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Some More Properties of Separation Axioms Via (1,2)S$_P$-Open Sets in Bitopological Spaces

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In this paper, investigate some properties of (1,2)S-T_0, (1,2)S_p-T_1, (1,2)S_p-T_2, (1,2)S_p-$T_{1/2}$ and establish the relationship between them.
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1. Introduction

Semi-open sets were established and explored by Levin [9] in 1963. In 1963, Kelly [6] introduced the concept of the bitopological spaces, so that a triplet (X,τ_1,τ_2) and X is a non-empty set, τ_1 and τ_2 are topologies on X. (1,2)α-open sets was defined and studied the concept of ultra-T_0, ultra-T_1, ultra-T_2 and ultra-$T_{1/2}$ in bitopological spaces by Lellis Thivagar [8]. The purpose of this paper is to investigate new separation axioms and to discuss its various aspects by using (1,2)S_p-open sets.

2. PRELIMINARIES

All over this paper by X indicate the bitopological spaces (X, τ_1, τ_2). If A is a subset of X, then the (1,2)S_p-closure and (1,2)S_p-interior of A in X are denoted by (1,2)S_p-Cl(A) and (1,2)S_p-Int(A) respectively.

Definition 2.1 [7] A subset A of X is called a

(i) (1,2)semi-open if $A \subseteq \tau_1\tau_2$-Cl(τ_1-Int(A)).

(ii) (1,2)pre-open if $A \subseteq \tau_1$-Int($\tau_1\tau_2$-Cl(A)).

(iii) (1,2)regular-open if A = τ_1-Int($\tau_1\tau_2$-Cl(A)).

Definition 2.2 [8] A subset A of X is called a

(i) (1,2)α-closed if τ_1Cl($\tau_1\tau_2$-Int(τ_1-Cl(A))) $\subseteq A$.

(ii) (1,2)semi-closed if $\tau_1\tau_2$-Int(τ_1-Cl(A)) $\subseteq A$.

(iii) (1,2)pre-closed if τ_1-Cl($\tau_1\tau_2$-Int(A)) $\subseteq A$.

(iv) (1,2)regular-closed if A = τ_1-Cl($\tau_1\tau_2$-Int(A)).

Definition 2.3 [5] A (1,2)semi-open set A of X is called (1,2)S_p-open set if for each $x \in A$, there exists a (1,2)pre-closed set F such that $x \in$ F $\subseteq A$.

Definition 2.4. [5] A point $x \in$ X is said to be a (1,2)S_p-interior point of A, if there exists a (1,2)S_p-open set U containing x such that $x \in U \subseteq A$. The union of all (1,2)S_p-open sets contained in A is said to be (1,2)S_p-interior of A and it is denoted by (1,2)S_p-Int(A).

Definition 2.5. [5] Let A be a set in a bitopological space X. A point $x \in$ X is said to be a (1,2)S_p-closure of A if and only if $A \cap U = \Phi$, for every (1,2)S_p-open set U containing x. The intersection of all (1,2)S_p-closed sets F containing A is called the (1,2)S_p-closure of A and is denoted by (1,2)S_p-Cl(A).

Definition 2.6. [2] Let X be defined by

(i) (1,2)S_p-N-D = {x: $x \in$ X and (1,2)S_p-d({x}) = Φ}.

(ii) (1,2)S_p-N-Shl = {x: $x \in$ X and (1,2)S_p-shl({x}) = Φ}.

(iii) (1,2)S_p-⟨x⟩ = (1,2)S_p-Cl({x}) ∩ (1,2)S_p-Ker({x}).

Lemma 2.7. [2] Let $x \in$ X. Then for any non-empty subset A of X, (1,2)S_p-Ker({A}) = {$x \in$ X / (1,2)S_p-Cl({x}) ∩ $A \neq \Phi$}.

Definition 2.8. [2] In X, a subset A of X is said to be weakly (1,2)S_p- separated from a subset B of X if there exists a (1,2)S_p-open set G of X such that $A \subseteq$ G and G ∩ B= Φ or A ∩ (1,2)S_p-Cl(B) = Φ.

Lemma 2.9. [2] In view of Definition 2.6 and Lemma 2.7we have for x, y in X,

(i) (1,2)S_p-Cl({x}) = {y: y is not weakly (1,2)S_p-separated from x}.

