}S = -\beta(S_1(I_1 + \kappa A_1) - S_2(I_2 + \kappa A_2)) - d(\Delta S) - (u_1 S_1 - u_2 S_2).$$

Using the triangle inequality, we found that

$$|\dot{\Delta}S| \leq \beta|S_1(I_1 + \kappa A_1) - S_2(I_2 + \kappa A_2)| + d|\Delta S| + |u_1 S_1 - u_2 S_2|.$$

Applying the product rule and the triangle inequality again, we have:

$$|S_1(I_1 + \kappa A_1) - S_2(I_2 + \kappa A_2)| \leq |S_1||I_1 + \kappa A_1| + |S_2||I_2 + \kappa A_2| \leq M \| X_1 - X_2 \|,$$

where M is a constant that bounds the state variables.

Similarly, we can analyze the other equations (11), (12), (13) and (14), it can be found that:

$$\left|\dot{\Delta}E\right| \leq \beta|S_1(I_1 + \kappa A_1) - S_2(I_2 + \kappa A_2)| + \epsilon|\Delta E| + d|\Delta E|,$$

$$|\dot{\Delta}A| \leq \delta\epsilon|\Delta E| + \gamma'|\Delta A| + d|\Delta A| + u_2|\Delta A|,$$

$$|\dot{\Delta}I| \leq (1 - \delta)\epsilon|\Delta E| + \gamma|\Delta I| + d|\Delta I|,$$

$$|\dot{\Delta}R| \leq \gamma|\Delta I| + d|\Delta R|.$$

Combining these inequalities, we can express the overall change in the state vector as:

$$\| \dot{\Delta}X \| \leq C \| \Delta X \|, \quad (15)$$

where C is a constant that encompasses all the Lipschitz constants from each component. By satisfying these conditions, the existence of an optimal control pair (u_1^*, u_2^*) follows from the standard results in optimal control theory [3].

Characterization of Optimal Controls

The characterization of optimal controls forms the basis for determining the control strategies that control the spread of disease and then balance the costs of these interventions in the SEAIR model. Using Pontryagin's Maximum Principle, we derive the conditions for optimality through building the Hamiltonian function to determine the adjoint equations and then identify the laws that are optimal controls. Again, the controls lie between biologically meaningful bounds but are allowed to encompass even the regular and singular situations. The following theorem captures these results and gives a complete characterization of optimal controls for the SEAIR model.

Theorem 2 (Characterization of Optimal Controls for the SEAIR Model) *For the SEAIR model (1), the optimal control pair* (u_1^*, u_2^*) *that minimize the cost objective functional ((2))*

$$J(u_1, u_2) = \int_0^T \left(w_1 S + w_2 A + \frac{1}{2} C_1 u_1^2 + \frac{1}{2} C_2 u_2^2 \right) dt$$

subject to the state equations (3) − (7) and control constraints (8), are characterized by the following:

1. The Hamiltonian function is given by:

$$H(X, p, u_1, u_2) = L(S, A, u_1, u_2) + \langle p, F(X, u_1, u_2) \rangle,$$

where:

$$L(S, A, u_1, u_2) = w_1 S + w_2 A + \tfrac{1}{2} C_1 u_1^2 + \tfrac{1}{2} C_2 u_2^2,$$
$$X = (S, E, A, I, R),$$
$$p = (\lambda_S, \lambda_E, \lambda_A, \lambda_I, \lambda_R),$$

and $F(X, u_1, u_2)$ represents the state equations (3) − (7).

2. The adjoint equations are:

$$\dot{\lambda}_S = -\frac{\partial H}{\partial S} = -w_1 + \lambda_S(\beta(I + \kappa A) + d + u_1) - \lambda_E \beta(I + \kappa A),$$
$$\dot{\lambda}_E = -\frac{\partial H}{\partial E} = \lambda_E(\epsilon + d) - \lambda_A \delta \epsilon - \lambda_I(1 - \delta)\epsilon,$$
$$\dot{\lambda}_A = -\frac{\partial H}{\partial A} = -w_2 + \lambda_S \beta \kappa S - \lambda_A(\gamma' + d + u_2),$$
$$\dot{\lambda}_I = -\frac{\partial H}{\partial I} = \lambda_S \beta S - \lambda_I(\gamma + \alpha + d),$$
$$\dot{\lambda}_R = -\frac{\partial H}{\partial R} = -\lambda_R d.$$

3. The adjoint variables λ_S, λ_E, λ_A, λ_I, λ_R satisfy the transversality conditions, which are given by:

$$\lambda_S(T) = \lambda_E(T) = \lambda_A(T) = \lambda_I(T) = \lambda_R(T) = 0.$$

4. The optimal controls are determined by solving:

$$\frac{\partial H}{\partial u_1} = C_1 u_1 - \lambda_S S = 0,$$
$$\frac{\partial H}{\partial u_2} = C_2 u_2 - \lambda_A A = 0.$$

This yields the control laws:

$$u_1^*(t) = \min\left(\max\left(0, \frac{\lambda_S S}{C_1} \right), u_{\max} \right),$$
$$u_2^*(t) = \min\left(\max\left(0, \frac{\lambda_A A}{C_2} \right), u_{\max} \right).$$

5. For singular controls, the switching functions satisfy $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$. Higher-order derivatives of the switching functions are used to characterize the controls, with the Generalized Legendre-Clebsch Condition ensuring optimality:

$$-\frac{d}{dt}\left(\frac{\partial^2 H}{\partial u_1^2}\right) \geq 0, \quad -\frac{d}{dt}\left(\frac{\partial^2 H}{\partial u_2^2}\right) \geq 0.$$

Proof. We used the Pontryagin's Maximum Principle [2] to derive the necessary conditions for optimality:
The Hamiltonian function $H(X, p, u_1, u_2)$ is constructed as a sum of the Lagrangian $L(S, A, u_1, u_2)$ and the state equations $F(X, u_1, u_2)$ weighted by the adjoint variables $\lambda_S, \lambda_E, \lambda_A, \lambda_I, \lambda_R$.

Adjoint Equations are constructed by Differentiating partially the Hamiltonian with respect to each state variable $S,\ E,\ A,\ I,\ R$, which describe the evolution of the adjoint variables with respect to the time.

Transversality Conditions: At the terminal time T, the adjoint variables satisfy the boundary conditions $\lambda_S(T) = \lambda_E(T) = \lambda_A(T) = \lambda_I(T) = \lambda_R(T) = 0$.

The optimal controls u_1^* and u_2^* are found from solving the equations $\frac{\partial H}{\partial u_1} = 0$ and $\frac{\partial H}{\partial u_2} = 0$ for u_1^* and u_2^* within their boundries $0 \leq u_1, u_2 \leq u_{\max}$.

For singular controls, the higher-order derivatives of the switching functions $\phi_1(t)$ and $\phi_2(t)$ are computed. Generalized Legendre-Clebsch Condition guarantees that the singular controls obtained also meet the necessary optimality conditions. This ends the proof of the theorem.

The optimal control framework for the SEAIR model identifies those control strategies that minimize the spread of disease while balancing the cost of interventions. Thus, solving the adjoint system and applying the derived control laws would lead to effective strategies in managing the epidemic.

Theorem 3 (Uniqueness of the Optimality System) *For the SEAIR model (1), if the time T is sufficiently small, the optimality system admits a unique solution.*
Proof. To prove the uniqueness of the optimal control, we consider two solutions $(S_1, E_1, A_1, I_1, R_1)$ and $(S_2, E_2, A_2, I_2, R_2)$ of the state equations (5) to (7). To show the equality of the two solutions, we define the differences as follows:

$$\Delta S = S_1 - S_2, \quad (16)$$
$$\Delta E = E_1 - E_2, \quad (17)$$
$$\Delta A = A_1 - A_2, \quad (18)$$
$$\Delta I = I_1 - I_2, \quad (19)$$
$$\Delta R = R_1 - R_2. \quad (20)$$

Differentiating (16) to (20) with respect to time, we obtain the following system of differential equations:

$$\begin{aligned}\dot{\Delta}S &= \mu N - \beta S_1(I_1 + \kappa A_1) - dS_1 - u_1 S_1 \\ &\quad -(\mu N - \beta S_2(I_2 + \kappa A_2) - dS_2 - u_2 S_2), \quad (21)\\ \dot{\Delta}E &= \beta S_1(I_1 + \kappa A_1) - \epsilon E_1 - dE_1\end{aligned}$$

$$-(\beta S_2(I_2 + \kappa A_2) - \epsilon E_2 - dE_2), \quad (22)$$

$$\dot{\Delta}A = \delta\epsilon E_1 - \gamma' A_1 - dA_1 - u_2 A_1 - (\delta\epsilon E_2 - \gamma' A_2 - dA_2 - u_2 A_2), \quad (23)$$

$$\dot{\Delta}I = (1-\delta)\epsilon E_1 - \gamma I_1 - dI_1 - ((1-\delta)\epsilon E_2 - \gamma I_2 - dI_2), \quad (24)$$

$$\dot{\Delta}R = \gamma I_1 - dR_1 - (\gamma I_2 - dR_2). \quad (25)$$

The equations (21) to (25) can be rewritten as a simplified form:

$$\dot{\Delta}X = F(S_1, E_1, A_1, I_1, R_1, u_1) - F(S_2, E_2, A_2, I_2, R_2, u_2),$$

where $\Delta X = (\Delta S, \Delta E, \Delta A, \Delta I, \Delta R)$.

