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A Two-Warehouse Inventory System with Advertisement and Time Dependent Demand and Preservation Investment Under the Trade Credit Policy

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This research paper investigates a constant non-instantaneous deteriorating two-warehouse inventory control model with delay in payment. The main objective of this research is to maximize total profit by establishing the optimal order quantity and cycle length. We elaborate contemplated time and advertisement dependent demand, credit policy, partial backlogged shortage and preservation technology. Preservation techniques are used to control the deterioration of the items in the warehouses. Moreover, numerical results have been performed to validate the work. Concavity has been discussed between the decision parameters with the help of various graphs. We solved the sensitivity analysis to examine the effect of many parameters on optimum solutions. This research effort has the potential to significantly assist organizations in (a) optimizing unit purchase costs and (b) decreasing the overall setup costs of inventory systems.
Keywords: preservation technology, two-warehouse, credit policy, advertisement-time demand.

1. Introduction

Inventory management refers to tracing essential materials, commodities, raw materials that a corporation consumes. Inventory and warehouse management both are almost same disciplines, as both help in dynamic inventory from supplier to last customer efficiently and effectively. Each involves shipping, storing and restoring stock level. During the storage of items in the warehouse, items are start deteriorate. Therefore, items need to keep secure. For this purpose, preservation technology has been applied. Many prominent researchers worked on the preservation technology model with different demand patterns. Bhawaria and Rathore [1] developed production model for bad quality items specially focusing on deteriorating items and preservation technology. They allowed the partial backlogging shortage. Bhawaria and Rathore [2] described EOQ model for bad quality products and (PT) preservation technology applied to control the deterioration of the items. Entire study carried out under the impact of inflation. Bhawaria et al. [3] determined reverse logistics model for non-instantaneous waste items with preservation technology (PT). In which demand was dependent on price and stock level. Rathore et al. [4] established optimum inventory policy for fragment quality products with ramp demand and two warehouse environments. Bhawaria et al. [5] introduced an inventory model for bad quality goods with carbon emission and time demand function. Sharma and Rathore [6] determined inventory model with hybrid type demand and shortage. They used preservation investment (PT) to stop the decaying rate of the goods. Rathore and Sharma [7] presented preservation investment inventory model for bad quality products with advertisement demand and trade credit policy. Sharma and Rathore [8] formulated inventory model under the impact of inflation and preservation investment and allowed shortage. Meedal and Rathore [9] established optimal inventory control model for decaying items with preservation policy. In this, demand was dependent on stock level.

Every big organization needs a warehouse to store the items and distribute them further out to consumers worldwide. Last few years, researchers and authors from various places of the world are spellbound by warehouse problems. In competitive market condition, retailers and wholesalers wish to procure extra goods athwart the warehouse efficiency. Present condition rises for many factors such as oneself warehouse efficiency restraints in plausible market, price discount, higher re-order cost, seasonal items etc. Excess stock facilitation store extra quantity of procured goods to handle this situation. In this reference, extra store facilitation is known as RW. First of all, stock of rented warehouse is cleared by transportation stock from rented to owned warehouse. This process is known as two warehouse inventory control channel. In the two-warehouse system research work started by Singh and Rathore [10] developed two-warehouse inventory model considering preservation technology. In their work used two types of warehouses first one is RW and second is OW. Shekhawat et al. [11] presented supply chain system and warehousing system with carbon tax policy under the constant demand function and investigation policy. A lot of research works in this area has been reported by Singh and Rathore [12], Rathore et al. [13], Yadav et al. [14] and many more.

Sharma and Mandal [15] established EOQ model with preservation technology and demand dependent on stock level. Entire study carried out under advance payment scheme. Pal et al. [16] investigated non-instantaneous low quality products two-warehouse problem under the impact of inflation. In this work price and time dependent demand function was used with shortages. Modak et al. [17] analyzed dynamic and pricing replenishment policy under the price and time dependent demand function. Preservation technology was applied to control the deterioration. Shan and Kittner [18] investigated allocation of policy for energy storage under the impact of inflation act. De et al. [19] contracted inventory control system for bad quality items under the impact of inflation in which demand function was dependent on price and stock. Hatibaruah and Saha [20] described inventory control model for low quality goods with preservation technology and ramp type demand. Entire study carried out under the impact of inflation [21-22]. Kumar et al. [23] investigate inventory management of automotive industries using ABC Analysis.

The principal aim of this research is to establish optimum order quantity trade cycle length wherein the whole benefit each unit time is maximize. To justify the validity of developed research work, numerical illustrations have solved. In figure 1, we provide the inventory process. The novelties of this model are explained as under:

- In this article, advertisement and time dependent demand rate is considered.
- Two warehouse inventory model for low quality products items is established and described under the trade credit scheme.
- Preservation investment is applied to control the deterioration of the goods.
- Trade credit based different cases; 3th objective cases (average profit of the inventory system) are constructed.

The formulization of the introduced model is envisaged in the following sections, section-1 introduction part, in section-2 assumption and notation, section-3 mathematical model formulation, performance measures in section 4. Solutions and numerical results are provided in sections 5 and 6, respectively. At last, conclusion is given in section 7.

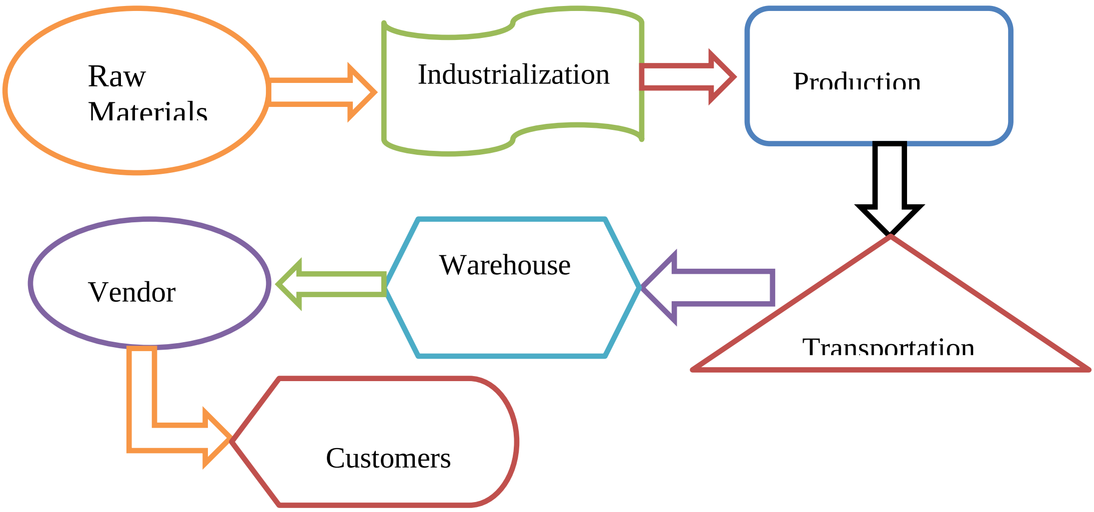

Fig. 1 Inventory Process

2. Notations and Assumptions

In this section, we develop the two-warehouse inventory model wherein planning horizon is infinite. The assumptions and notations utilized in formulating the model are presented below:

Assumptions

- The demand rate of the items is depends on advertisement and time, i.e.$D(A,t) = A^a(x + yt)$ where $x > 0, y > 0$.

- Lead time is zero and rate of replenishment is instant.
- Partially shortage arises which defined as $e^{-\lambda}(T-t)$, where $\lambda > 0$.
- Deterioration rate of items is involved after item is kept in the space.
- OW store capacity is fixed and RW storage capacity is limitless.
- When $M \leq T$, credit period settled at M. after the prefix time, retailer must pay interest with rateI_p. Again $M \geq T$ the retailer doesn't requirement to paid interest. In the condition, the retail traders earn revenue due to selling as well as interest.