(ii) (1,2)S_p-Ker({x}) = {y: x is not weakly (1,2)S_p-separated from y}.

Remark 2.10. [1] Every (1,2)S_p-closed set is (1,2)S_pg-closed set but the converse is not true.

Lemma 2.11. [1] For a bitopological space X, either every singleton set {x} is (1,2)S_p-closed or its complement $\{x\}^c$is (1,2)S_pg-closed.

Definition 2.12. [1] A subset A of X is called a (1,2)$S_p$generalized-closed (shortly (1,2)S_pg-closed) set in case (1,2)S_p-Cl(A) ⊆ U whenever A ⊆ U and U ∈ (1,2)S_p-O(X).

The family of all (1,2)S_pg-closed set is denoted by (1,2)S_pG-CL(X).

Definition 2.13. [3] A subset A of X is said to be (1,2)S_p-locally closed (shortly (1,2)S_p-LC) if $A = \mathrm{C} \cap \mathrm{D}$, where C is a (1,2)$S_p$-open set and D is a (1,2)S_p-closed set in X. The family of all (1,2)S_p-locally closed set is denoted by (1,2)S_p-LC(X).

Definition 2.14. [4] A subset N of a space X is said to be (1,2)S_p-neighbourhood (shortly (1,2)S_p-nbhd) of a point x ∈ X, if there exists a (1,2)S_p-open set U such that $x \in U \subseteq N$.

3. (1,2)S_P-T_0, (1,2)S_P-T_1 AND (1,2)S_P-T_2 SPACES

Definition 3.1. A bitopological space (X,τ_1,τ_2) is called a (1,2)S_p-T_0 space if and only if for every distinct points x, y ∈ X there exists a (1,2)S_p-open set containing x but not y or a (1,2)S_p-open set containing y but not x.

Example 3.2. Let X = {a, b, c, d}. τ_1 = {Φ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}} and τ_2 = {Φ, X}. Hence, (1,2)S-O(X) = {Φ, X, {a}, { b}, { c }, {a, b}, {a , c}, {a, d}, {b, c}, {b , d}, {c, d}, {a, b, c}, { a, b, d}, { a, c, d}, {b, c, d}}. (1,2)P-CL(X) = {X, Φ, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}. (1,2)S_p-O(X) = {Φ, X, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, X is a (1,2)S_p-T_0 space.

Theorem 3.3. A bitopological space (X,τ_1,τ_2) is (1,2)S_p-T_0 if and only if any of the following conditions hold:

(i) for arbitrary x, y ∈ X, $x \neq y$, either x is weakly (1,2)S_p-separated from y or y is weakly (1,2)S_p-separated from x.

(ii) y ∈ (1,2)S_p-Cl({x}) implies x ∉ (1,2)S_p-Cl({y}).

Proof.

(i) Obvious from the Definitions of 2.8 and 3.1.

(ii) By assumption, y ∈ (1,2)S_p-Cl({x}) and so y is not weakly (1,2)S_p-separated from x. As X is a (1,2)S_p-T_0 space, x should be weakly (1,2)S_p-separated from y, that is x ∉ (1,2)S_p-Cl({y}).

Corollary 3.4. A bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_0 space if and only if one of the following conditions hold:

(i) for all x, y ∈ X, y ∈ (1,2)S_p-Ker({x}) implies x ∉ (1,2)S_p-Ker({y}).

(ii) for all x, y ∈ X, if $x \neq y$, (1,2)S_p-Ker({x}) ≠ (1,2)S_p-Ker({y})

Theorem 3.5. A bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_0 space if and only if [{y} ∩ (1,2)S_p-Cl({x})] ∩ [{x} ∩ (1,2)S_p-Cl({y})] is degenerate.

Proof. Let X be $(1,2)S_p$-T_0 space. Then we have any one of the two cases viz, x is weakly $(1,2)S_p$-separated from y and y is weakly $(1,2)S_p$-separated from x.

Case (i): If x is weakly $(1,2)S_p$-separated from y, then $\{x\} \cap (1,2)S_p$-Cl($\{y\}$) = Φ and $\{y\} \cap (1,2)S_p$-Cl($\{x\}$) is a degenerate set.