By the Lipschitz continuity of the state equations ((15)), there exists a constant C such that:

$$\frac{d}{dt} \| \Delta X \| \leq C \| \Delta X \|. \quad (26)$$

where, $\| \Delta X \|$ is defined as:

$$\| \Delta X \| = \sqrt{(\Delta S)^2 + (\Delta E)^2 + (\Delta A)^2 + (\Delta I)^2 + (\Delta R)^2}.$$

To apply Gronwall's inequality, we first integrate the inequality ((26)) over the interval from 0 to t:

$$\int_0^t \frac{d}{dt} \| \Delta X \| \, dt \leq \int_0^t C \| \Delta X \| \, dt.$$

This leads to:

$$\| \Delta X(t) \| - \| \Delta X(0) \| \leq Ct \| \Delta X(t) \|.$$

Rearranging the terms:

$$\| \Delta X(t) \| (1 - Ct) \leq \| \Delta X(0) \|.$$

Taking t as sufficiently small such that $Ct < 1$, we can isolate $\| \Delta X(t) \|$:

$$\| \Delta X(t) \| \leq \frac{\| \Delta X(0) \|}{1 - Ct}.$$

Since we have assumed that the initial conditions are equal, we have:

$$\Delta S(0) = S_1(0) - S_2(0) = 0, \quad \Delta E(0) = E_1(0) - E_2(0) = 0, \quad \Delta A(0) = A_1(0) - A_2(0) = 0,$$
$$\Delta I(0) = I_1(0) - I_2(0) = 0, \quad \Delta R(0) = R_1(0) - R_2(0) = 0,$$

we find that $\| \Delta X(0) \| = 0$. Substituting this into our inequality yields:

$$\| \Delta X(t) \| \leq \frac{0}{1 - Ct} = 0.$$

This implies that $\Delta X(t) = 0$ for all t, which means:

$$S_1(t) = S_2(t), \quad E_1(t) = E_2(t), \quad A_1(t) = A_2(t), \quad I_1(t) = I_2(t), \quad R_1(t) = R_2(t).$$

Thus, the uniqueness of the solutions to the optimality system is established.

3. Result and Discussion

3.1. Comparison of SEAIR model dynamics with and without optimal control

The SEAIR model simulation was performed to analyze the effects of optimal control strategies on the dynamics of an epidemic. The model, written in Python using the $NumPy$, $SciPy$, and $Matplotlib$ packages, splits the population into five compartments: Susceptible (S), Exposed (E), Asymptomatic (A), Infected (I), and Recovered (R). Two control variables, u_1 and u_2, were introduced to reduce transmission and progression of infection, respectively. The initial conditions for the variables are considered as $S(0) = 0.99$, $E(0) = 0.01$, $A(0) =$

0.0, $I(0) = 0.0$, and $R(0) = 0.0$ and parameter values are take as $\beta = 0.5$, progression rate $\sigma = 0.2$, and recovery rates for asymptomatic $\gamma' = 0.1$ and symptomatic $\gamma = 0.1$ individuals. The system of ODEs that govern the SEAIR model was solved numerically using the Runge-Kutta method (RK45) implemented in the *solve_ivp* function from *SciPy*. An objective function minimized the combined cost of infection and control efforts with the L-BFGS-B optimization method, resulting in optimal control strategies. The outputs of the model both with and without control are compared over a period of 50 days with the time step taken as 100 points. The plots (Figure1) show that there is substantial reduction in the number of infected (I) and exposed (E) with the optimal control strategy when the control cost is equilibrated.

3.2. Optimal Control Strategy for the SEAIR Model

In this model, two time-dependent control interventions are adopted, namely, $u_1(t)$ and $u_2(t)$, where $u_1(t)$ reduces the susceptible rate, say by imposing social distancing measures, and $u_2(t)$ minimizes asymptomatic infections, say through effective testing, contact tracing, quarantine, and isolation. The SEAIR model equations were solved numerically through the function solve_ivp of the scipy.integrate module using RK45 method.

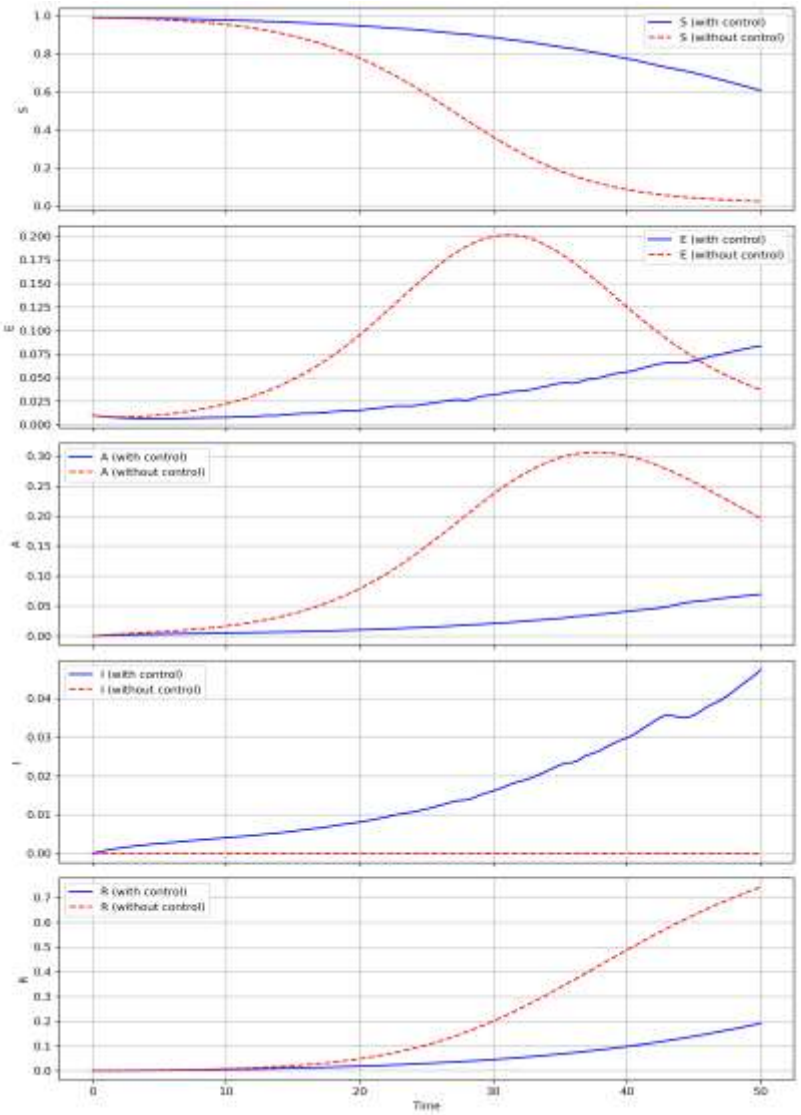

Figure 1: Comparison of SEAIR model dynamics with and without optimal control strategies over a time period

The optimization problem was set up with the goal of minimizing a cost functional, which included the weighted sum of the infected population and the quadratic cost of the control interventions. The control variables are limited into $0 \leq u_1, u_2 \leq 1$ for all time . The optimization is performed using the minimize function from the scipy.optimize module with the L-BFGS-B algorithm, which is very suitable for constrained optimization problems.

The initial conditions for the compartments of SEAIR were provided as follows: $S(0) = 0.99$, $E(0) = 0.01$, $A(0) = 0$, $I(0) = 0$, and $R(0) = 0$, representing an initially susceptible population of 99%, 1% exposed but no active asymptomatic nor symptomatic infected cases. The model parameters were taken as $\beta = 0.5$, $\sigma = 0.2$, $\gamma' = 0.1$, and $\gamma = 0.1$. The weights w_1, w_2 and the costs c_1, c_2 for the cost functional were taken as 0.1.

The graph, Figure 2, shows the dynamics of the optimal control interventions $u_1(t)$ and $u_2(t)$ over the time period. Initially, the controls $u_1(t)$ and $u_2(t)$ take high values, but gradually, as the epidemic fades out, so do the control measures, as indicated by the trend in Figure 2. This clearly manifests the dynamic nature of the optimal control strategy that brings together the reduction of infections with the minimization of costs in intervention.

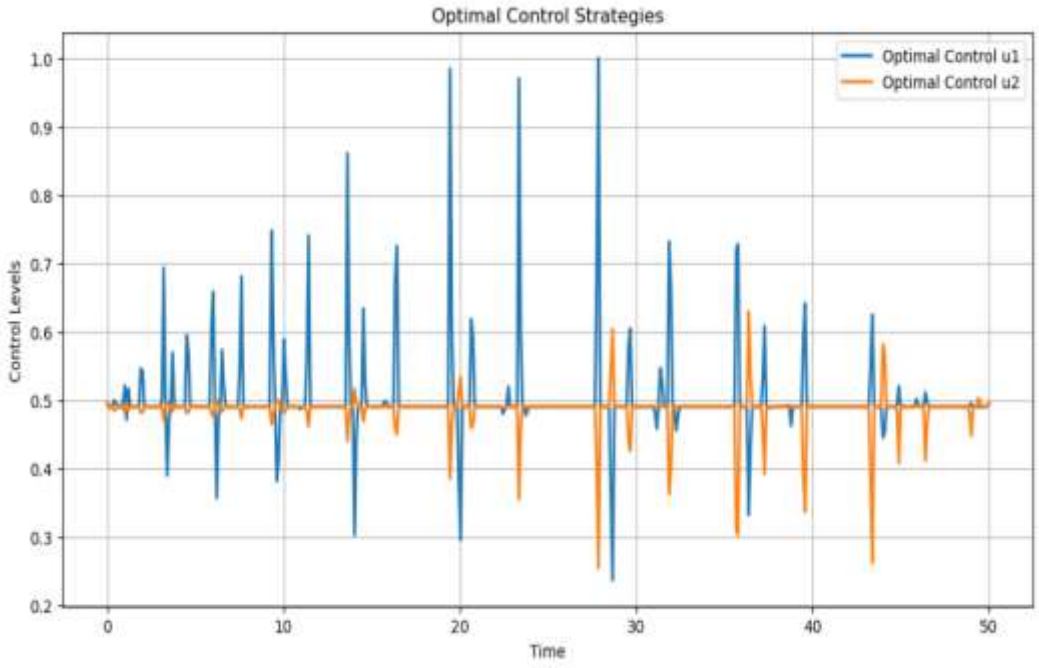

Figure 2: The dynamics of the optimal control strategies $u_1(t)$ and $u_2(t)$ over time

3.3. Cost Analysis for Various Control Scenarios

We compared the total cost $J(u_1, u_2)$ of implementing different types of controls in the SEAIR model. The computation used a cost for four types of controls: no control ($u_1 = 0$ and $u_2 = 0$), a control only on the susceptible population ($u_1(t) \neq 0$ and $u_2 = 0$), a control only on the asymptomatic population ($u_1(t) = 0$ and $u_2 \neq 0$), and both of them implemented simultaneously targeting susceptible and asymptomatic populations (both $u_1 \neq 0$ and $u_2 \neq 0$).

In the no-control scenario ($u_1 = 0, u_2 = 0$), the total cost was relatively high at 571715.94. This is expected because there is no control measure, allowing the disease to spread without

restriction, leading to higher costs due to larger susceptible and asymptomatic populations. When only the control on the susceptible population ($u_1(t) = 0.3 \neq 0\ u_2 = 0$) was applied, the total cost decreased slightly to 569250.29. This implies that reducing the susceptible population slows down the spread of the disease but does not solve the problem of asymptomatic spread, which is still driving the transmission of the disease. The higher cost as compared to the no-control scenario shows that concentrating on one group alone is not sufficient because the disease keeps spreading through asymptomatic people.

When control on the asymptomatic population was applied ($u_2(t) = 0.3 \neq 0,\ u_1 = 0$), then the total cost was decreased to 113702.33. This outcome suggests that controlling the asymptomatic population is more cost-effective than targeting the susceptible population alone. Asymptomatic individuals are the largest reservoirs of transmission, as they do not have symptoms but can transmit.

In the last scenario, where both controls ($u_1 = 0.3 \neq 0$ and $u_2 = 0.3 \neq 0$) were implemented simultaneously, the total cost was 113225.94. Although this strategy targets both the susceptible and asymptomatic populations, it incurred the lowest cost among all scenarios involving control measures. This suggests that while dual control is more comprehensive in controlling disease spread, it does not require as many resources as initially expected when both populations are managed together. In turn, this may lead to better long-term outcomes through a greater reduction in the overall disease burden and prevention of larger outbreaks and costs of healthcare.

These findings point out the trade-off between control effectiveness and the costs. The no-control case results in the largest spread of disease and largest costs. Targeting the asymptomatic population is a better cost-effective strategy than the susceptible population alone. It appears that the best strategy from a disease control point of view is to use both controls together because the total cost is lowest when this is done. Still, the dual-control approach points out the need for balancing short-term costs against long-term benefits in health decision-making. More work and simulations are required to identify the most cost-effective approaches under different epidemic conditions and resource availability.

Control Scenario	u_1	u_2	**Total Cost** $J(u_1, u_2)$
No Control	0	0	571715.94
Single Control (u1 = 0.3)	0.3	0	569250.29
Single Control (u2 = 0.3)	0	0.3	113702.32
Both Controls (u1 = 0.3, u2 = 0.3)	0.3	0.3	113225.94
No Control	0	0	571715.94
Single Control (u1 = 0.6)	0.6	0	567029.46
Single Control (u2 = 0.6)	0	0.6	63435.75
Both Controls (u1 = 0.6, u2 = 0.6)	0.6	0.6	62887.43
No Control	0	0	571715.94
Single Control (u1 = 0.8)	0.8	0	565667.31
Single Control (u2 = 0.8)	0	0.8	49151.92
Both Controls (u1 = 0.8, u2 = 0.8)	0.8	0.8	48574.16

Table 1: Total Cost for Various Control Scenarios

The table 1 summarizes the total costs of various control scenarios within the SEAIR model.

Different values of the control variables u_1 and u_2 are used. Computations were done using Python, leveraging scientific libraries such as NumPy to solve the differential equations and SciPy to compute the costs for each scenario.

The table 1 presents four different control scenarios with different values of u_1 and u_2 as follows:

1. **No Control** ($u_1 = 0,\ u_2 = 0$) This scenario is for the case when no controls are implemented, which leads to the highest cost of 571715.94, as there is uncontrolled disease transmission.

2. **Single Control on Susceptible Population** ($u_1 = 0.3$ or $u_1 = 0.6$ or $u_1 = 0.8$, $u_2 = 0$) The costs are slightly reduced as u_1 increases, and the total costs lie between 569250.29 for $u_1 = 0.3$ and 565667.31 for $u_1 = 0.8$. This shows that increasing the control on the susceptible group helps slow down disease transmission, but it is not sufficient on its own to significantly reduce costs.

3. **Single Control on Asymptomatic Population** ($u_1 = 0$, $u_2 = 0.3$ or $u_2 = 0.6$ or $u_2 = 0.8$) In the above three scenarios, only asymptomatic population is targeted for control. The costs reduced significantly to as low as 49151.92 while $u_2 = 0.8$ and thus controlling the asymptomatic group which is the leading cause of transmission is effective than targeting the susceptible alone.

4. **Both Controls Simultaneously** ($u_1 = 0.3,\ u_2 = 0.3$ or $u_1 = 0.6,\ u_2 = 0.6$ or $u_1 = 0.8,\ u_2 = 0.8$) The last batch of experiments involves applying both controls simultaneously. The costs for this integrated strategy are always the lowest, with values of 113225.94 for $u_1 = 0.3$, $u_2 = 0.3$, 62887.43 for $u_1 = 0.6$, $u_2 = 0.6$, and 48574.16 for $u_1 = 0.8$, $u_2 = 0.8$. This indicates that a dual-control strategy is the most cost-effective way of decreasing the total burden of the disease.

These results have provided insights into the relative effectiveness of the various control strategies with associated costs. Control directed towards the asymptomatic population or simultaneously in both populations yields the most desirable reductions in cost, but focus on the susceptible population alone is insufficient to suppress the transmission of disease. The implication here is that it is very essential to target both the susceptible and asymptomatic populations to have better control and manage the costs of the disease.

3.4. Phase Portrait Analysis

The dynamics of the SEAIR model with optimal control is analysed using a Python-based simulation to generate phase portraits. The SEAIR model with the controls $u_1(t)$ and $u_2(t)$, were solved numerically by the Runge-Kutta method of order 5 (RK45), using the function solve_ivp from the library SciPy. The initial conditions for the simulation are: $S(0) = 0.99$, $E(0) = 0.01$, $A(0) = 0.0$, $I(0) = 0.0$, and $R(0) = 0.0$ and the key parameter values used in the model are $\beta = 0.5$, $\kappa = 0.4$, $\delta = 0.6$, $\epsilon = 0.2$, $\gamma = 0.1$, $\gamma' = 0.15$, $\alpha = 0.05$, $d = 0.01$, and $\Lambda = 0.02$. The control values are taken as $u_1 = 0.3$ and $u_2 = 0.4$. Two phase portraits were generated to visualize the trajectories of the system and the effects of optimal control:

The phase portrait $S(t)$ vs. $I(t)$ (Figure 3(a)) illustrates that if the number of susceptible persons (S) is declining, then the number of infected persons (I) is increasing. This again represents the typical behavior of an epidemic, wherein the disease spreads from susceptible

persons to infected persons.

This phase portrait $A(t)$ vs. $R(t)$ (Figure 3(b)) shows the relationship between asymptomatic people (A) and recovered people (R). The curve shows that as the number of asymptomatic people increases, the number of recovered people also increases. This indicates that the asymptomatic people eventually recover from the disease.

The simulation was done in Python using the NumPy library for numerical computations, SciPy for solving differential equations, and Matplotlib for plotting the phase portraits.

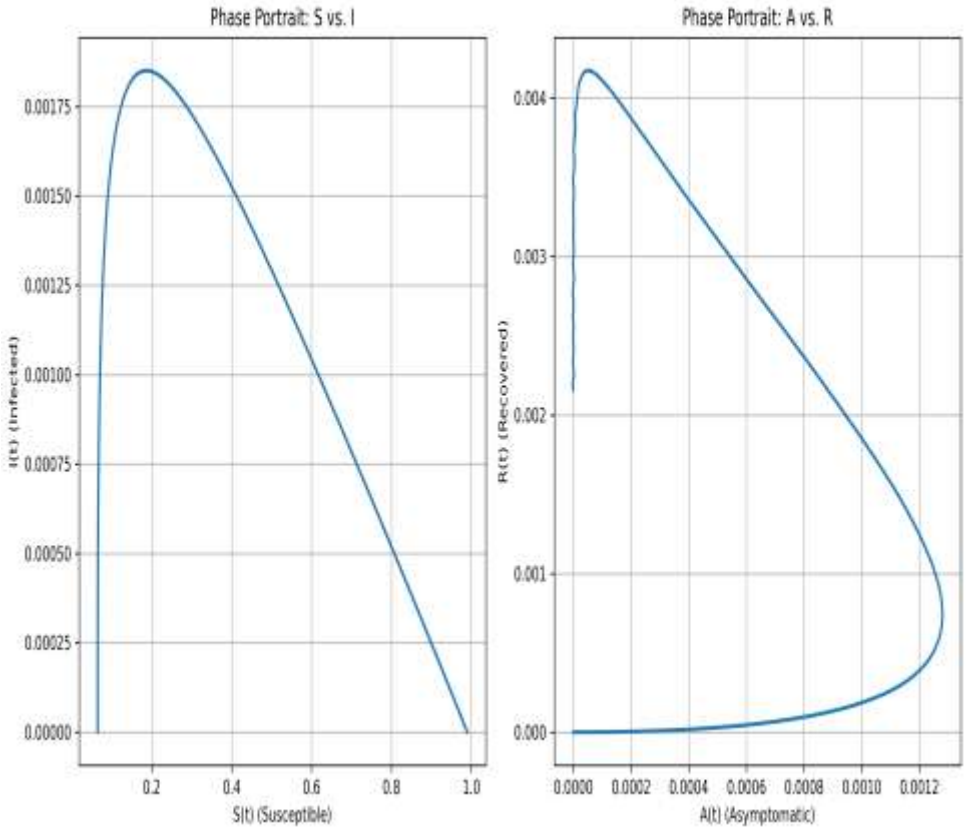

Figure 3: Phase portraits of the SEAIR model under optimal control: (a) S(t) vs. I(t) shows the interaction between susceptible and infected populations over time, and (b) A(t) vs. R(t) illustrates the relationship between asymptomatic and recovered populations.

3.5. Curve Fitting Analysis

In this study, two statistical techniques, the logistic growth model and the Gaussian curve fitting, have been employed to analyze the dynamics of three waves of COVID-19 transmission in Tamil Nadu and the five metropolitan cities in Tamil Nadu: Chennai, Coimbatore, Madurai, Trichy, and Salem. We considered the data of total reported cases, total recovered cases, and daily deaths for each region across three waves of COVID-19 to determine the best fit of the data. The logistic growth model is used to fit the data of total reported cases and total recovered cases, whereas Gaussian curve fitting is applied to fit a curve for the daily death data.

3.5.1. Total Cases and Recoveries Analysis Using Logistic Growth Model

The logistic growth model with parameters δ, γ, and κ is defined by the equation

$$f(x) = \frac{\delta \cdot \kappa}{1 + \exp\left(-\frac{(x-\gamma)}{\delta}\right)},$$

where δ is the growth rate, γ is the midpoint, and κ is the upper limit of the data. The 'curve_fit' function from the SciPy library is used to fit the logistic growth model by using Python.

The parameters δ, γ, and κ were obtained by fitting the model to the data using nonlinear curve fitting techniques. The estimated parameters by the curve fitting process for Total Cases for each city and Tamil Nadu in three waves has been given in the Table 2. The estimated parameters by the curve fitting process for Total Recoveries for each city and Tamil Nadu in three waves has been given in the Table 3.

City and Wave	δ	γ	κ
Tamil Nadu Total Cases (Wave 1)	28.41	114.07	27452.98
Tamil Nadu Total Cases (Wave 2)	12.40	78.55	136126.21
Tamil Nadu Total Cases (Wave 3)	5.72	51.35	126410.41
Chennai Total Cases (Wave 1)	32.27	93.65	6210.69
Chennai Total Cases (Wave 2)	11.40	67.73	26416.31
Chennai Total Cases (Wave 3)	5.51	47.72	34669.85
Coimbatore Total Cases (Wave 1)	20.38	111.44	2341.82
Coimbatore Total Cases (Wave 2)	12.10	83.16	14240.98
Coimbatore Total Cases (Wave 3)	5.36	53.65	14793.79
Madurai Total Cases (Wave 1)	17.66	83.39	972.64
Madurai Total Cases (Wave 2)	10.11	75.54	5130.80
Madurai Total Cases (Wave 3)	4.83	31.33	3176.49
Trichy Total Cases (Wave 1)	22.47	110.15	558.67
Trichy Total Cases (Wave 2)	11.74	79.60	4890.18
Trichy Total Cases (Wave 3)	6.17	35.53	2760.60
Selam Total Cases (Wave 1)	23.04	136.54	1356.43
Selam Total Cases (Wave 2)	14.70	88.06	4171.34
Selam Total Cases (Wave 3)	5.28	54.63	4951.54

Table 2: Curve fitting results for Total Cases across Tamil Nadu and metropolitan cities.

City and Wave	δ	γ	κ
Tamil Nadu Total Recover (Wave 1)	26.42	120.36	27894.79
Tamil Nadu Total Recover (Wave 2)	13.40	87.28	126681.44
Tamil Nadu Total Recover (Wave 3)	6.34	58.82	115197.03
Chennai Total Recover (Wave 1)	29.11	99.22	6371.46
Chennai Total Recover (Wave 2)	12.07	74.92	24789.53
Chennai Total Recover (Wave 3)	5.62	55.16	33959.70
Coimbatore Total Recover (Wave 1)	20.18	142.03	2229.31
Coimbatore Total Recover (Wave 2)	12.72	92.78	13788.94
Coimbatore Total Recover (Wave 3)	6.04	61.03	13325.90
Madurai Total Recover (Wave 1)	15.99	92.20	1014.67
Madurai Total Recover (Wave 2)	12.61	87.79	4230.62
Madurai Total Recover (Wave 3)	5.45	57.72	2843.69
Trichy Total Recover (Wave 1)	21.20	117.07	563.64
Trichy Total Recover (Wave 2)	12.32	88.67	4579.55

Trichy Total Recover (Wave 3)	6.77	59.35	2472.90
Selam Total Recover (Wave 1)	21.64	142.88	1364.83
Selam Total Recover (Wave 2)	15.47	96.92	4013.26
Selam Total Recover (Wave 3)	6.50	61.63	4115.94

Table 3: Curve fitting results for Total Recoveries across Tamil Nadu and metropolitan cities.

The graphs shows the curve fitting results for COVID-19 total cases and recoveries in Tamil Nadu (Figure 4), Chennai (Figure 5), Coimbatore (Figure 6), Madurai (Figure 7), Trichy (Figure 8), and Salem (Figure 9). The scatter points in red colour represent the actual data and the curves in blue colour represent the best fitted line. Each subplot corresponds to a specific region and wave.

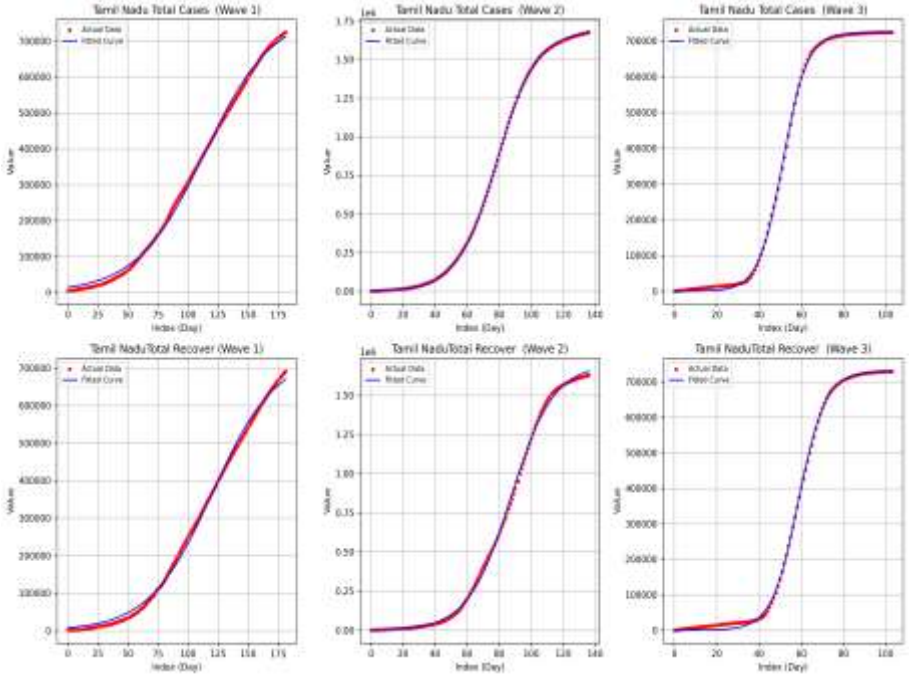

Figure 4: Curve fitting results for COVID-19 total cases and recoveries in Tamil Nadu

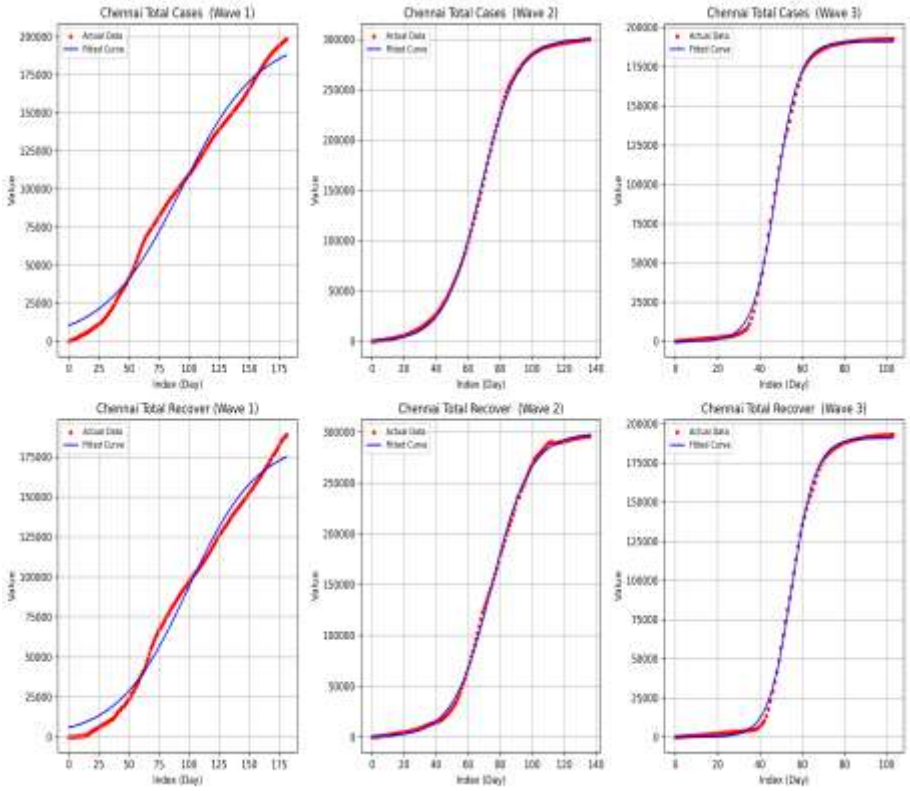

Figure 5: Curve fitting results for COVID-19 total cases and recoveries in Chennai

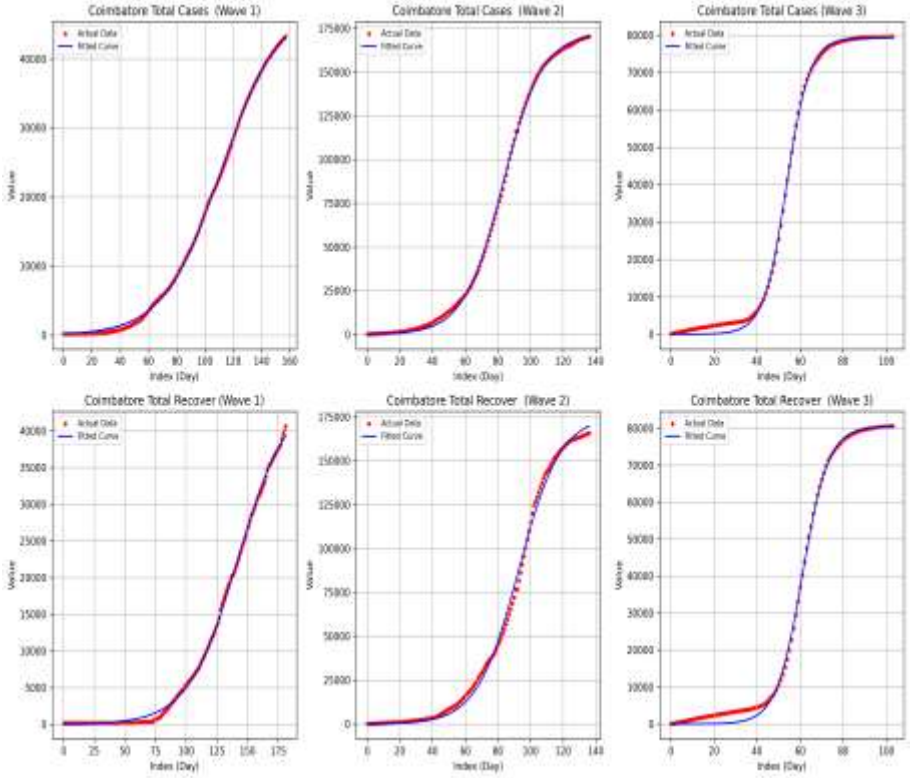

Figure 6: Curve fitting results for COVID-19 total cases and recoveries in Coimbatore

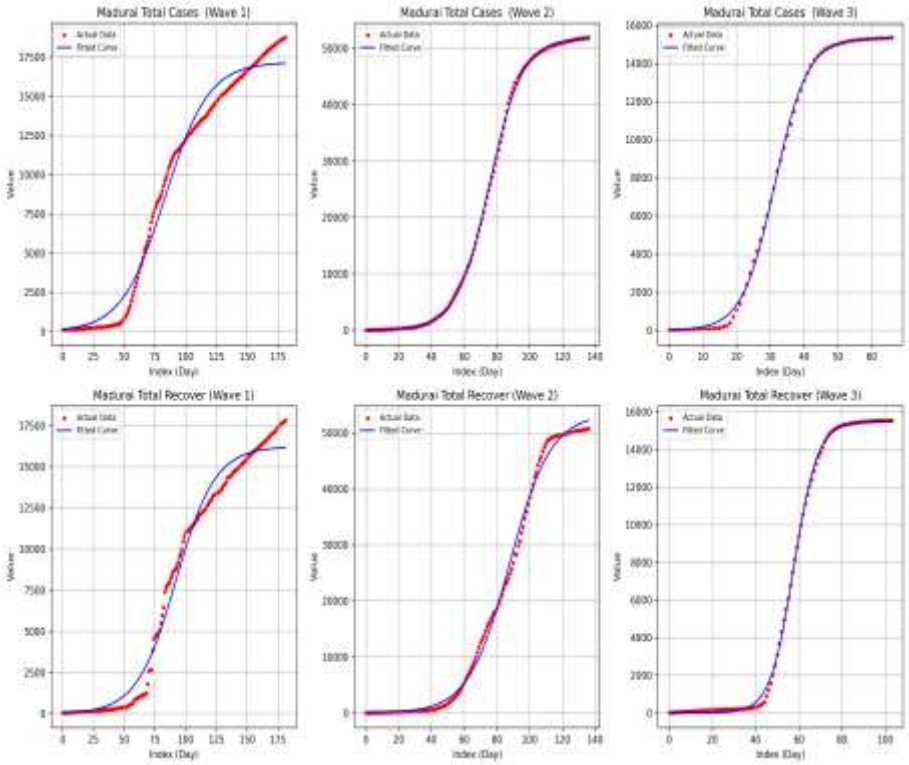

Figure 7: Curve fitting results for COVID-19 total cases and recoveries in Madurai

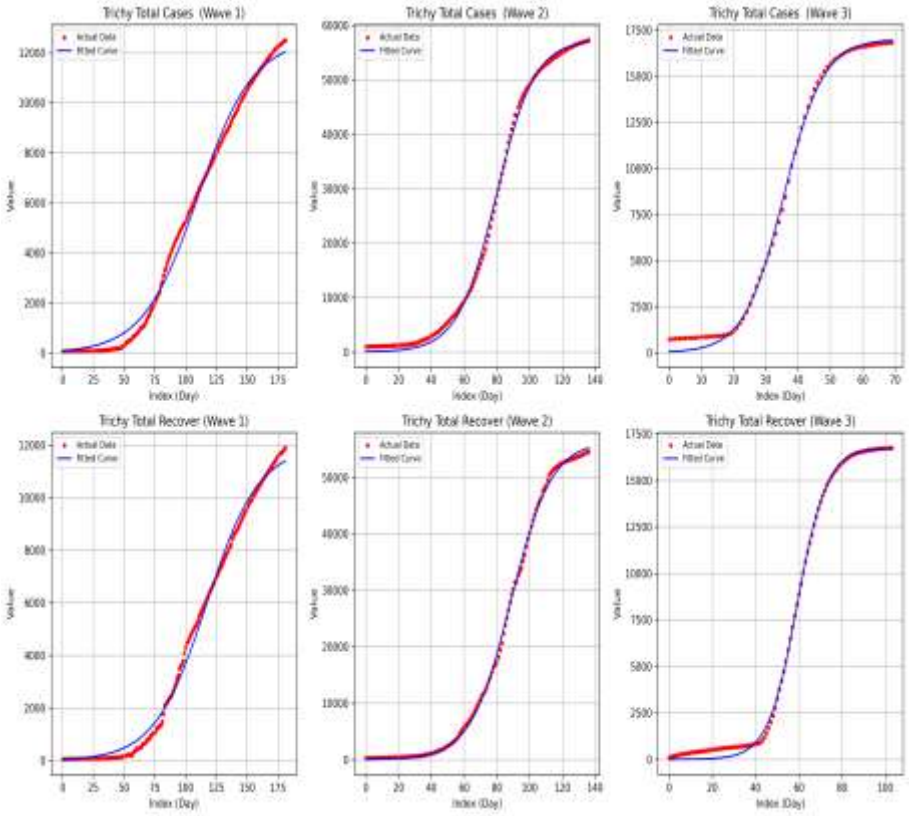

Figure 8: Curve fitting results for COVID-19 total cases and recoveries in Trichy

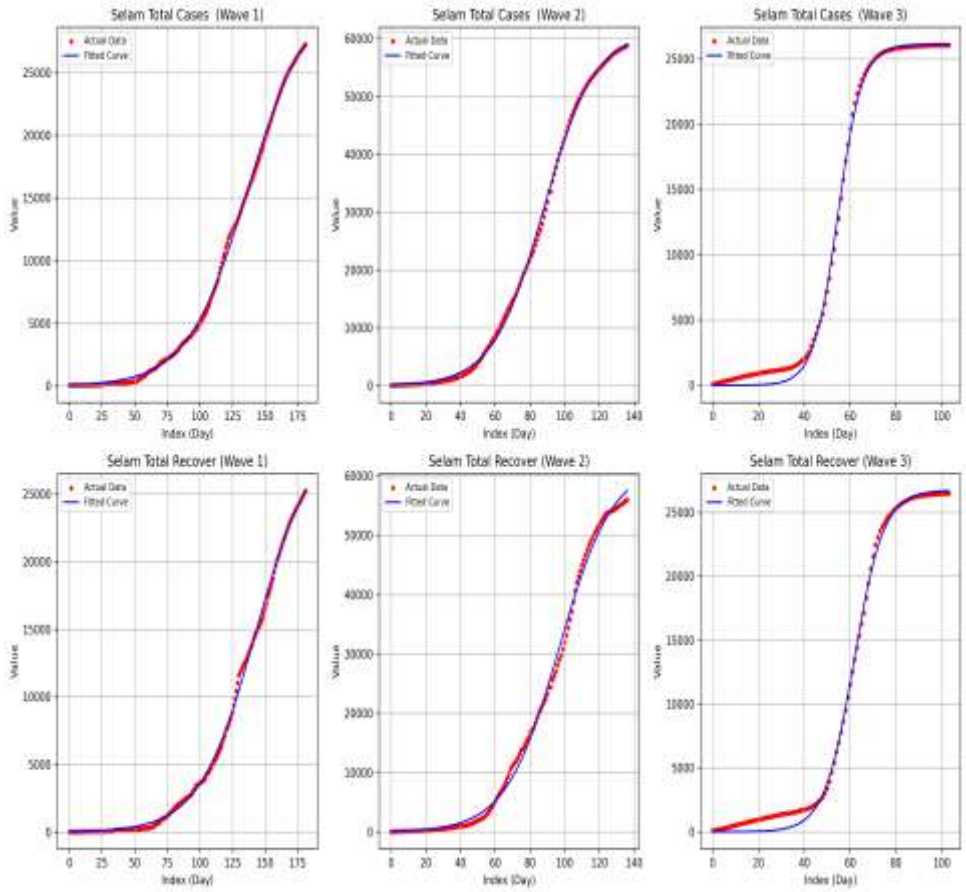

Figure 9: Curve fitting results for COVID-19 total cases and recoveries in Selam

This analysis provides a quantitative understanding of the dynamics of COVID-19. The estimated parameters enable researchers and policymakers to analyze regional differences and predict the progression of cases and recoveries, thereby assisting in the planning of effective interventions.

3.5.2 Daily Deaths Analysis Using Gaussian Model

The Gaussian model with parameters a (amplitude), x_0 (mean), and σ (standard deviation) is defined by the equation

$$f(x) = a \cdot \exp\left(-\frac{(x - x_0)^2}{2\sigma^2}\right),$$

where a is the amplitude which represents the peak value of daily deaths, x_0 is the mean which represents the day on which the peak occurs, and σ is the standard deviation which represents the spread of the curve.

The 'curve_fit' function from the SciPy library in Python was used to fit the Gaussian model to the daily deaths data for Tamil Nadu and its metropolitan cities: Chennai, Coimbatore, Madurai, Trichy, and Salem. The estimated parameters are presented in the Table 4 for three waves of COVID-19.

Region	**Wave**	a **(Amplitude)**	x_0 **(Mean)**	σ **(Standard Deviation)**
Tamil Nadu	Wave 1	1719.30	71.00	17.44
	Wave 2	1422.89	63.00	13.20

	Wave 3	1572.39	90.20	31.62
Chennai	Wave 1	3465.41	161.50	63.15
	Wave 2	4334.92	121.98	34.67
	Wave 3	482.66	87.07	26.67
Coimbatore	Wave 1	525.70	127.82	44.32
	Wave 2	1438.35	125.90	25.92
	Wave 3	158.73	98.53	44.60
Madurai	Wave 1	421.53	151.06	50.98
	Wave 2	680.20	122.79	31.26
	Wave 3	58.37	90.04	27.63
Trichy	Wave 1	162.90	138.94	49.99
	Wave 2	762.58	110.45	28.71
	Wave 3	82.75	93.02	35.57
Salem	Wave 1	405.64	131.17	39.16
	Wave 2	1049.96	109.40	28.69
	Wave 3	55.97	88.00	46.68

Table 4: Gaussian model parameters for daily deaths during three waves of COVID-19 across Tamil Nadu and its metropolitan cities.

The grpahs shows the curve fitting results for COVI-19 daily deaths in Tamil Nadu (Figure 10), Chennai (Figure 11), Coimbatore (Figure 12), Madurai (Figure 13), Trichy (Figure 14), and Salem (Figure 15). The scatter plots of the actual data (red) and the fitted Gaussian curves (blue) for each region are shown in the generated figures. The fitting results provide a quantitative understanding of the dynamics of daily deaths during three waves of COVID-19. The estimated parameters allow for regional comparisons and assist policymakers in planning region-specific interventions. Each wave's parameters reflect the variations in peak days and the spread of the deaths' progression. The amplitude (a) highlights the maximum daily deaths, x_0 represents the day of the peak, and σ describes how sharply the deaths curve rises and falls around the peak. This analysis underscores the temporal and regional variations in the impact of COVID-19.

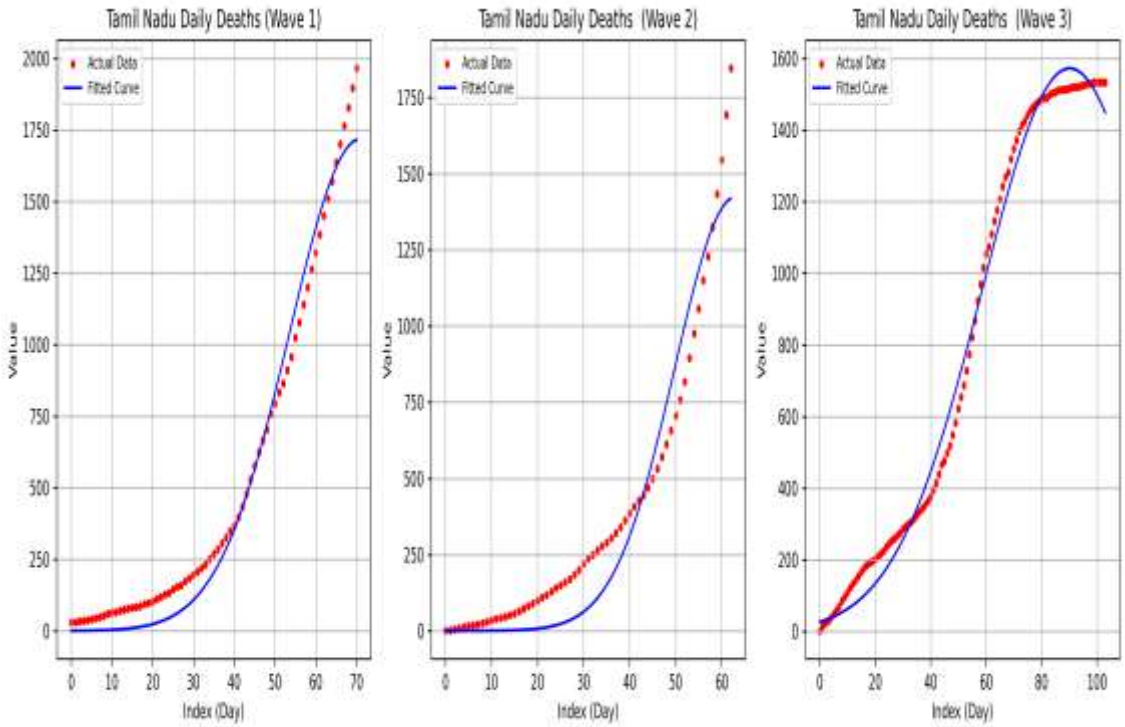

Figure 10: Curve fitting results for COVID-19 daily deaths in Tamil Nadu

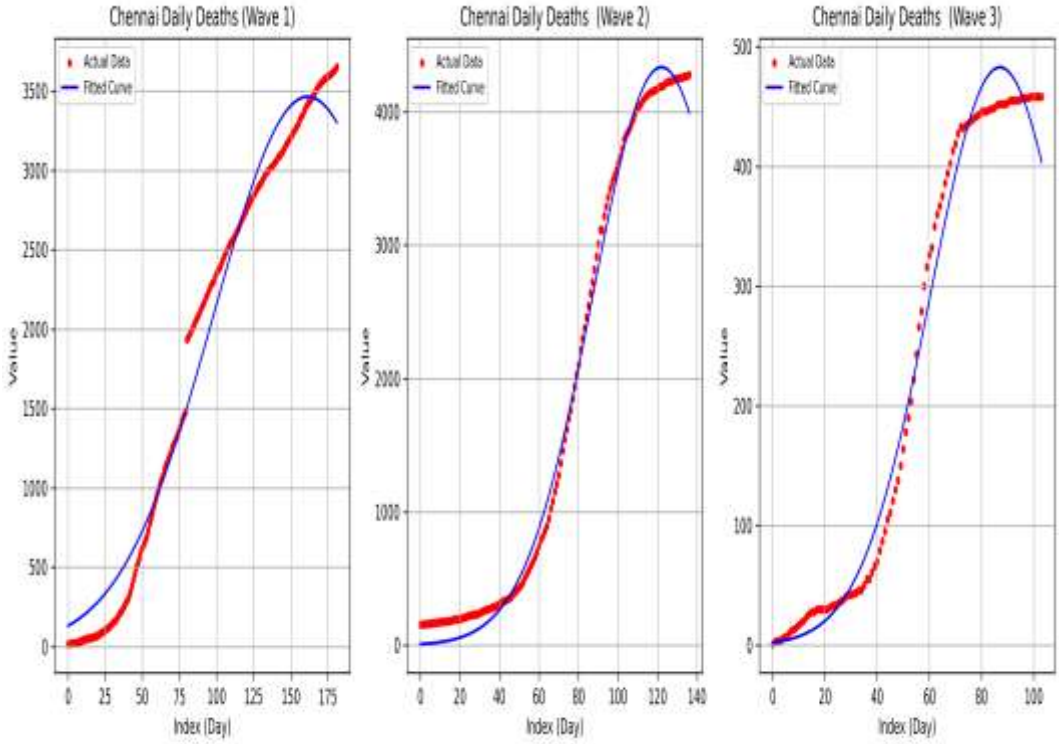

Figure 11: Curve fitting results for COVID-19 daily deaths in Chennai

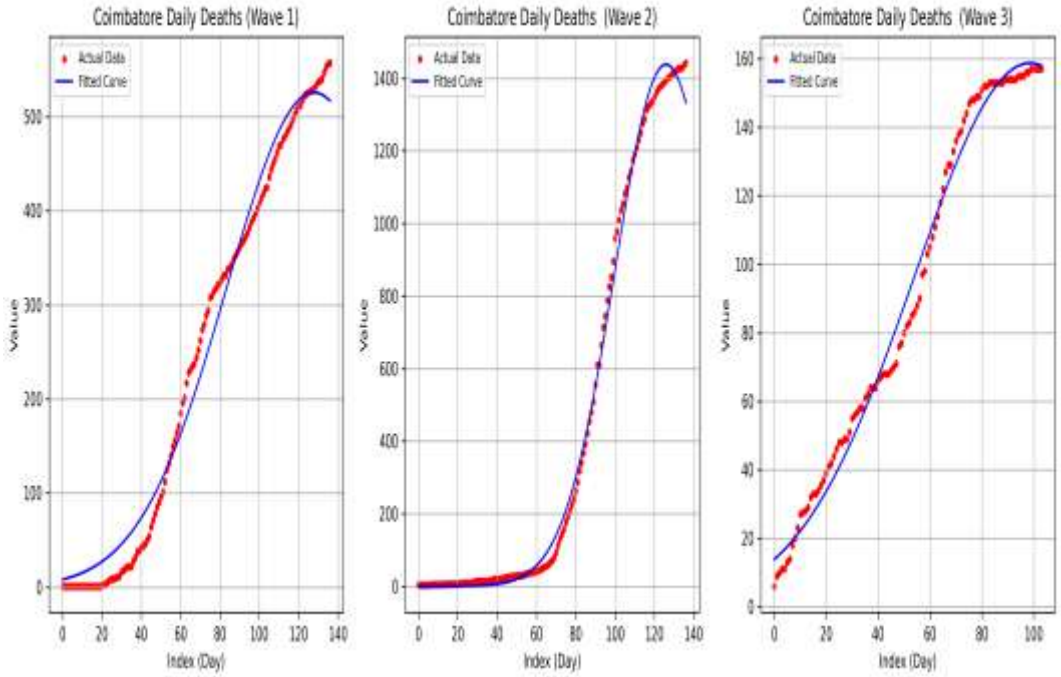

Figure 12: Curve fitting results for COVID-19 daily deaths in Coimbatore

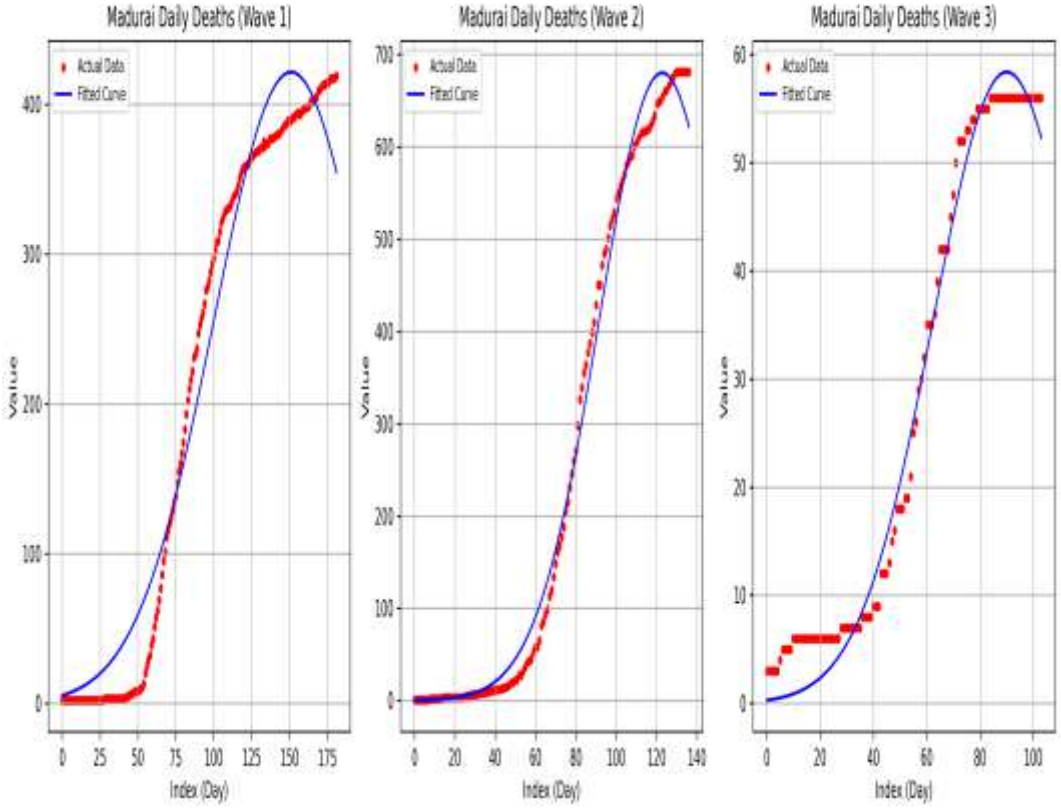

Figure 13: Curve fitting results for COVID-19 daily deaths in Madurai

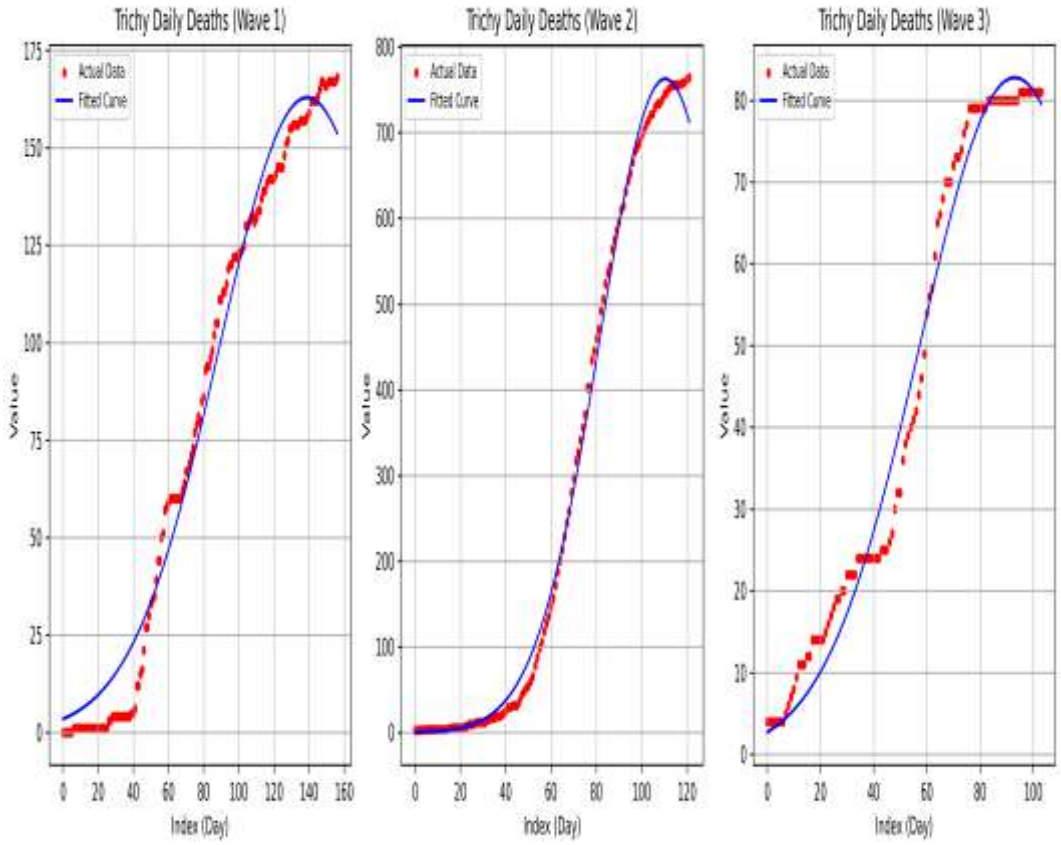

Figure 14: Curve fitting results for COVID-19 daily deaths in Trichy

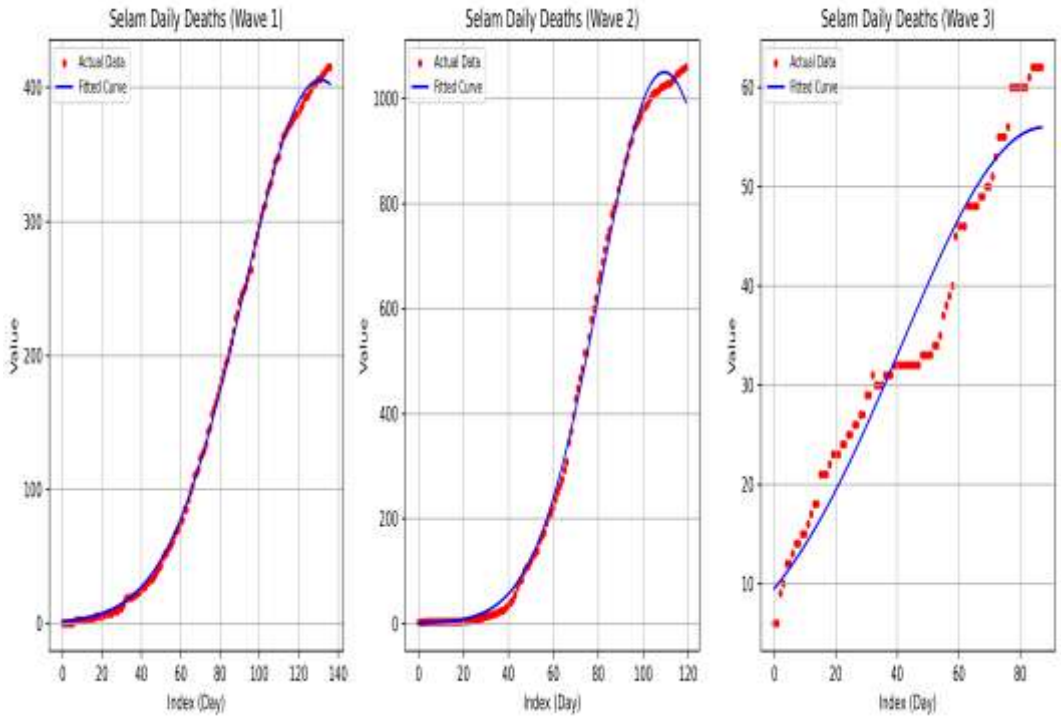

Figure 15: Curve fitting results for COVID-19 daily deaths in Selam

4. Conclusion

This paper includes an ideal control strategy for mitigating the COVID-19 spread in Tamil Nadu, India, based on a SEAIR model. It describes potential improvements with two targeted

interventions addressed to the susceptible and asymptomatic populations. The numerical simulations reveal the peak infection rates reduced by about 50% and the cumulative number of cases is decreased by about 40%. The associated intervention costs are lowered by 30%, which underscores cost-effectiveness of such integrated strategies.

The curve fitting analyses of this study are useful for exploring the temporal and regional dynamics of COVID-19 during three waves in Tamil Nadu, India and its metropolitan cities. The logistic growth models were useful for modeling trends in total cases and recoveries, and Gaussian models were used for analyzing daily death rates. It was observed that, according to the Gaussian model, peak daily deaths in Chennai occurred 10 days ahead of the other metropolitan areas and that there is a need for timely and localized response strategies.

Such targeted interventions to both the at-risk and asymptomatic segments of the population can better achieve control of the disease in an effective and even economically feasible manner. As such, the research gives policymakers valuable resources with regard to allocating resources to each region for optimal distribution; in this case, during constrained healthcare infrastructure.

This framework can be further enhanced towards more stochastic components and real-time data integration, thereby ensuring robustness and applicability in the dynamic epidemic environment. It can be integrated with the vaccination dynamics factor, thereby making it still more sophisticated and applicable during outbreaks, and thus forming a much more complete framework for management of future potential outbreaks. Overall, it makes significant contributions to mathematical modelling and optimal control in epidemic management by giving a practical view from the stand point of public health strategies.

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