Notations

Parameters		Description
$I_R(t)$	:	Ranted warehouse inventory level at time t.
$I_0(t)$	:	Owned warehouse inventory level at time t.
z_m	:	Maximum inventory level (units)
A	:	Ordering cost (Rs/order)
PC	:	Purchasing Cost
γ	:	Backlogging cost/unit/year
C_1	:	Alternative Cost
τ_θ	:	Items deterioration in OW & RW, constant rate

$\tau_\theta = (\theta - m(\xi))$, where $m(\xi) = e^{-b\xi}$.

w	:	Unlimited storage capacity in RW & OW
h	:	Carrying cost per unit item per unit time in owned Warehouse
f	:	Carrying cost per unit item per unit time in OW, $f > h$.
ξ	:	Preservation technology cost
M	:	Trade credit period given by supplier.
I_p	:	Paid interest
I_e	:	Earned Interest
t_D	:	Starting deterioration time
t_w	:	Zero inventory level time in OW
$\boldsymbol{t_R}$	:	Zero inventory level in RW
$\boldsymbol{b(t)}$	:	Items backlogging rate

$\boldsymbol{l(t)}$: Lost sale

$\boldsymbol{TP_i(t_R, T, t_w)}$: Total Profit

3. Mathematical Model Formulations

In this analysis, odd dealer buy 'Q' unit of items. Then after that fulfill backorder quantity of last cycle, in hand production level reach '$\boldsymbol{z_m}$'. After filling the backlogged quantity, units are stock in OW and remaining units are kept in another warehouse. We assumed that deterioration of the items is not start immediately, so, at time $\boldsymbol{t_D}$ deterioration start of the items. At initial (e.g. at time t=0) odd dealer fills the demand of customers thru rented warehouse till stocks. In the RW fatigued during interval $[0, \boldsymbol{t_R}]$. After time$\boldsymbol{t_r}$, RW stock level perfectly fatigued, the demand is converse thru owned warehouse during interval $[\boldsymbol{t_R}, \boldsymbol{t_D}]$. Conversely, stock level decreases due to demand during interval $[0, \boldsymbol{t_D}]$. But during the interval $\boldsymbol{t_D \le t \le t_w}$ inventory level drops due to impact of deterioration rate and demand. In this study, the different three cases are considered for analysis purpose.

Case-1: When $\boldsymbol{t_D < t_R < t_w}$

(a) Case 1.1: $\boldsymbol{0 < M \le t_D < t_R < t_w}$

(b) Case 1.2: $\boldsymbol{t_D < M \le t_R < t_w}$

(c) Case 1.3: $\boldsymbol{t_D < t_R < M \le t_w}$

(d) Case 1.4: $\boldsymbol{t_R < t_w < M \le T}$

(e) Case1.5: $\boldsymbol{t_R < t_w < T \le M}$

Case-2: When $\boldsymbol{t_R < t_D < t_w}$

(a) Case-2.1 $\boldsymbol{0 < M \le t_R \le t_D < t_w}$

(b) Case-2.2 $\boldsymbol{t_R < M \le t_D < t_w}$

(c) Case-2.3 $\boldsymbol{t_R < t_D < M \le t_w}$

(d) Case-2.4 $\boldsymbol{t_D < t_w < M \le T}$

(e) Case-2.5 $\boldsymbol{t_D < t_w < T \le M}$

Case-3: When $\boldsymbol{t_R < t_w < t_D}$

(a) Case-3.1 $\boldsymbol{0 < M \le t_R \le t_w \le t_D}$

(b) Case-3.2 $\boldsymbol{t_R < t_w < M \le t_D}$

(c) Case-3.3 $\boldsymbol{t_R < t_w < M \le t_D}$

(d) Case-3.4 $\boldsymbol{t_w \le t_D < T \le M}$

(e) Case-3.5 $\boldsymbol{t_w \le t_D < T \le M}$

3.1. Case-1. $t_D < t_R < t_w$

In this case, during the interval $[0, t_D]$ deterioration not considered, so rented warehouse demand is only cause for fatiguing inventory level as shown in fig. 2. Also, in the RW inventory level has not changed. Due to demand and deterioration inventory level diminishes in the RW during time interval $[t_D, t_R]$. Consequently, inventory level of RW finished at time t_R. So, during interval $[t_R, t_w]$ demand fulfill by OW. Resultant, this period OW stock level degrade due to demand and deterioration and reduce to zero at time t_w. In the interval $[t_w, T]$, demand exists with backlogged. So the governing differential equations are-

$$\frac{dI_R(t)}{dt} = -D(A,t) \qquad 0 \le t \le t_D \qquad (1)$$

$$\frac{dI_R(t)}{dt} + \tau_{\theta_r} I_R(t) = -A^a(x+yt) \qquad t_D \le t \le t_R \qquad (2)$$

$$\frac{dI_o(t)}{dt} + \tau_{\theta_o} I_o(t) = 0 \qquad t_D \le t \le t_R \qquad (3)$$

$$\frac{dI_o(t)}{dt} + \tau_{\theta_o} I_o(t) = -A^a(x+yt) \qquad t_R \le t \le t_w \qquad (4)$$

$$\frac{db(t)}{dt} = A^a(x+yt)e^{-\lambda(T-t)} \qquad t_w \le t \le T \qquad (5)$$

On solving equations (1) to (5) with boundary conditions $I_R(0) = z_m - w, I_R(t_R) = 0, I_o(t_D) = w, I_o(t_w) = 0 \; and \; b(t_w) = 0$, we have

$$I_R(t) = z_m - w - A^a\left(xt + y\frac{t^2}{2}\right) \qquad (6)$$

$$I_R(t) = \frac{A^a}{\tau_{\theta_r}}\left[e^{\tau_{\theta_r}(t_r - t)}\left\{x + y\left(t_r - \frac{1}{\tau_{\theta_r}}\right)\right\} - \left\{x + y\left(t - \frac{1}{\tau_{\theta_r}}\right)\right\}\right] \qquad (7)$$

$$I_o(t) = we^{\tau_{\theta_o}(t_D - t)} \qquad (8)$$

$$I_o(t) = \frac{A^a}{\tau_{\theta_r}}\left[e^{\tau_{\theta_o}(t_w - t)}\left\{x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right\} - \left\{x + y\left(t - \frac{1}{\tau_{\theta_o}}\right)\right\}\right] \qquad (9)$$

$$b(t) = A^a\left[\left\{x\frac{e^{-\lambda(T-t)}}{\lambda} + y\left\{\frac{e^{-\lambda(T-t)}}{\lambda}\left(t - \frac{1}{\lambda}\right)\right\}\right\} - \left\{x\frac{e^{-\lambda(T-t_w)}}{\lambda} + y\left\{\frac{e^{-\lambda(T-t_w)}}{\lambda}\left(t_w - \frac{1}{\lambda}\right)\right\}\right\}\right] \qquad (10)$$

Now the lost sale amount is presented by

$$l(t) = \int_{t_w}^{t} A^a(x+yt)\{1 - e^{-\lambda(T-t)}\}dt \qquad (11)$$

$$l(t) = A^a x(t - t_w) - \frac{A^a y}{\lambda}\left[e^{-\lambda(T-t)}\left(t - \frac{1}{\lambda}\right) + e^{-\lambda(T-t_w)}\left(\frac{1}{\lambda} - t_w\right)\right] \quad (12)$$

Now using $I_R(t)$ at $t = t_D$ from equations (2) & (4) maximum inventory level we have.

$$z_m = w + A^a\left(xt_D + \frac{yt_D^2}{2}\right) + \frac{A^a}{\tau_{\theta_r}}\left[e^{\tau_{\theta_r}(t_R - t_D)}\left\{x + y\left(t_R - \frac{1}{\tau_{\theta_r}}\right)\right\} - \left\{x + y\left(t_D - \frac{1}{\tau_{\theta_r}}\right)\right\}\right]$$
(13)

Again $I_o(t)$ at $t = t_R$ from equations (6) & (8) we have

$$we^{\tau_{\theta_o}(t_D - t_R)} = \frac{A^a}{\tau_{\theta_o}}\left[e^{\tau_{\theta_o}(t_w - t_R)}\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right) - \left(x + y\left(t_R - \frac{1}{\tau_{\theta_o}}\right)\right)\right] \quad (14)$$

Putting t=T in the equation (10), we get

$$b(T) = \frac{A^a}{\lambda}\left[\left\{x + y\left(T - \frac{1}{\lambda}\right)\right\} - \left\{\left(xe^{-\lambda(T-t_w)}\right) + y\left\{e^{-\lambda(T-t_w)}\left(t_w - \frac{1}{\lambda}\right)\right\}\right\}\right]$$
(15)

Now, the total quantity is given by

$$Q = z_m + b(T)$$

$$Q = w + A^a\left(x\,t_D + \frac{yt_D^2}{2}\right) + \frac{A^a}{\tau_{\theta_r}}\left[e^{\tau_{\theta_r}(t_R - t_D)}\left\{x + y\left(t_R - \frac{1}{\tau_{\theta_r}}\right)\right\} - \left\{x + y\left(t_D - \frac{1}{\tau_{\theta_r}}\right)\right\}\right] + \frac{A^a}{\lambda}\left[\left\{x + y\left(T - \frac{1}{\lambda}\right)\right\} - \left\{\left(xe^{-\lambda(T-t_w)}\right) + y\left\{e^{-\lambda(T-t_w)}\left(t_w - \frac{1}{\lambda}\right)\right\}\right\}\right]$$
(16)

3.2 Case-2 $t_R \le t_D < t_w$

In this condition preservation technology of the rented warehouse perhaps is good facility. These type of situations demonstrated. Then, the governing differential equations are-

$$\frac{dI_R(t)}{dt} = -A^a(x + yt) \qquad 0 \le t \le t_R \quad (17)$$

$$\frac{dI_o(t)}{dt} = -A^a(x + yt) \qquad t_R < t \le t_D \quad (18)$$

$$\frac{dI_o(t)}{dt} + \tau_{\theta_o} I_o(t) = -A^a(x + yt) \qquad t_D < t \le t_w$$
(19)

$$\frac{db(t)}{dt} = A^a(x + yt)e^{-\lambda(T-t)} \qquad t_w < t \le T$$
(20)

Solutions of equations (17) to (20) with the boundary conditions $I_R(t_R) = 0, I_o(t_R) = w, I_o(t_w) = 0, b(t_w) = 0$

$$I_R(t) = A^a\left[x(t_R - t) + \frac{y(t_R^2 - t^2)}{2}\right]$$
(21)

$$I_o(t) = w + A^a\left[x(t_R - t) + \frac{y(t_R^2 - t^2)}{2}\right] \tag{22}$$

$$I_o(t) = \frac{A^a}{\tau_{\theta_o}}[e^{-\tau_{\theta_o}(t_w - t)}\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right) - \left(x + y\left(t - \frac{1}{\tau_{\theta_o}}\right)\right)] \tag{23}$$

$$b(t) = \frac{A^a e^{-\lambda(T-t)}}{\lambda}\left\{x + y\left(t - \frac{1}{\lambda}\right)\right\} - \frac{A^a e^{-\lambda(T-t_w)}}{\lambda}\left\{x + y\left(t_w - \frac{1}{\lambda}\right)\right\} \tag{24}$$

Lost sale at time t are presented as-

$$l(t) = \int_{t_w}^{t} A^a(x + yt)(1 - e^{-\lambda(T-t)})\, dt$$
(25)

$$l(t) = A^a\left[x(t - t_w) + \frac{x\left(e^{-\lambda(T-t_w)} - e^{-\lambda(T-t)}\right)}{\lambda} + \frac{y}{2}\left(t^2 - t_w^2\right) + y\left\{\left(t_w - \frac{1}{\lambda}\right)e^{-\lambda(T-t_w)} - e^{-\lambda(T-t)}\left(t - \frac{1}{\lambda}\right)\right\}\right]$$
(26)

Using continuity of $I_o(t)$ at $t = t_D$ from equations (22) & (23), we have

$$w + A^a\left[x(t_R - t) + \frac{y(t_R^2 - t^2)}{2}\right] = \frac{A^a}{\tau_{\theta_o}}\left[e^{-\tau_{\theta_o}(t_w - t)}\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right) - \left(x + y\left(t - \frac{1}{\tau_{\theta_o}}\right)\right)\right]$$

(27)

Using the condition at time $t = 0$ when $I_R(t) = z_m - w$ in equation (22), we have

$$z_m = w + A^a\left[x(t_R) + \frac{y(t_R^2)}{2}\right] \tag{28}$$

The backlogging quantity per cycle is at time t=T, in Eq. (24)

$$b(T) = \frac{A^a}{\lambda}\left[x + y\left(T - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}\left(x + y\left(t_w - \frac{1}{\lambda}\right)\right)\right]$$
(29)

Therefore, order quantity is

$$Q = z_m + b(T)$$

$$Q = w + A^a\left[x(t_R) + \frac{y(t_R^2)}{2}\right] + \frac{A^a}{\lambda}\left[x + y\left(T - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}\left(x + y\left(t_w - \frac{1}{\lambda}\right)\right)\right]$$
(30)

3.3 Case-3 $t_R \le t_w \le t_d$

In this case, deterioration of the items is start when stock level has been used interval ($t_w \le t_D$). This type of characteristic may be presented and the governing equations are-

$$\frac{dI_R(t)}{dt} = -A^a(x + yt) \qquad 0 \le t \le t_R \tag{31}$$

$$\frac{dI_o(t)}{dt} = -A^a(x + yt) \qquad t_R < t \le t_w \tag{32}$$

$$\frac{db(t)}{dt} = A^a(x + yt)e^{-\lambda(T-t)} \qquad t_w < t \le T$$
(33)

Now solution of the equations (31) to (33) with boundary conditions $I_R(t_R) = 0, I_o(t_R) = w, I_o(t_w) = 0, b(t_w) = 0$. Are-

$$I_R(t) = \frac{yA^a}{2}(t_R^2 - t^2) \tag{34}$$

$$I_o(t) = w + \frac{yA^a}{2}(t_R^2 - t^2)$$
(35)

$$b(t) = \frac{A^a}{\lambda}[x(e^{-\lambda(T-t)} - e^{-\lambda(T-t_w)}) + y(e^{-\lambda(T-t)}\left(t - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})))] \tag{36}$$

Lost sale amount is given by-

$$l(t) = \int_{t_w}^{t} A^a(x + yt)(1 - e^{-\lambda(T-t)})dt$$
(37)

$$l(t) = A^a\left[\left\{x(t - t_w) - \frac{x}{\lambda}\left(e^{-\lambda(T-t)} - e^{-\lambda(T-t_w)}\right)\right\} + \frac{y}{2}\left(-t_w^2 + t^2\right) - y\left\{\left(e^{-\lambda(T-t)}\left(t - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}\left(t_w - \frac{1}{\lambda}\right)\right)\right\}\right]$$
(38)

Putting t=0 when $I_R(t) = z_m - w$ and from equation 34 we obtain

$$z_m = w + \frac{yA^a}{2}t_R^2$$
(39)

When t=T in equation 36 we have

$$b(T) = \frac{A^a}{\lambda}[x(1 - e^{-\lambda(T-t_w)}) + y\left(t - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})] \tag{40}$$

Therefore

$$Q = z_m + b(T)$$

$$Q = w + \frac{yA^a}{2}t_R^2 + \frac{A^a}{\lambda}[x(1 - e^{-\lambda(T-t_w)}) + y\left(t - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})] \tag{41}$$

4. Performance Measure

In order to evaluate the effectiveness and efficiency of the proposed system, we provide ordering costs for all three cases as under

- Ordering Cost

Ordering cost = A (42)

For case-1: $t_D < t_R < t_w$

- Holding Cost in RW

$$HC = f\left\{\int_0^{t_D} I_R(t)dt + \int_{t_D}^{t_R} I_R(t)dt\right\} \tag{43}$$

$$HC = f\left\{\left(z_m t_D - w t_D - A^a\left(x\frac{t_D^2}{2} + y\frac{t_D^3}{6}\right)\right) + \frac{A^a}{\tau_{\theta_r}}\left\{\left(x t_D^2 + y\frac{t_D^2}{2} - \frac{t_D}{\tau_{\theta_r}}\right) - \left(x t_R + y\frac{t_R^2}{2} - \frac{t_R}{\tau_{\theta_r}}\right) + \frac{1}{\tau_{\theta_r}}\left(x + y\left(t_R - \frac{1}{\tau_{\theta_r}}\right)\right)\left(e^{\tau_{\theta_r}(t_R - t_D)} - 1\right)\right\}\right\} \tag{44}$$

- Holding Cost in OW

$$HC = h\left[\int_o^{t_D} w dt + \int_{t_D}^{t_R} I_o(t)dt + \int_{t_R}^{t_w} I_o(t)dt\right] \tag{45}$$

$$HC = h\left[w t_D + \frac{w}{\tau_{\theta_o}}\left(1 - e^{\tau_{\theta_o}(-t_R + t_D)}\right) + \frac{A^a}{\tau_{\theta_o}}\left\{-\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right)\frac{1}{\tau_{\theta_o}}\left(e^{\tau_{\theta_o}(-t_R + t_w)} - 1\right) + x(t_R - t_w) + \frac{y}{2}\left(t_R^2 + t_w^2\right) - \frac{1}{\tau_{\theta_o}}(t_R + t_w)\right\}\right] \tag{46}$$

- Backlogging Cost

$$BC = \gamma\int_{t_w}^{T} b(t)dt \tag{47}$$

$$BC = A^a\gamma\left[\frac{x\left(1 - e^{-\lambda(T - t_w)}\right)}{\lambda^2} + \frac{y}{\lambda^2}\left(T - \frac{2}{\lambda}\right) - \frac{y e^{-\lambda(T - t_w)}}{\lambda^2}\left(t_w - \frac{2}{\lambda}\right) + \left(x + y\left(t_w - \frac{1}{\lambda}\right)\right)\frac{e^{-\lambda(T - t_w)}}{\lambda}(t_w - T)\right] \tag{48}$$

- Lost sale cost

$$LSC = LSC\int_{t_w}^{T} A^a(x + yt)\left(1 - e^{\lambda(T - t)}\right)dt \tag{49}$$

$$LSC = A^a LSC\left[x\left(T - t_w - \frac{1}{\lambda} + \frac{e^{-\lambda(T - t_w)}}{\lambda}\right) + \frac{y(T^2 - t_w^2)}{2} - y\left(\frac{T}{\lambda} - \frac{t_w e^{-\lambda(T - t_w)}}{\lambda} - \frac{1}{\lambda^2} + \frac{e^{-\lambda(T - t_w)}}{\lambda^2}\right)\right] \tag{50}$$

- Purchasing Cost

$$PC = PCQ \tag{51}$$

- Sales Revenue

$$SR = \left\{\int_0^{t_w} A^a(x + yt)dt + b(T)\right\} \tag{52}$$

$$SR = \left[A^a\left(x t_w + y\frac{t_w^2}{2}\right) + \frac{A^a}{\lambda}\left[\left\{x + y\left(T - \frac{1}{\lambda}\right)\right\} - \left\{\left(x e^{-\lambda(T - t_w)}\right) + y\left\{e^{-\lambda(T - t_w)}\left(t_w - \right.\right.\right.\right.\right.$$

$\frac{1}{\lambda}\Big)\Big\}\Big\}\Big]\Bigg]$ (53)

- Preservation Technology Cost

$$\boldsymbol{PTC = \xi \int_{t_D}^{t_R} dt} \quad (54)$$

$$\boldsymbol{PTC = \xi(t_R - t_D)} \quad (55)$$

In this case, $\boldsymbol{t_D < t_R < t_w}$ following distinct sub cases occur. Now the all situations described for every subcase is given below as

(a) Case 1.1 $\boldsymbol{0 < M \le t_D < t_R < t_w}$

In this case, retailer interest earn on shortage amount on last cycle and complete the demand of exiting cycle up to period$\boldsymbol{t = M}$. Now as per situation earned interest is solved as

Situation 1. Interest earned by shortage quantity

$$\boldsymbol{IE_{111} = I_e \int_0^M b(T)dt}$$

$$\boldsymbol{IE_{111} = \frac{I_e M A^a}{\lambda}\left[\left(x + y\left(T - \frac{1}{\lambda}\right)\right) - \left(xe^{-\lambda(T-t_w)} + y\left(e^{-\lambda(T-t_w)}\left(t_w - \frac{1}{\lambda}\right)\right)\right)\right]}$$

Situation-2 Earned interest form sales in interval [0, M].

$$\boldsymbol{IE_{112} = I_e \int_0^M A^a(x + yt)dt}$$

$$\boldsymbol{IE_{112} = I_e M A^a\{x + \frac{yM}{2}\}}$$

The total earned interest is given by

$$\boldsymbol{TIE_{11} = I_e M A^a[\frac{1}{\lambda}\{\left(x + y\left(T - \frac{1}{\lambda}\right)\right) - \left(xe^{-\lambda(T-t_w)} + y\left(e^{-\lambda(T-t_w)}\left(t_w - \frac{1}{\lambda}\right)\right)\right\} + x + \frac{yM}{2}]}$$

Paid interest calculated as

Situation 1. interest paid in interval $[\boldsymbol{M, t_w}]$ in the owned warehouse.

$$\boldsymbol{IP_{111} = I_p pc[\int_M^{t_D} w dt + \int_{t_D}^{t_R} I_o(t)dt + \int_{t_R}^{t_w} I_o(t)dt]}$$

$$\boldsymbol{IP_{111} = I_p pc\left[w\left\{(t_D - M) + \frac{1}{\tau_{\theta_o}}\left(1 - e^{\tau_{\theta_o}(t_D - t_R)}\right)\right\} + \frac{A^a}{\tau_{\theta_o}}\left\{\frac{\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right)\left(e^{\tau_{\theta_o}(t_w - t_R)}\right)}{\tau_{\theta_o}} - \left(x(t_w - t_R) + y(t_w - t_R)(t_w - t_R) - \left(\frac{1}{\tau_{\theta_o}}\right)\right)\right\}\right]}$$

Situation-2 Paid interest during interval [M, $\boldsymbol{t_R}$] in RW

$$IP_{112} = pcI_p\{\int_M^{t_D} I_R(t)dt + \int_{t_D}^{t_R} I_R(t)dt\}$$

$$IP_{112} = pcI_p\left\{(z-M)(t_d-M) - A^a\left\{\frac{x}{2}\left(t_D^2 - M^2\right) + \frac{y}{6}\left(t_D^3 - M^3\right) + \frac{A^a}{\tau_{\theta_r}}\left(x(t_R - t_D) + y\left(\frac{t_R^2 - t_D^2}{2} - \frac{t_R - t_D}{\tau_{\theta_r}}\right)\right) + \left(x + y\left(t_R - \frac{1}{\tau_{\theta_r}}\right)\right)\left(\frac{1}{\tau_{\theta_r}} - \frac{e^{\tau_{\theta_r}(t_R - t_D)}}{\tau_{\theta_r}}\right)\right\}\right\}$$

Now the total paid interest is given by

$$TIP_{11} = IP_{111} + IP_{112}$$

(b) Case-1.2 $t_D < M \le t_R < t_w$

In this case the mathematical part of earned interest is same according to case-1.1.

Situation-1 interest paid in OW during [M, t_w]

$$IP_{121} = pcI_p\{\int_M^{t_R} I_o(t)dt + \int_{t_R}^{t_w} I_o(t)dt\}$$

$$IP_{121} = pcI_p\left\{\frac{w}{\tau_{\theta_o}}\left(e^{\tau_{\theta_o}(t_D - M)} - e^{\tau_{\theta_o}(t_D - t_R)}\right) + \frac{A^a}{\tau_{\theta_o}}\left\{\left(x + y\left(t_D - \frac{1}{\tau_{\theta_o}}\right)\right)\left(e^{\tau_{\theta_o}(t_w - t_R)} - 1\right)\frac{1}{\tau_{\theta_o}} - \left(x(t_w - t_R) + y\left(\frac{(t_w - t_R)^2}{2} - \frac{t_w - t_R}{\tau_{\theta_o}}\right)\right)\right\}\right\}$$

Situtation-2 earned interest from sales

$$IP_{122} = pcI_p\{\int_M^{t_R} I_R(t)dt\}$$

$$IP_{122} = \frac{-pcI_pA^a}{\tau_{\theta_r}}\left[\left\{x(t_R - M) + y\left(\frac{t_R^2 - M^2}{2} - \frac{t_w - t_R}{\tau_{\theta_o}}\right)\right\} + \left\{1 - e^{\tau_{\theta_o}(t_D - M)}\right\}\frac{1}{\tau_{\theta_r}}\left\{x + y\left(t_R - \frac{1}{\tau_{\theta_r}}\right)\right\}\right]$$

The total paid interest is given by

$$IP_{12} = IP_{121} + IP_{122}$$

(c) Case-1.3 $t_D < t_R < M \le t_w$

In this case calculation of paid interest and earned interest same to case 1.1, so both interests will be same.

(d) Case-1.4 $t_R < t_w < M \le T$

$$TIP_{14} = 0$$

Situation-1 earned interest from shortage is given by

$$IE_{141} = I_e \int_0^M b(T)dt$$

$$IE_{141} = \frac{I_e M A^a}{\lambda}\left[\left(x + y\left(T - \frac{1}{\lambda}\right)\right) - \left\{e^{\lambda(T-t_w)}\left(1 + y\left(t_w - \frac{1}{\lambda}\right)\right)\right\}\right]$$

Situation-2 earned interest from sales is given by

$$IE_{142} = A^a I_e \int_0^{t_w}(x + yt)dt$$

$$IE_{142} = A^a I_e\left[xt_w + \frac{yt_w^2}{2}\right]$$

Situation-3 earned interest from sales up t=T is given by

$$IE_{143} = I_e\left[\iint_{t_w 0}^{M t_w} A^a(x + yt)dt\right]$$

$$IE_{143} = I_e A^a\left[xt_w(M - t_w) + \frac{yt_w^2(M - t_w)}{2}\right]$$

The total earned interest is given by

$$TIE_{14} = IE_{141} + IE_{142} + IE_{143}$$

(e) Case-1.5 $t_R < t_w < T \le M$

$$TIP_{15} = 0$$

TIE_{15} In this case calculation of earned interest as it is case 1.4, this is given in above case.

For case-2 $t_R \le t_D < t_w$

- Holding Cost in RW

$$HC = f\int_0^{t_R} I_R(t)dt \tag{56}$$

$$HC = fA^a\left[x\left(t_R - \frac{t_R^2}{2}\right) + y\left(t_R^2 - \frac{t_R^3}{3}\right) - xt_R - \frac{yt_R^2}{2}\right] \tag{57}$$

- Holding Cost in OW

$$HC = h\{\int_0^{t_R} w dt + \int_{t_R}^{t_D} I_o(t)dt + \int_{t_d}^{t_w} I_o(t)dt\} \tag{58}$$

$$HC = -\frac{A^a t_R^2}{2}\left(x + \frac{yt_R}{3}\right) + w(t_D - t_R) + A^a\left\{\frac{x}{2}\left(t_R^2 - t_D^2\right) + \frac{y}{6}\left(t_R^3 - t_D^3\right)\right\} + \frac{A^a}{\tau_{\theta_o}}\left[x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\left(e^{-\tau_{\theta_o}(t_w - t_D)} - 1\right) - \left\{x + y\left(\frac{t_w + t_D}{2} - \frac{1}{\tau_{\theta_o}}\right)\right\}(t_w - t_D)\right] \tag{59}$$

- Backlogging Cost

$$BC = \gamma\int_{t_w}^{T} b(t)dt \tag{60}$$

$$BC = \frac{\gamma A^a}{\lambda}[\left\{\frac{x}{\lambda} + \frac{y\left(T - t_w - \frac{1}{\lambda}\right)}{\lambda}\right\}\left(1 - e^{-\lambda(T-t_w)}\right) + \frac{1}{\lambda}(-1 + e^{-\lambda(T-t_w)} - (T - t_w)(x + y(t_w - \frac{1}{\lambda}))))] \tag{61}$$

- Lost Sale Cost

$$LSC = C_1 \int_{t_w}^{T} \{1 - e^{-\lambda(T-t)}\}\{A^a(x + yt)\}dt \quad (62)$$

$$LSC = C_1 A^a [x(T - t_w) + \frac{y(T^2 - t_w^2)}{2} - \frac{x}{\lambda}\left(1 - e^{-\lambda(T-t_w)}\right) + \frac{y}{\lambda}\{-\left(T - \frac{1}{\lambda}\right) + e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})\}] \quad (63)$$

- Purchasing Cost

$$PC = pcQ$$

$$PC = pc[z + b(t)] \quad (64)$$

- Preservation technology Cost

$$PTC = \xi \int_{t_R}^{t_D} dt \quad (65)$$

$$PTC = \xi(t_D - t_R) \quad (66)$$

- Sales Revenue

$$SR = \{\int_0^{t_w} A^a(x + yt)dt + b(T)\} \quad (67)$$

$$SR = A^a t_w [x + \frac{y t_w}{2} + \frac{1}{\lambda}\left\{\left(x + y\left(T - \frac{1}{\lambda}\right)\right) - e^{-\lambda(T-t_w)}\left(x + y\left(t_w - \frac{1}{\lambda}\right)\right)\right\}] \quad (68)$$

(a) Case-2.1 $0 < M \le t_R \le t_D < t_w$

In case-2.1, earned interest is similar as case-1.1.

Situation 1. From M to t_w current value of paid interest for fresh items in (OW).

$$IP_{211} = pcI_p\left[\int_M^{t_R} I_o(t)dt + \int_{t_R}^{t_D} I_o(t)dt + \int_{t_D}^{t_w} I_o(t)dt\right]$$

$$IP_{211} = pcI_p\left[w(t_R - M) + xA^a(t_R - M)t_R - \frac{x(t_R - M)^2}{2} + \frac{y t_R^2(t_R - M)}{2} - \frac{y(t_R - M)^3}{6} + w(t_D - t_R) + A^a\{x t_R(t_D - t_R)\} - \frac{xA^a(t_D - t_R)^2}{2} + \frac{yA^a t_R^2(t_D - t_R)}{2} - \frac{y(t_D - t_R)^3}{6} + \frac{A^a}{\tau_{\theta_o}}\left\{-\frac{x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)}{\tau_{\theta_o}} + \frac{e^{\tau_{\theta_o}(t_w - t_D)}\left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right)}{\tau_{\theta_o}} - \left(x(t_w - t_D)\right) + \frac{y(t_w - t_D)}{2} - \frac{y(t_w - t_D)}{\tau_{\theta_o}}\right\}\right]$$

Situation-2 paid interest on stored items in RW

$$IP_{212} = pcI_p\left[\int_M^{t_R} I_R(t)dt\right]$$

$$IP_{212} = A^a pcI_p \left[x\left(\frac{t_R^2}{2} - t_R + \frac{M^2}{2}\right) + \frac{t_R^3}{3} - \frac{yM\left(t_R^2 - \frac{M^3}{3}\right)}{2} \right]$$

Here, whole interest need to pay is given below-

$$TIP_{21} = IP_{211} + IP_{212}$$

(b) Case-2.2 $t_R < M \leq t_D < t_w$

In case 2.2 earned interest is similar to case 2.1. So, the paid interest is-

$$TIP_{22} = pcI_p \left\{ \int_M^{t_D} I_o(t)dt + \int_{t_D}^{t_w} I_o(t)dt \right\}$$

$$TIP_{22} = pcI_p \left[w(t_D - M) + A^a \left\{ xt_r(t_D - M) - \frac{x(t_D - M)^2}{2} + \frac{y}{2}\left(t_r^2(t_D - M) - \frac{(t_D - M)^3}{3}\right) \right\} + \frac{A^a}{\tau_{\theta_o}} \left\{ \left(x + y\left(t_w - \frac{1}{\tau_{\theta_o}}\right)\right)\left(\frac{e^{\tau_{\theta_o}(t_w - t_D)}}{\tau_{\theta_o}} - \frac{1}{\tau_{\theta_o}}\right) - \left(x(t_w - t_D) + y\left(\frac{(t_w - t_D)^2}{2} - \frac{(t_w - t_D)}{\tau_{\theta_o}}\right)\right) \right\} \right]$$

(c) Case-2.3 $t_R < t_D < M \leq t_w$

In case 2.3, earned interest is same as case 2.1

In this case, whole paid interest is only un-purchased amount in OW.

Here, paid interest is-

$$TIP_{23} = pcI_p \int_M^{t_w} I_o(t)dt$$

$$TIP_{23} = pcI_p \left[w(t_w - M) + A^a \left\{ x\left(t_r - \frac{(t_w - M)^2}{2}\right) + \frac{y}{2}\left(t_r^2(t_w - M) - \frac{(t_w - M)^3}{3}\right) \right\} \right]$$

(d) Case-2.4 $t_d < t_w < M \leq T$

In this case retailers no need to pay interest, so

$$TIP_{24} = 0$$

And the earned interest and paid is the same as case 1.4.

(e) Case-2.5 $t_D < t_w < T \leq M$

Here, paid interest is-

$$TIP_{25} = 0$$

And both interest same as case 1.4.

For Case-3 $t_R \leq t_w \leq t_D$

- Holding Cost in RW

$$HC = f\int_0^{t_R} I_R(t)dt \tag{69}$$

$$HC = \frac{fyA^a t_R^3}{3} \tag{70}$$

- Holding Cost in OW

$$HC = h[\int_0^{t_D} w dt + \int_{t_D}^{t_R} I_o(t)dt + \int_{t_R}^{t_w} I_o(t)dt] \quad (71)$$

$$HC = h[w\left\{t_D + t_w - t_R + \frac{1}{\tau_{\theta_o}} - \frac{e^{\tau_{\theta_o}(t_D - t_R)}}{\tau_{\theta_o}}\right\} + \frac{yA^a(t_R^3 - t_w^3)}{6}] \quad (72)$$

- Backlogging Cost

$$BC = \gamma \int_{t_w}^{T} b(t)dt \quad (73)$$

$$BC = \frac{\gamma A^a}{\lambda}[\left\{\frac{x}{\lambda}\left(1 - e^{-\lambda(T-t_w)}\right) - (T - t_w)e^{-\lambda(T-t_w)}\right\} + y\{\frac{1}{\lambda}\left(T - \frac{1}{\lambda} - \left(t_w - \frac{1}{\lambda}\right)e^{-\lambda(T-t_w)}\right) - \frac{1}{\lambda^2}\left(1 - e^{-\lambda(T-t_w)}\right) - e^{-\lambda(T-t_w)}(T - t_w)(t_w - \frac{1}{\lambda})\}] \quad (74)$$

- Lost Sale Cost

$$LSC = c_1 \int_{t_w}^{T} (e^{-\lambda(T-t)})A^a(x + yt)dt \quad (75)$$

$$LSC = A^a c_1[x(T - t_w) + \frac{y(T-t_w)^2}{2} + \frac{x}{\lambda}\left(-1 + e^{-\lambda(T-t_w)}\right) + \frac{y}{\lambda}\{-\left(T - \frac{1}{\lambda}\right) + e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})\}] \quad (76)$$

- Purchasing Cost

$$PC = PCQ \quad (77)$$

$$PC = PC[z + b(T)] \quad (78)$$

- Preservation Technology Cost

$$PTC = \int_{t_w}^{t_D} \xi dt \quad (79)$$

$$PTC = \xi(t_D - t_w) \quad (80)$$

- Sales Revenue

$$SR = \{\int_0^{t_w}(x + yt)A^a dt + b(T)\} \quad (81)$$

$$SR = A^a[\left(xt_w + \frac{yt_w^2}{2}\right) + \frac{1}{\lambda}[x(1 - e^{-\lambda(T-t_w)}) + y\left(t - \frac{1}{\lambda}\right) - e^{-\lambda(T-t_w)}(t_w - \frac{1}{\lambda})]] \quad (82)$$

(a) Case-3.1 $0 < M \le t_R \le t_w \le t_d$

Earned interest of case 3.1 is same to same case-1.1, 1.2, 1.3, 2.1, 2.2, & 2.3.

Situation-1 payable interest in OW during interval [M, t_w]

$$= PCI_p\left[\int_M^{t_R} I_o(t)dt + \int_{t_R}^{t_w} I_o(t)dt\right]$$

$$= pcI_p\left[w(t_w - M) + A^a\left(x - \frac{1}{2} + \frac{y}{2} + \frac{yt_w}{2}\right)t_R^2 - A^a M\left(1 + \frac{y}{2}\right)t_R + \frac{A^a M^2}{2}\left(1 + \frac{My}{3}\right) + \right.$$

$\frac{A^a y}{2}\left(t_R^3 + \frac{t_w^3}{3}\right)\Big]$

Situation-2 Payable interest during interval [M, t_R]

$$= pcI_p\left[\int_M^{t_R} I_R(t)dt\right]$$

$$= \frac{pcI_p y A^a}{2}\left[\frac{2}{3}t_R^3 + Mt_R^2 + \frac{M^3}{3}\right]$$

Here, in this case the entire paid interest is given below.

$$TIP_{31} = situation\ 1 + Situation\ 2$$

(b) Case-3.2 $t_r < t_w < M \le t_D$

In this scenario retailer earn interest up to paid $\mathbf{0}$ to M. After M, odd dealers need to pay interest with special interest as per condition on un-purchased items. Interest calculated in case-3.1 is same to same as case- 1.1, 1.2, 1.3, 2.1, 2.2, & 2.3.

Now the earned interest is given below

$$TIE_{32} = Case\ 2.1$$

Now, again total payable interest is given below

$$TIP_{32} = pcI_p\left[\int_M^{t_w} I_0(t)dt\right]$$

$$TIP_{32} = pcI_p\left[A^a\left\{\frac{yt_R^2}{2}(t_w - M) - \frac{y}{6}\left(t_w^3 - M^3\right)\right\} + w(t_w - M)\right]$$

(c) Case-3.3 $t_R < t_w < M \le t_D$

In this case retailers earn interest up to interval [0, M]. Conditions are similar as previous case.

(d) Case-3.4 $t_w \le t_D < T \le M$

In case 3.4 M is longer than T. So, retailers not require paying interest.

(e) Case-3.5 $t_w \le t_D < T \le M$

In case 3.5, earn and paid interest calculation is as it is case-3.4.

Retailer's Profit:

Total benefit is

$$TP(\xi, t_R, T) = \frac{1}{T}[SR - PC - HC\ in\ RW - HC\ in\ OW - PTC - BC - OC - IP + IE]$$

$$TP(\xi, t_R, T) = \begin{Bmatrix} TP_1(\xi, t_R, T)\ if\ t_D < t_R < t_w \\ TP_2(\xi, t_R, T)\ if\ t_R < t_D < t_w \\ TP_3(\xi, t_R, T)\ if\ t_R < t_w < t_D \end{Bmatrix}$$

Case-1 $t_D < t_R < t_w$

In case-1, total profit function is-

$$TP_1(\xi, t_R, T) = \begin{cases} TP_{11}(\xi, t_R, T) \; if \; 0 < M \leq t_D < t_R < t_w \\ TP_{12}(\xi, t_R, T) \; if \; t_D < M \leq t_R < t_w \\ TP_{13}(\xi, t_R, T) \; if \; t_D < t_R < M \leq t_w \\ TP_{14}(\xi, t_R, T) \; if \; t_R < t_w < M \leq T \\ TP_{15}(\xi, t_R, T) \; if \; t_R < t_w < T \leq M \end{cases}$$

Where sub case is-

$$TP_{11}(\xi, t_R, T) = \frac{1}{T}[SR - PC - HC \; in \; RW - HC \; in \; OW - PTC - BC - OC - IP_{11} + IE]$$

Case-2 $t_R \leq t_D < t_w$

In this case following cases arises which can be written as

$$TP_2(\xi, t_R, T) = \begin{cases} TP_{21}(\xi, t_R, T) \; if \; 0 < M \leq t_R < t_D < t_w \\ TP_{22}(\xi, t_R, T) \; if \; t_R < M \leq t_D < t_w \\ TP_{23}(\xi, t_R, T) \; if \; t_R < t_D < M \leq t_w \\ TP_{24}(\xi, t_R, T) \; if \; t_D < t_w < M \leq T \\ TP_{25}(\xi, t_R, T) \; if \; t_D < t_w < T \leq M \end{cases}$$

Where sub case is-

$$TP_{21}(\xi, t_R, T) = \frac{1}{T}[SR - PC - HC \; in \; RW - HC \; in \; OW - PTC - BC - OC - IP_{21} + IE]$$

Case-3 $t_R < t_w \leq t_D$

In this case following cases arise and these cases are given below

$$TP_3(\xi, t_R, T) = \begin{cases} TP_{31}(\xi, t_R, T) \; if \; 0 < M \leq t_R < t_w \leq t_D \\ TP_{32}(\xi, t_R, T) \; if \; t_R < M \leq t_w \leq t_D \\ TP_{33}(\xi, t_R, T) \; if \; t_R < t_w < M \leq t_D \\ TP_{34}(\xi, t_R, T) \; if \; t_w \leq t_D < M \leq T \\ TP_{35}(\xi, t_R, T) \; if \; t_w \leq t_D < T \leq M \end{cases}$$

Where sub case is

$$TP_{31}(\xi, t_R, T) = \frac{1}{T}[SR - PC - HC \; in \; RW - HC \; in \; OW - PTC - BC - OC - IP_{31} + IE]$$

5. Solution Process

$\frac{\partial TP(\xi, t_R, T)}{\partial \xi} = 0$, $\frac{\partial TP(\xi, t_R, T)}{\partial t_r} = 0$, $\frac{\partial TP(\xi, t_R, T)}{\partial T} = 0$.

Hessian Matrix of the total profit function is as follows.

$$TP(\xi, t_R, T) = \begin{bmatrix} \frac{\partial^2 TC(\xi, t_R, T)}{\partial \xi^2} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial \xi \partial T} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial \xi \partial t_R} \\ \frac{\partial^2 TC(\xi, t_R, T)}{\partial T \partial \xi} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial T^2} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial T \partial t_R} \\ \frac{\partial^2 TC(\xi, t_R, T)}{\partial t_R \partial \xi} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial t_R \partial T} & \frac{\partial^2 TC(\xi, t_R, T)}{\partial {t_R}^2} \end{bmatrix}$$

6. Numerical illustration

In this section, we provide the numerical results in form of graphs and tables for all three cases. To solve the numerical illustration and graphical part 'Mathematica software' has been used.

For case-1:

We have taken the appropriate values of the different parameters as $x = 600, y = 11, b = 5, t_D = 3, f = 15, A^a = 6,\ \theta = 0.02, LSC = 5, PC = 20, h = 5, A = 12, \gamma = 2,\ I_P = 0.3,\ t_w = 150, M = 30, I_e = 0.5, W = 90, c_1 = 13, \lambda = 0.52$

Optimal values of the parameters and total profit.

$t_R = 1359.05, T = 798.139, \xi = 20.296, TP = 7.7499 \times 10^8$

For case-2:

We have taken the appropriate values of the different parameters as $x = 650, y = 11, b = 5.1, t_D = 3.1, f = 15, A^a = 6,\ \theta = 0.02, LSC = 5, PC = 21, h = 5, A = 12, \gamma = 2.01,$ $I_P = 0.3,\ t_w = 150, M = 30, I_e = 0.6, W = 95, c_1 = 15, \lambda = 0.5$

The optimum values of the decision parameters and profit.

$t_R = 1489.05, T = 798.119, \xi = 20.125, TP = 7.7548 \times 10^8$

For case-3:

We have taken the appropriate values of the different parameters as $x = 630, y = 9, b = 5.2, t_D = 3.1, f = 12, A^a = 7,\ \theta = 0.021, LSC = 5, PC = 21, h = 5, A = 12, \gamma = 2.02,\ I_P = 0.3,\ t_w = 160, M = 30, I_e = 0.6, W = 92, c_1 = 15, \lambda = 0.5$

The optimum values of the decision parameters and profit.

$t_R = 1480.05, T = 796.219, \xi = 21.127, TP = 7.6734 \times 10^8$

Here $z_m = 138.10928 \times 10^7, B(T) = 107981.251, Q = 108.119 \times 10^7$

6.1 Graphical Presentation

We provide various graphs for all three cases considered in pervious section. Figs. 5-6, 7-8 and 9-10 are plotted for case 1, case 2 and case 3, respectively. We observe from all these figures that concavity increase with TP and time. After some time, it shows the decreasing trend.

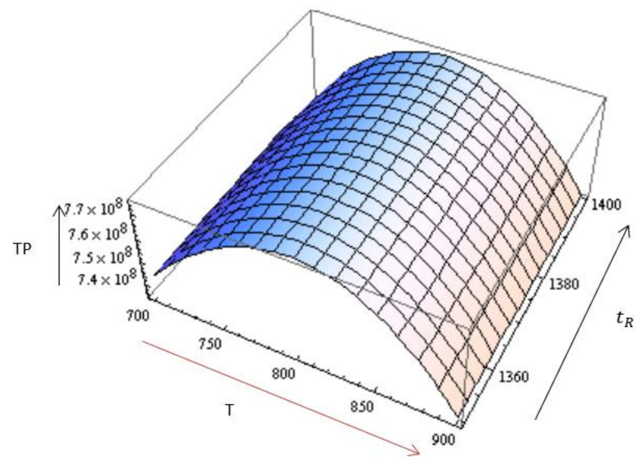

Fig-5. Concavity between T & $\boldsymbol{t_R}$ with TP

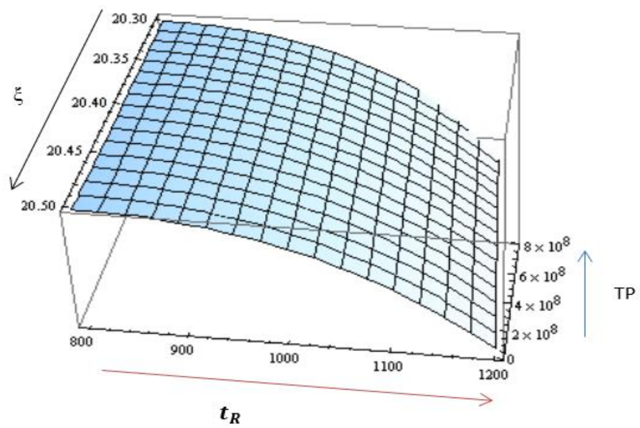

Fig-6. Concavity between ξ & $\boldsymbol{t_R}$ with TP

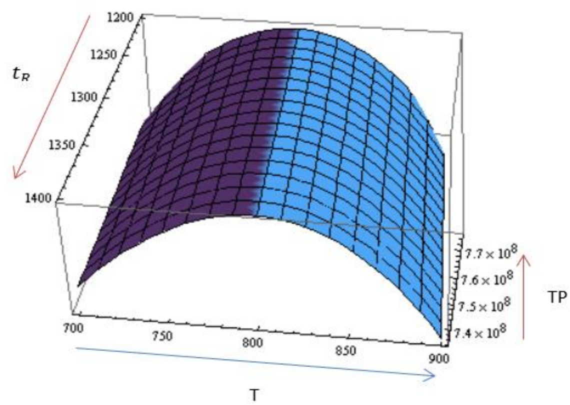

Fig-7. Concavity between T & $\boldsymbol{t_R}$ with TP

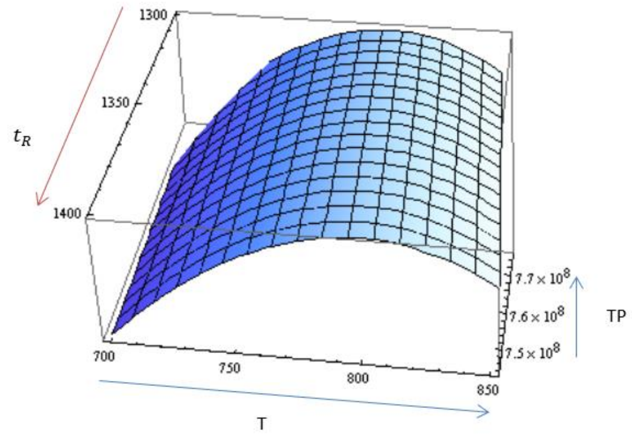

Fig-8. Concavity between T & $\boldsymbol{t_R}$ with TP

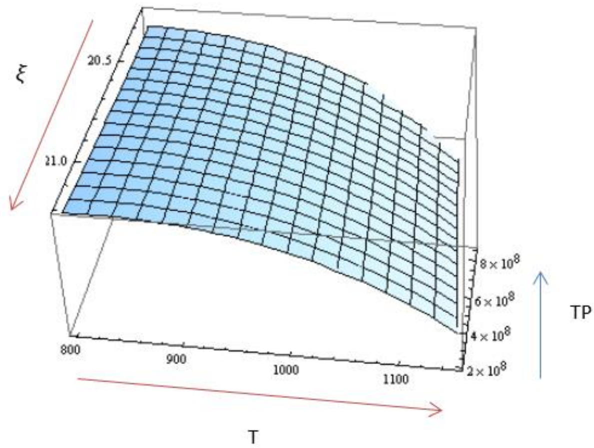

Fig-9. Concavity between T & $\boldsymbol{\xi}$ with TP

6.2 Sensitivity Analysis

Now, we provide the sensitivity analysis for all three cases in the form of various tables. Tables 1, 3 and 5 shows the variations with respect to various parameters such as I_e, I_p, M, h, A^a and y on t_R, T, ξ and TP$\times 10^8$. Moreover, their analysis has been done in tables 2, 4 and 5, respectively.

Table-1 Sensitivity Analysis for case-1 $t_D < t_R < t_w$

NO.	Parameters	Changes	t_R	T	ξ	TP$\times 10^8$
1	I_e	0.4	1481.74	949.205	19.7677	10.0501
		0.5	1359.05	798.139	20.296	7.7499
		0.6	1248.86	673.652	20.3229	5.59430
		0.2	1359.08	798.185	20.305	6.6345

2	I_p	0.3	1359.05	798.139	20.296	7.7499
		0.4	1358.94	798.128	20.274	7.7236
3	M	25	1358.12	797.456	20.147	7.6245
		30	1359.05	798.139	20.296	7.7499
		35	1360.56	798.459	20.312	7.8765
4	h	4	1288.78	786.678	20.128	7.3452
		5	1359.05	798.139	20.296	7.7499
		6	1467.62	806.693	20.435	7.8674
5	A^a	5	1256.08	754.678	19.673	5.659
		6	1359.05	798.139	20.296	7.7499
		7	1434.54	876.438	21.121	8.5623
6	y	10	1365.54	802.243	20.367	7.9834
		11	1359.05	798.139	20.296	7.7499
		12	1345.89	792.436	20.117	7.453

Table-2 Analysis of above table for case-1 $t_D < t_R < t_w$

No	Parameters	Changes	t_R	T	ξ	TP× 10^8
1.	I_e	↓	↑	↑	↓	↑
		↑	↓	↓	↑	↓
2.	I_p	↓	↑	↑	↑	↑
		↑	↓	↓	↓	↓
3.	**M**	↓	↓	↓	↓	↓
		↑	↑	↑	↑	↑
4.	**h**	↓	↓	↓	↓	↓
		↑	↑	↑	↑	↑
5.	A^a	↓	↓	↓	↓	↓
		↑	↑	↑	↑	↑
6.	**y**	↓	↑	↑	↑	↑
		↑	↓	↓	↓	↓

Note- Here upper arrow showing increment and down arrow showing decrement.

Table-3 Sensitivity Analysis for case-2 $t_R < t_D < t_w$

No.	Parameters	Changes	t_R	T	ξ	TP× 10^8
1	θ	0.01	1359.03	798.139	20.2866	7.7547
		0.02	1489.05	798.119	20.125	7.7548
		0.03	1359.02	798.139	20.3066	7.7648
2	M	29	1358.57	797.603	20.2967	7.6058
		30	1489.05	798.119	20.125	7.7548
		31	1359.48	798.675	20.2965	7.7767
3	h	4	1192.87	768.003	20.304	6.5434
		5	1489.05	798.119	20.125	7.7548

		6	1508.28	819.657	20.2917	8.6753
4	λ	0.4	1512.19	798.139	20.2966	7.7685
		0.5	1489.05	798.119	20.125	7.7548
		0.6	1244.61	798.139	20.2966	7.7456

Table-4 Analysis for case-2 $t_R < t_D < t_w$

No.	Parameters	Changes	t_R	T	ξ	TP× 10^8
1	θ	↓	↓	↑	↑	↓
		↑	↓	↑	↑	↑
2	M	↓	↓	↓	↑	↓
		↑	↓	↑	↑	↑
3	h	↓	↓	↓	↑	↓
		↑	↑	↑	↑	↑
4	λ	↓	↑	↑	↑	↑
		↑	↓	↑	↑	↓

Note- Here upper arrow shows increment and down arrow shows decrement.

Table-5 Sensitivity Analysis for case-3 $t_R < t_w < t_D$

No.	Parameters	Changes	t_R	T	ξ	TP× 10^8
1	b	5.1	1374.62	800.609	20.296	7.5367
		5.2	1480.05	796.219	21.1270	7.6734
		5.3	1405.35	805.313	20.2949	7.5743
2	LSC	4.5	1250.74	675.685	20.3347	7.5468
		5.0	1480.05	796.219	21.1270	7.6734
		5.1	1467.54	931.061	20.2659	7.7649
3	t_w	150	1372.29	798.139	20.2966	7.3945
		160	1480.05	796.219	21.1270	7.6734
		170	1346.14	798.139	20.2966	7.4656
4	w	91	1395.06	795.153	20.1460	7.6547
		92	1480.05	796.219	21.1270	7.6734
		93	1485.82	798.118	22.1520	7.7895

Table-6 Analysis for case-3 $t_R < t_w < t_D$

No.	Parameters	Changes	t_R	T	ξ	TP× 10^8
1	b	↓	↓	↑	↓	↓
		↑	↓	↑	↓	↓
2	LSC	↓	↓	↓	↓	↓
		↑	↓	↑	↓	↑
3	t_w	↓	↓	↓	↓	↓
		↑	↓	↓	↓	↓
4	w	↓	↓	↓	↓	↓
		↑	↑	↑	↑	↑

Note- Here upper arrow showing increment and down arrow showing decrement.

7. Conclusion

In the present research, we described inventory control model for non-instantaneous deteriorating products under the two-warehouse considering trade-credit policy, preservation technology, partial backlogging. And preservation technology provides the extra protection to deteriorating items in the warehouse systems. In this research work we have mentioned all conditions. We calculated numerical examples for different cases and validate our research work. Graphical parts of different cases demonstrate the concavity of this model. This inventory model can be applied in the companies, industry related to laptop, mobile phones and garments items etc. In future research work, this work may be extended by involving stock dependent demand, time demand and price dependent demand etc. And this research can be expended by taking various features discount policy, imperfect production, cash follow analysis. Also this research may be expended in fuzzy environment.

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