Case (ii): If y is weakly $(1,2)S_p$-separated from x, then $\{y\} \cap (1,2)S_p$-Cl($\{x\}$) = Φ and $\{x\} \cap (1,2)S_p$-Cl($\{y\}$) is a degenerate set. Hence, $[\{y\} \cap (1,2)S_p$-Cl($\{x\}$)$] \cap [\{x\} \cap (1,2)S_p$-Cl($\{y\}$)$]$ is a degenerate set.

Conversely, Suppose, $[\{y\} \cap (1,2)S_p$-Cl($\{x\}$)$] \cap [\{x\} \cap (1,2)S_p$-Cl($\{y\}$)$]$ is a degenerate set. Then it is either Φ or a singleton set. If it is a singleton set, it's value is either $\{x\}$ or $\{y\}$. Also, if it is $\{x\}$, then y is weakly $(1,2)S_p$-separated from x. If it is $\{y\}$, then x is weakly $(1,2)S_p$-separated from y. Hence, (X,τ_1,τ_2) is a $(1,2)S_p$-T_0 space.

Theorem 3.6. A bitopological space (X,τ_1,τ_2) is a $(1,2)S_p$-T_0 space if and only if $(1,2)S_p$-d($\{x\}$) $\cap$ $(1,2)S_p$-shl($\{x\}$) = Φ.

Proof. Let X be a $(1,2)S_p$-T_0 space. Suppose $(1,2)S_p$-d($\{x\}$) $\cap$ $(1,2)S_p$-shl($\{x\}$) $\neq \Phi$. Then let $z \in$ $(1,2)S_p$-d($\{x\}$) and $z \in (1,2)S_p$-shl($\{x\}$), $z \neq x$, $z \in$ $(1,2)S_p$-Cl($\{x\}$) and $z \in$ $(1,2)S_p$-Ker($\{x\}$). Thus, z is not weakly $(1,2)S_p$-separated from x and x is not weakly $(1,2)S_p$-separated from z, which is a contradiction. $(1,2)S_p$-Cl($\{x\}$) and $z \notin$ $(1,2)S_p$-Ker($\{x\}$). Thus, if z is not weakly $(1,2)S_p$-separated from x, then x is weakly $(1,2)S_p$-separated from z. Hence, X is a $(1,2)S_p$-T_0 space.

Conversely, $(1,2)S_p$-d($\{x\}$) $\cap$ $(1,2)S_p$-shl($\{x\}$) = Φ. Then there exists a $z \neq x$, $z \in$ $(1,2)S_p$-Cl($\{x\}$) and $z \notin$ $(1,2)S_p$-Ker($\{x\}$). Thus, if z is not weakly $(1,2)S_p$-separated from x, then x is weakly $(1,2)S_p$-separated from z. Hence, X is a $(1,2)S_p$-T_0 space.

Corollary 3.7. If a bitopological space (X,τ_1,τ_2) is a $(1,2)S_p$-T_0 space, then for any $z \in$ X, $(1,2)S_p$-$\langle x \rangle = \langle x \rangle$.

Definition 3.8. A bitopological space (X,τ_1,τ_2) is called a $(1,2)S_p$-T_1 space if and only if for every distinct points x, $y \in$ X there exists a $(1,2)S_p$-open set containing x but not y and a $(1,2)S_p$-open set containing y but not x.

Definition 3.9. A bitopological space (X,τ_1,τ_2) is called a $(1,2)S_p$-T_2 space if and only if for all distinct points x, $y \in$ X there exist two disjoint $(1,2)S_p$-open sets U and V such that $x \in U$ and $y \in V$.

Example 3.10. Let X = {a, b, c}. τ_1 = {Φ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}} and τ_2= {Φ, X}. Hence, (1,2)S-O(X) = {Φ, X, {a}, {b}, { c }, {a, b}, {a , c}, {b, c}}. (1,2)P-CL(X) = {X, Φ, {a}, {b}, { c }, {a, b}, {a , c}, {b, c}}. $(1,2)S_p$-O(X) = {Φ, X, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. Here, X is $(1,2)S_p$-T_2 space.

Remark 3.11. Every $(1,2)S_p$-T_2 space is $(1,2)S_p$-T_1 space.

Remark 3.12. Every $(1,2)S_p$-T_1 space is $(1,2)S_p$-T_0 space, but the converse is not true as shown in the following example.

Example 3.13. Let X = {a, b, c} with two topologies τ_1 = {Φ, X, {a}, {c}, {a, c}} and τ_2 = {Φ, X}. Hence, (1,2)S-O(X) = {Φ, X, {a}, {c}, {a, b}, {a, c}, {b, c}}, (1,2)P-CL(X) = {X, Φ, {b}, {a, b}, {b, c}}, (1,2)S_p-O(X) = {Φ, X, {a, b}, {b, c}}. Here, X is a (1,2)S_p-T_0 space but not a (1,2)S_p-T_1 space.

Theorem 3.14. A bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_0 space if and only if the (1,2)S_p-closure of distinct points are distinct.

Proof. Assume that (1,2)S_p-closure of distinct points are distinct and for $x \neq y$, (1,2)S_p-Cl($\{x\}$) $\neq$ (1,2)S_p-Cl($\{y\}$). Then there exists atleast one point $z \in$ X such that z $\in$ (1,2)S_p-Cl($\{x\}$) but $z \notin$ (1,2)S_p-Cl($\{y\}$). Let $z \in$ (1,2)S_p-Cl($\{y\}$). Then (1,2)S_p-Cl($\{x\}$) $\subseteq$ (1,2)S_p-Cl($\{y\}$). Which implies $z \in$ (1,2)S_p-Cl($\{y\}$), which is a contradiction. Hence $x \in$ [(1,2)S_p-Cl({y})]^c. Therefore, X is a (1,2)S_p-T_0 space.

Conversely, let a bitopological space X be a (1,2)S_p-T_0. Take $x, y \in$ X and $x \neq y$. Then there exists a (1,2)S_p-open set U such that $x \in U$ and $y \notin U$ which implies $y \in U^c = F$ (say). Now, (1,2)S_p-Cl($\{y\}$) = ∩ {F / $\{y\} \subseteq F$ and F is a (1,2)S_p-closed set} which implies that $y \in$ (1,2)S_p-Cl($\{y\}$) and $x \notin$ (1,2)S_p-Cl({y}). Therefore, (1,2)S_p-closure of a distinct points are distinct in X.

Theorem 3.15. A bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_1space if and only if every singleton subset $\{x\}$ of X is a (1,2)S_p-closed set.

Proof. Let $x, y \in$ X and $x \neq y$. If $\{x\}$ and $\{y\}$ are the (1,2)S_p-closed sets of x and y respectively such that $\{x\} \neq \{y\}$, then $\{x\}^c$and $\{y\}^c$are (1,2)S_p-open sets such that $y \in \{x\}^c$ and $x \notin \{x\}^c$ also $x \in \{y\}^c$ and $y \notin \{y\}^c$. Hence, X is a (1,2)S_p-T_1.

Conversely, let a bitopological space (X,τ_1,τ_2) be a (1,2)S_p-T_1space. Then for any $x, y \in$ X and $x \neq y$, there exist two (1,2)S_p-open sets U than V such that $x \in V$, $y \notin V$ and $y \in U$, $x \notin U$ implies $U \subset \{x\}^c$. Also, $\cup\{U \,/\, y \neq x\} \subset \{x\}^c$ and $\{x\}^c \subset \cup\{U \,/\, y \neq x\}$ implies $\{x\}^c = \cup\{U \,/\, y \neq x\}$ and is a (1,2)S_p-open set. Hence, every singleton subset $\{x\}$ of a bitopological space X is a (1,2)S_p-closed set.

Theorem 3.16. If a bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_1 space if and only if the intersection of (1,2)S_p-neighborhoods of an arbitrary points of X is a singleton set.

Proof. Let a bitopological space X be (1,2)S_p-T_1 space. Also, let $x \in$ X and N_x be the (1,2)S_p-neighborhood of x. Choose any point y of X and $y \neq x$ and since X is (1,2)S_p-T_1 space, then there exists a (1,2)S_p-open set containing x but not y and hence y does not belong to the intersection of neighborhoods of x, that is $y \notin N_x$. Since y is an arbitrary, N_x has no point other than x.

Conversely, suppose the intersection of all (1,2)S_p-neighborhood of x is a singleton set $\{x\}$, then it does not contain any other point other than say, y. Similarly, there must exist a (1,2)S_p-neighborhood of y which does not contain x. Hence, by definition 3.8, X is (1,2)S_p-T_1.

Proposition 3.17. Let X be $(1,2)S_p$-T_1 space then any subset A of X is a $(1,2)S_p$-locally closed set.

Proof. Let $A \subseteq$ X and X be $(1,2)S_p$-T_1. Then by Theorem 3.16, for each $x \in$ X, there exists a $(1,2)S_p$-open set U_x such that $U_x \cap A = \{x\}$. If we take $U = \cup\{U_x \ / \ x \in A\}$, then $A = U \cap$ $(1,2)S_p$-Cl(A). Hence, A is a $(1,2)S_p$-locally closed set.

Theorem 3.18. A bitopological space (X,τ_1,τ_2) is $(1,2)S_p$-T_1 if and only if one of the following condition holds good:

(i) for arbitrary x, $y \in$ X, $x \neq y$, x is weakly $(1,2)S_p$-separated from y.

(ii) for all $x \in$ X, $(1,2)S_p$-Cl($\{x\}$) $=$ ($\{x\}$).

(iii) for all $x \in$ X, $(1,2)S_p$-d($\{x\}$) $= \Phi$ either $(1,2)S_p$-N-D = X.

(iv) for all $x \in$ X, $(1,2)S_p$-Ker($\{x\}$) $= \{x\}$.

(v) for all $x \in$ X, $(1,2)S_p$-shl($\{x\}$) $= \Phi$ either $(1,2)S_p$-N-shl = X.

(vi) for all x, y $\in$ X, $x \neq y$, $(1,2)S_p$-Cl($\{x\}$) $\cap$ $(1,2)S_p$-Cl($\{y\}$) = Φ.

(vii) for all x, $y \in$ X, $x \neq y$, $(1,2)S_p$-Ker($\{x\}$) $\cap$ $(1,2)S_p$-Ker($\{y\}$) = Φ.

Proof. (i) Let X be $(1,2)S_p$-T_1 space. Then for every distinct points x, $y \in$ X there exists a $(1,2)S_p$-open set containing of x but not y and a $(1,2)S_p$-open set containing y but not x. This implies that a $(1,2)S_p$-open set U of X containing x, that is $x \subseteq U$ and $U \cap y = \Phi$ or $\{x\} \cap$ $(1,2)S_p$-Cl($\{y\}$) = Φ. Hence, x is weakly $(1,2)S_p$-separated from y.

(ii) Let X be $(1,2)S_p$-T_1 space. Then for every distinct points x, $y \in$ X, there exists a $(1,2)S_p$-open set containing x but not y and a $(1,2)S_p$-open set containing y but not x. If X is weakly $(1,2)S_p$-separated from y, then for $y \neq x$, $y \notin$ $(1,2)S_p$-Cl($\{x\}$) implies $x \notin$ $(1,2)S_p$-Ker($\{y\}$). Thus, $(1,2)S_p$-Ker($\{y\}$) = $\{y\}$ and $(1,2)S_p$-Cl($\{x\}$) $= \{x\}$.

Conversely, let $(1,2)S_p$-Cl($\{x\}$) $= \{x\}$ and $(1,2)S_p$-Ker($\{y\}$) = $\{y\}$. Then $x \notin$ $(1,2)S_p$-Ker($\{y\}$) and $y \notin$ $(1,2)S_p$-Cl($\{x\}$) which implies $x \neq y$. Thus, x is weakly $(1,2)S_p$-separated from y. Hence, X is $(1,2)S_p$-T_1 space.

(iii) Also, (iv) and (v) are obvious.

(vi) Let X be $(1,2)S_p$-T_1 space. By (ii), for every x, $y \in$ X, $(1,2)S_p$-Cl($\{x\}$) $=$ ($\{x\}$) and $(1,2)S_p$-Cl($\{y\}$) $=$ ($\{y\}$). Hence, when $x \neq y$, $(1,2)S_p$-Cl($\{x\}$) $\cap$ $(1,2)S_p$Cl($\{y\}$) $= \Phi$.

(vii) Let X be $(1,2)S_p$-T_1 space. By (iv), for every x, $y \in$ X, $(1,2)S_p$-Ker($\{x\}$) = $\{x\}$ and $(1,2)S_p$-Ker($\{y\}$) = $\{y\}$. Hence, when $x \neq y$, $(1,2)S_p$-Ker($\{x\}$) $\cap$ $(1,2)S_p$-Ker($\{y\ \}$) = Φ.

Definition 3.19. For a bitopological space (X,τ_1,τ_2):

(i) a point $x \in$ X is called a $(1,2)S_p$-neat point if it has X as its only $(1,2)S_p$-neighborhood.

(ii) a bitopological space X is called a (1,2)S_p-symmetric if $x \in$ (1,2)S_p-Cl($\{y\}$) implies $y \in$ (1,2)S_p-Cl($\{x\}$).

Example 3.20. Let X = {a, b, c} with two topologies τ_1 = {Φ, X, {a}, {b}, {a, b}} and τ_2= {Φ, X, {b}, {b, c}} Hence, (1,2)S-O(X) = {Φ, X, {a}, {b}, {a, b}, {b, c}}. (1,2)P-CL(X) = {X, Φ, { c }, {a, c}, {b, c}}. (1,2)S_p-O(X) = {Φ, X, {b, c}}. Here, (1,2)S_p- N({a}) = X ⊆ X, (1,2)S_p- N({b}) = {b, c} ⊆ X and (1,2)S_p- N({b}) = {b, c} ⊆ X. Hence, {a} is a (1,2)S_p-neat point.

Example 3.21. Let X = {a, b, c, d} with two topologies τ_1 = {Φ, X, {a}, {b, c}, {a, b, c}, {b, c, d}} and τ_2= {Φ, X} Hence, (1,2)S-O(X) = {Φ, X, {a}, {b, c}, {a, b, c}, {b, c, d}}, (1,2)P-CL(X) = {X, Φ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}, (1,2)S_p-O(X) = {Φ, X, {a}, {b, c}, {a, b, c}, {b, c, d}}, (1,2)S_p-CL(X) = {X, Φ, {a}, {d}, {a, b}, {b, c, d}}. Here, let x = b, y = c then b ∈ (1,2)S_p-Cl({c}) = {b, c, d} implies c ∈ (1,2)S_p-Cl({b}) = {b, c, d}. Hence, X is (1,2)S_p-symmetric.

Lemma 3.22. A bitopological space (X,τ_1,τ_2) is (1,2)S_p-symmetric if and only if $\{x\}$ is (1,2)S_pg-closed set for each $x \in$ X.

Proof. Let $\{y\}$ be (1,2)S_pg-closed set. Suppose $x \in$ (1,2)S_p-Cl($\{y\}$) and $y \notin$ (1,2)S_p-Cl($\{x\}$). Then $\{y\} \subseteq$ [X − (1,2)S_p-Cl($\{x\}$)]. Since $\{y\}$ is (1,2)S_pg-closed. (1,2)S_pCl($\{y\} \subseteq$ [X − (1,2)S_pCl($\{x\}$)], which is a contradiction. Hence, $x \notin$ (1,2)S_pCl($\{y\}$). Thus, X is (1,2)S_p-symmetric.

Conversely, let a bitopological space X be a (1,2)S_p-symmetric, that is if $x \in A$, A is (1,2)S_p-open. Then (1,2)S_p-Cl($\{x\}$) ⊆ X − A implies (1,2)S_p-Cl($\{x\}$) ∩ (X − A) ≠ Φ. Now, let $y \in$ (1,2)S_p-Cl($\{x\}$) ∩ (X − A), then $x \in$ (1,2)S_p-Cl($\{y\}$) ⊆ (X − A) and hence $x \in$ (X − A), which is a contradiction. Thus, $\{x\}$ is (1,2)S_pg-closed set for each $x \in$ X.

Corollary 3.23. If a bitopological space (X,τ_1,τ_2) is a (1,2)S_p-T_1 space, then it is a (1,2)S_p-symmetric space.

Proof. Let X be a (1,2)S_p-T_1 space. Then by Theorem 3.15, every singleton set is (1,2)S_p-closed. Again by Lemma 2.11, every (1,2)S_p-closed set is (1,2)S_pg-closed. Hence, by Lemma 3.22, X is a (1,2)S_p-symmetric space.

4. (1,2)S_p-$T_{1/2}$ SPACES

Definition 4.1. A bitopological space (X,τ_1,τ_2) is called (1,2)S_p-$T_{1/2}$ space if and only if every (1,2)S_pg-closed subset of X is a (1,2)S_p-closed set.

Theorem 4.3. A bitopological space (X,τ_1,τ_2) is a (1,2)S_p-$T_{1/2}$ space if and only if either $\{x\}$ is (1,2)S_p-open or (1,2)S_p-closed.

Proof. Let X be a (1,2)S_p-$T_{1/2}$ space. If $\{x\}$ is not a (1,2)S_p-closed set, then by Lemma 2.11, A = X − $\{x\}$ is (1,2)S_pg-closed which implies A is (1,2)S_p-closed. Hence, $\{x\}$ is (1,2)S_p-open.

Conversely, let $A \subset$ X be a (1,2)S_pg-closed set with $x \in$ (1,2)S_p-Cl(A). If $\{x\}$ is (1,2)S_p-open then $\{x\} \cap A \neq \Phi$. Otherwise, if $\{x\}$ is (1,2)S_p-closed, then $A \cap$ (1,2)S_p-Cl($\{x\}$) $\neq \Phi$. In both the cases, $x \in A$ and A is (1,2)S_p-closed.

Remark 4.4. Every (1,2)S_p-T_1 space is a (1,2)S_p-$T_{1/2}$ space.

Theorem 4.5. Every (1,2)S_p-$T_{1/2}$ space is a (1,2)S_p-T_0 spac.

Proof. Let X be a (1,2)S_p-$T_{1/2}$ space. Assume that X is not a (1,2)S_p-T_0 space. Then there exist $x, y \in$ X, $x \neq y$ such that (1,2)S_p-Cl($\{x\}$) = (1,2)S_p-Cl($\{y\}$). Let $A =$ (1,2)S_p-Cl($\{x\}$) $\cap \{x\}^c$.

Claim 1: A is (1,2)S_pg-closed.

Let $A \subseteq U$, where U is (1,2)S_p-open. To prove that prove (1,2)S_p-Cl(A) $\subseteq U$. It is enough to prove that (1,2)S_p-Cl($\{x\}$) $\subseteq U$. That is to prove that $x \in U$. If not, let $x \notin U$, then $x \in U^c$which implies (1,2)S_p-Cl($\{x\}$) $\subseteq U^c$. Then there exists a $y \in$ (1,2)S_p-Cl($\{x\}$) $\subseteq U^c$ implies $y \in$ (1,2)S_p-Cl($\{x\}$) $\cap \{x\}^c = A \subseteq U$. Hence, $y \in$ U $\cap\, U^c$ which is a contradiction. Thus $x \in U$. Hence, A is (1,2)S_pg-closed.

Claim 2: A is not (1,2)S_p-closed.

Let $x \in U$ and U is (1,2)S_p-open. By assumption, $x \in$ (1,2)S_p-Cl($\{x\}$) = (1,2)S_p-Cl($\{y\}$) which implies $\{y\} \cap U \neq \Phi$ implies that $\{y\} \subset$ (1,2)S_p-Cl($\{y\}$) $\cap\, U$. That is $\{y\} \subset$ (1,2)S_p-Cl($\{x\}$) $\cap$ U. Then $\{y\} \cap \{x\}^c \subset$ [(1,2)S_p-Cl($\{x\}$) $\cap\, U$] $\cap \{x\}^c$ = [(1,2)S_p-Cl($\{x\}$) $\cap \{x\}^c$] $\cap\, U = A \cap U$. Thus, $x \in$ (1,2)S_p-Cl($\{A\}$), but $x \notin A$. Hence, A is not (1,2)S_p-closed and X is not a (1,2)S_p-$T_{1/2}$ space, which is a contradiction. Hence, X is a (1,2)S_p-T_0 space.

Remark 4.6. The Converse of the above Theorem 4.5, is not true as shown in the following example.

Example 4.7. In Example 3.2, (1,2)S_p-CL(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. (1,2)S_pG-CL(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. Here, X is (1,2)S_p-T_0 space. But every (1,2)S_pg-closed set is not (1,2)S_p-closed set. Hence, X is not a (1,2)S_p-$T_{1/2}$ space.

Definition 4.8. For a subset A of X, $A \in$ (1,2)S_pG-L$\mathbb{C}^*$(X) if there exists a (1,2)S_pg-open set U and a (1,2)S_p-closed set V of X such that $A = U \cap V$.

Definition 4.9. For a subset A of X, $A \in$ (1,2)S_pG-L$\mathbb{C}^{**}$(X) if there exists a (1,2)S_p-open set U and a (1,2)S_pg-closed set V of X such that $A = U \cap V$.

Remark 4.10. For a bitopological spaces (X,τ_1,τ_2),

(i) (1,2)S_p-LC(X) $\subseteq$ (1,2)S_pG-L$\mathbb{C}^*$(X) $\subseteq$ (1,2)S_pG-LC(X).

(ii) (1,2)S_p-LC(X) $\subseteq$ (1,2)S_pG-L$\mathbb{C}^{**}$(X) $\subseteq$ (1,2)S_pG-LC(X).

The equality sign holds good only if X is a (1,2)S_p-$T_{1/2}$ space.

Proof. Let (X,τ_1,τ_2) be a bitopological space. Then (1,2)S_p-O(X) $\subseteq$ (1,2)S_pG-O(X) and

(1,2)S_p-CL(X) ⊆ (1,2)S_pG-CL(X). If X is (1,2)S_p-$T_{1/2}$ space, then every (1,2)S_pg-closed set is a (1,2)S_p-closed set. Hence, the equality holds.

Remark 4.11. The reverse inclusions of Remark 4.10, (i) and (ii) are not always true as shown in the following example.

Example 4.12. (i) Let X = {a, b, c, d} with two topologies τ_1 = {Φ, X, {a, c}, {a, c, d}} and τ_2 = {Φ, X, {b}, {b, c}}. (1,2)S-O(X) = {Φ, X, {a, c}, {a, c, d}}. (1,2)P-CL(X) = {X, Φ, {a}, {b}, {c}, {d}, {a, b}, {a, d}, {b, d}, {c, d}, {a, b, d}}. (1,2)S_p-O(X) = {Φ, X, {a, c}, {a, c, d}}. (1,2)S_p-CL(X) = {X, Φ, {b}, {b, d}}. (1,2)S_pG-CL(X) = {X, Φ, {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}}. (1,2)S_pG-O(X) = {Φ, X, {a}, {c}, {d}, {a, c}, {a, d}, {c, d}, {a, c, d}}. (1,2)S_pG-L$\mathbb{C}^*$(X) = {X, Φ, {a}, {b}, {c}, {d}, {a, c}, {a, d}, {b, d}, {c, d}, {a, c, d}}. (1,2)S_pG-LC(X) = {X, Φ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, c}, {b, c, d}, {a, c, d}}. Here, (1,2)S_pG-LC(X) ⊈ (1,2)S_pG-L$\mathbb{C}^*$(X) ⊈ (1,2)S_p-LC(X).

(ii) In Example 3.2, (1,2)S_p-O(X) = {Φ, X, {a, d}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}. (1,2)S_p-CL(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}}. (1,2)S_pG-CL(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c}}. (1,2)S_pG-O(X) = {Φ, X, {d}, {a, d}, {b, d}, {c, d}, {a, b, d}, {a , c, d}, {b, c, d}}. (1,2)S_p-LC(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, d}, {a, c, d}, {b, c, d}}. 1,2)S_pG-LC(X) = {X, Φ, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. (1,2)S_pG-L$\mathbb{C}^{**}$(X) = {X, Φ, {a}, {b}, {c}, {a, b}, {a, c}, {a, d}, {b, c}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}, {b, c, d}}. Here, (1,2)S_pG-LC(X) ⊈ (1,2)S_pG-L$\mathbb{C}^{**}$(X) ⊈ (1,2)S_p-LC(X).

References

1. Dhanalakshmi.S, Maheswari.M and Durga Devi.N: "Some properties of (1,2)S_P-generalized closed sets in bitopological spaces", International Journal of Mathematical Sciences, Volume. 20, No. 3-4, (2021), pp.345-350.
2. Dhanalakshmi.S, Maheswari.M and Durga Devi.N: "New Characterization of (1,2)S_P-Kernel in bitopological spaces", Ratio Mathematica, Volume. 45, (2023), pp.121-126.
3. Dhanalakshmi.S and Durga Devi.N: "Some generalization of (1,2)S_P-Locally closed sets in bitopological spaces", J. Math. Comput. Sci 11, No. 5, (2021), pp.5931-5936.
4. Durga Devi.N: "Some new separation axioms, Multifunctions and Approximations in bitopological spaces", Manonmaniam Sundaranar University Ph.D., