Exploration of Symmetric G-Complete G-Metric Spaces via Binary Operators: Theoretical Insights and Examples

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Using a new binary operator, we prove some fixed-point theorems in a G-Complete G-Metric space.
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1. Introduction

Metric spaces hold a prominent position in fixed point theory. A landmark conclusion recognized as the Banach Contraction Principle (BCP)[1] in fixed point theory. After this result there are number of generalizations of this result are available by using various metric spaces. One of these spaces, interest of our study is the G-metric space. This robust generalization namely G metric space is introduced by Z. Mustafa, B. Sims [8]. A new binary operator is introduced in [10] and recently in [3] this binary operator used for proving some results in context of G- metric spaces. One can found more interesting concepts and results of G-Metric in [2], [6], [7], [9].

Conversely, a number of authors present different notions that they use to prove their results like commuting mapping [5], weak commutativity [11]. In [12], R.B. Singh constructed a sequence in G metric space with three dose functions using the concept of D – Contraction and fixed function. In this paper we have established a fixed-point theorem using the notion of complete G-metric space with a binary operator. We also furnished an example to verify our result.

2. Preliminaries

Definition 1.1 [8] Let X be a nonempty set, and let the function $G: X \times X \times X \to [0, \infty)$

satisfy the following properties:

(G1) $G(x,y,z) = 0$ if $x = y = z$ whenever $x,y,z \in X$;

(G2) $G(x,x,y) > 0$ whenever $x,y \in X$ with $x \neq y$;

(G3) $G(x,x,y) \leq G(x,y,z)$ whenever $x,y,z \in X$ with $z \neq y$;

(G4) $G(x,y,z) = G(x,z,y) = G(y,z,x) = \cdots,$ (symmetry in all three variables);

(G5) $G(x,y,z) \leq [G(x,a,a) + G(a,y,z)]$.

for any points $x,y,z,a \in X$. Then (X,G) is called a G-metric space.

Proposition 1.1 [8] Let (X,G) be a G-metric space, then for any $x,y,z \in X$ such that $G(x,y,z) = 0$, we have that $x = y = z$.

Definition 1.2 [8] Let (X,G) be a G-metric space, and let $\{x_n\}$be a sequence of points of X, therefore, we say that (x_n) is G-convergent to $x \in X$ if $\lim_{m,n\to\infty} G(x,x_n,x_m) = 0$, that is, for any $\varepsilon > 0$,there exists $N \in \mathbb{N}$ such that $G(x,x_n,x_m) < \varepsilon$, for all, $n,m \geq N$. We call x the limit of the sequence and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.

Proposition 1.2 [8] Let (X,G) be a G-metric space. Define on X the metric $d_G b_y d_G(x,y) = G(x,y,y) + G(x,y,y)$ whenever $x,y \in X$. Then for a sequence $(x_n) \subseteq X$, the following are equivalent

(i) (x_n) is G-convergent to $x \in X$.

(ii) $\lim_{m,n\to\infty} G(x,x_n,x_m) = 0$.

(iii) $\lim_{n\to\infty} d_G(x,x_n) = 0$.

(iv) $\lim_{n\to\infty} G(x,x_n,x_n) = 0$.

(v) $\lim_{n\to\infty} G(x_n,x,x) = 0$.

Definition 1.3 [8] Let(X,G) be a G-metric space. A sequence $\{x_n\}$ is called a G-Cauchy sequence if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$ for all $n,m,l \geq m,l \geq N$, that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to +\infty$.

Proposition 1.3 [8] In a G-metric space (X,G), the following are equivalent

(i) The sequence$(x_n) \subseteq X$ is G-Cauchy.

(ii) For each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $m,n \geq N$.

Definition 1.4 [8] A G-metric space (X,G) is said to be symmetric if

$$G(x,y,y) = G(x,x,y) \text{ for all } x,y \in X .$$

Definition 1.5 [8] A G-metric space (X,G) is G-complete if every G-Cauchy sequence of elements of $(X,G)$is G-convergent in (X,G).

We conclude this introductory part with:

Definition 1.6 [8] A self-mapping T defined on a G-metric space $(X, G)$is said to be orbitally continuous if and only if

$$\lim_{i\to\infty} T^{ni}x = x * \in X \Longleftrightarrow Tx *= \lim_{i\to\infty} T\, T^{ni}x.$$

Theorem 1.1 [8]A G-metric G on a G-metric space (X, G) is continuous on its three variables.

Definition 1.7 [4] A pair of self-mapping (A, B)on a metric space (X, d) is said to be weakly commuting if $d(ABx, BAx) \leq d(Bx, Ax)$ for all x in X. Obviously, commuting mappings are weakly commuting but the converse is not necessarily true.

Definition 1.8 [4] A pair of self-mappings (A,B) of a metric space (X,d) is said to be compatible if $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t \in X$ Obviously, weakly commuting

mapping is compatible.

but the converse is not necessarily true.

In what follows, **N** is the set of all-natural numbers and $\mathbf{R}^+$ is the set of all positive real numbers.

Let $\lozenge\colon R^+ X\, R^+ \to R^+$ be a binary operator satisfying the following conditions:

(1) $\lozenge$ is associative and commutative ;

(2) $\lozenge$ is continuous ;

Some typical examples of $\lozenge$ are:

Example:

$$a \lozenge b = max\{a, b\}; a \lozenge b = a + b\,;\ a \lozenge b = ab + a + b\,; a \lozenge b = \frac{a}{b}(b \neq 0);$$

and $a \lozenge b = \frac{ab}{max\{a,b,1\};}$ for each $a, b, \in R^+$

Definition 1.9 [4] The binary operation is said to satisfy α -property if there exists a positive real number α such that

$$a \lozenge b\, \alpha \leq \max\{a, b\} \text{ where } 0 < \alpha < 1$$

Motivated by Sedghi and Shobe [11], in the present paper, using a new binary operator we prove a common fixed-point theorem on G-complete G-Metric Space.

2. Main Results :

Theorem 2.1. Let (X, G)be a symmetric G-complete G-metric space and T be a mapping from X to itself. Suppose that T satisfies the following condition:

$$G\,(Tx, Ty, Tz) \leq \left(\frac{G(Tx,y,z)+G\,(x,Ty,z)\lozenge G\,(x,y,Tz)}{G\,(x,Tx,Tx)+G\,(y,Ty,Ty)\,\lozenge\, G\,(z,Tz,Tz)+1}\right) G\,(x, y, z) \qquad (2.1)$$

for all $x, y, z \in X$, then

(a) T has at least one fixed point $\xi \in X$;

(b) for any$x \in X$, the sequence $\{T^n x\}$ G-converges to a fixed point;

(c) if $\xi, k \in X$, are two distinct fixed points, then

$$G(\xi, k, k) = G\,(\xi, \xi, k) \geq \frac{1}{1+\alpha}$$

Proof. Let $x_0 \in X$ be arbitrary and construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. Moreover, we may assume, without loss of generality that $x_n \neq x_m$ for $n \neq m$.

For the triplet $x_n, x_{n+1,} x_{n+1}$ and by setting

$d_n = G(x_n, x_{n+1}, x_{n+1})$, we have;

$$0 < d_n = G(x_n, x_{n+1}, x_{n+1}) = G\big(Tx_{n-1,} Tx_{n,} Tx_{n,}\big)$$
$$\leq \left(\frac{G(x_n, x_n, x_n) + G\big(x_{n-1,} x_{n+1}, x_n\big) \lozenge G\big(x_{n-1,} x_{n,} x_{n+1,}\big)}{G\big(x_{n-1,} x_n, x_n\big) + G\big(x_{n,} x_{n+1}, x_{n+1}\big) \lozenge\ G\big(x_{n,} x_{n+1,} x_{n+1,}\big) + 1}\right) G\big(x_{n-1,} x_{n,} x_{n,}\big) dn$$
$$\leq \left(\frac{\alpha d_{n-1} + \alpha d_n}{d_{n-1} + \alpha d_n + 1}\right) d_{n-1}$$

$b_n = \frac{\alpha d_{n-1} + \alpha d_n}{d_{n-1} + \alpha d_n + 1}$

Where $\alpha = 1$

we get, iteratively

$$d_n \leq b_n d_{n-1}$$

$$\leq b_n b_{n-1} d_{n-2}$$

$$\vdots$$

$$\leq b_n b_{n-1} b_1 d_0.$$

<u>Claim</u>: The sequence $\{b_n\}$ is a non-increasing sequence of positive reals.

It is clear that for any natural number$n \in \mathbb{N}$, $0 < b_n < 1$, and so $d_n < d_{n-1}$. We then have the following consecutive equivalences:

$$d_n \leq d_{n-1} \Longleftrightarrow d_n + d_{n+1} \leq d_{n-1} + d_n$$
$$\Longleftrightarrow 1 + \frac{1}{d_{n-1} + d_n} \leq 1 + \frac{1}{d_n + d_{n+1}}$$
$$\Longleftrightarrow \frac{d_{n-1} + d_n + 1}{d_{n-1} + d_n} \leq \frac{d_n + d_{n+1} + 1}{d_n + d_{n+1}}$$

$$\Leftrightarrow \frac{1}{b_n} \leq \frac{1}{b_{n+1}}$$

Hence

$$b_n b_{n-1} \cdots b_1 b_1^n \to 0 \text{ as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} b_n b_{n-1} \cdots b_1 = 0, \text{as } n \to \infty.$$

hence $\lim_{n\to\infty} d_n = 0$

For any $m, n \in \mathbb{N}, m > n$, m, since we have

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} G\left(x_{n+i}, x_{n+i+1}, x_{n+i+1}\right)$$

which translates to

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} b_{n+i}$$

and we obtain

$$G(x_n, x_m, x_m) \leq \sum_{i=0}^{m-n} [(b_{n+i} \ldots.. b_i) d_0]$$

Put $b_k = \alpha_k \ldots \ldots \alpha_1$ and observe that

$$\sum_{k=0}^{\infty} b_k < \infty,$$

$$\lim_{k\to\infty} \frac{b_{k+1}}{b_k} = 0, \text{ i.e. the series}$$

Therefore

$$\sum_{i=0}^{m-n} (b_{n+i} \ldots\ldots b_1) \to 0 \text{ as } m \to \infty$$

In other words, $\{x_n\}$ is a G-Cauchy sequence so G-converges to some $\xi \in X$

Claim: ξ is a fixed point of T.

For the triplet$(x_{n+1}, T\xi, T\xi)$in (2.1), we get

$$G(x_{n+1}, T\xi, T\xi) \leq \left(\frac{G(x_{n+1}, \xi, \xi) + \alpha G(x_n, T\xi, \xi)}{G(x_n, x_{n+1}, x_{n+1}) + \alpha G(\xi, T\xi, \xi) + 1}\right) G(x_n, \xi, \xi) \qquad (2.2)$$

On taking the limit on both sides of (2.2), and using the fact that the function G is continuous, we have

$G(\xi, T\xi, T\xi) = 0$, thus $T\xi = \xi$. If κ is a fixed point of T with $\kappa \neq \xi$, then

$$G(\xi, \kappa, \kappa) = G(T\xi, T\kappa, T\kappa) \leq (1+\alpha)[G(\xi, \kappa, \kappa)]^2$$

Therefore $G(\xi, \kappa, \kappa) = G(\xi, \xi, \kappa) \geq \frac{1}{1+\alpha}$

Corollary 2.1. Theorem 2.1 remains true if contraction condition 2.1 are replaced by any of the following condition :

$$G\,(Tx, Ty, Tz) \leq \frac{G\,(x,Ty,z) \lozenge G\,(x,y,Tz)}{G\,(x,Tx,Tx)+G\,(y,Ty,Ty) \lozenge G\,(z,Tz,Tz)+1} G\,(x, y, z) \tag{2.3}$$

Proof: Corollaries corresponding to the contraction condition of 2.3 can be deduced directly from theorem 2.1 by choosing

o < $d_n = G\left(x_{n,} x_{n+1}, x_{n+1}\right)$

$= G\left(Tx_{n-1,} Tx_{n,}, Tx_{n,}\right)$ respectively.

Example 2.1. Let $X = \{0,1/2,1\}$ and let $G: X^3 \rightarrow [0, \infty)$ be defined by

$$G(0,1,1) = 6 = G(1,0,0), G\left(0, \frac{1}{2}, \frac{1}{2}\right) = 4 = G\left(\frac{1}{2}, 0,0\right)$$

$$G\left(\frac{1}{2}, \frac{1}{2}, 1\right) = 5 = G\left(1, \frac{1}{2}, \frac{1}{2}\right), G\left(0, \frac{1}{2}, 1\right) = \frac{15}{2}$$

$$G(x, x, x) = 0 \quad \forall\, x \in X$$

(X, G) is a symmetric G-complete G-metric space.

Let $T: X \rightarrow X$ be defined by $T_0 = 0, T_{\frac{1}{2}} = \frac{1}{2}, T_1 = 0$.

$$G\left(T_0, T_{\frac{1}{2}}, T_{\frac{1}{2}}\right) = G\left(0, \frac{1}{2}, \frac{1}{2}\right) = 4; G\left(T_{0,} T_{1,} T_1\right) = G(0,0,0) = 0$$

$$G\left(T_{\frac{1}{2}}, T_1, T_1\right) = G\left(\frac{1}{2}, 0,0\right) = 4; G\left(T_{0,} T_{\frac{1}{2},} T_1\right) = G\left(0, \frac{1}{2}, 0\right) = 4.$$

We have

$$4 = G\left(T_0, T_{\frac{1}{2}}, T_{\frac{1}{2}}\right) = G\left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$\leq \frac{G\left(T_{0,} \frac{1}{2}, \frac{1}{2}\right) + G\left(0, T_{\frac{1}{2}}, \frac{1}{2}\right) \lozenge G\left(0, \frac{1}{2}, T_{\frac{1}{2}}\right)}{G(0, T_0, T_0) + G\left(\frac{1}{2}, T_{\frac{1}{2}}, T_{\frac{1}{2}}\right) \lozenge G\left(\frac{1}{2}, T_{\frac{1}{2}}, T_{\frac{1}{2}}\right) + 1} \times G\left(0, \frac{1}{2}, \frac{1}{2}\right)$$

$$\leq \frac{4 + 4 \;\lozenge 4}{1} 4 = (1+\alpha)16$$

$$0 = G(T_0, T_1, T_1) = G(0,0,0)$$

$$0 \leq \left(\frac{G(T_0, T_1, T_1) + G(0, T_1, 1) \lozenge G(0, 1, T_1)}{G(0, T_0, T_0) + G(1, T_1, T_1) \lozenge G(1, T_1, T_1) + 1}\right) \times G(0,1,1)$$

$$\leq \left(\frac{G(0,1,1) + G(0,0,1) \lozenge G(0,1,0)}{G(0,0,0) + G(1,0,0) \lozenge G(1,0,0) + 1}\right) \times G(0,1,1)$$

$$0 \leq \frac{36(1+\alpha)}{6\alpha + 1}$$

$$4 = G\left(T_{\frac{1}{2}}, T_{1,} T_1\right) = G\left(\frac{1}{2}, 0,0\right)$$

$$\leq \left(\frac{G\left(\frac{1}{2}, 1,1\right) + G\left(\frac{1}{2}, T_1, 1\right) \lozenge G\left(\frac{1}{2}, 1, T_1\right)}{G\left(\frac{1}{2}, T_{\frac{1}{2}}, T_{\frac{1}{2}}\right) + G(1, T_1, T_1) \lozenge G(1, T_1, T_1) + 1}\right) \times G\left(\frac{1}{2}, 1,1\right)$$

$$\leq \left(\frac{5 + \alpha \frac{15}{2}}{\alpha\, 6 + 1}\right) \times \ 5$$

$$4 = G\left(T_0, T_{\frac{1}{2}}, T_1\right) = G\left(0, \frac{1}{2}, 0\right)$$

$$\leq \left(\frac{G\left(T_0, \frac{1}{2}, 1\right) + G\left(0, \frac{1}{2}, 1\right) \lozenge G\left(0, \frac{1}{2}, T_1\right)}{G(0, T_0, T_0) + G\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \lozenge G(1, T_1, T_1) + 1}\right) \times G\left(0, \frac{1}{2}, 1\right)$$

$$\leq \left(\frac{\frac{15}{2} + \frac{15}{2}}{\alpha 6 + 1}\right) \frac{15}{2}$$

Theorem 2.2. Let (X, G) be a symmetric G-complete G-metric space where T is an orbitally continuous mapping from X to itself. If it is the case that T satisfies the following condition :

$$G\,(Tx, Ty, Tz) \leq a_1\, G\,(x, y, z) + \ a_2\,[G\,(x, Tx, Tx) \lozenge G\,(y, Ty, Ty) \lozenge G\,(z, Tz, Tz)] + a_3[G(Tx, y, z) \lozenge G(x, Ty, z) \lozenge G(x, y, Tz)]$$

For all $x, y, z \in X$ where $a_i = \ a_i\,(x, y, z), i = 1, \ldots \ldots 3,$ are nonnegative functions such that

$$0 < \lambda < 1;$$

$a_1\,(x, y, z) + \ 3\alpha\, a_2\,(x, y, z) + 4\,\alpha\, a_3\,(x, y, z) < \lambda$.

Then T has a fixed point in X.

Proof: Let $x_0 \in X$ be arbitrary and construct the sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. We have, for the triplet (x_n, x_{n+1}, x_{n+1}) and by setting $d_n = G(x_n, x_{n+1}, x_{n+1})$,

we have :

$$d_{n+1} = G\left(Tx_n, Tx_{n+1}, Tx_{n+1}\right) \leq a_1 d_n + a_2 \alpha[d_n + 2d_{n+1}] + a_3 \alpha[2d_n + 2d_{n+1}]$$

$$d_{n+1} \leq \frac{a_1 + \alpha\, a_2 + 2\alpha\, a_3}{1 - (2\,\alpha\, a_2 + 2\alpha\, a_3)} d_n \tag{2.4}$$

By usual procedure from (2.4), since

$a_1\,(x, y, z) + 3\alpha\, a_2\,(x, y, z) + 4\,\alpha\, a_3\,(x, y, z) < 1$,

it follows that $\{T^n\, x_0\}$ is a G-cauchy sequence. By the definition G-completeness of X, there exists $x^* \in X$ such that $T^n\, x_0$. G-converges to x^*. Since X is G-complete and x^* is fixed point by orbitally continuity of T.

3. Discussion

In this paper, we introduce a novel inequality derived using binary operators within the framework of fixed-point theorems. We have also established some corollaries based on our main theorem and provided illustrative examples in support of our findings. This work has the potential to open new avenues in the study of geometric spaces and fixed-point theory, with possible applications extending to broader fields, including societal issues.

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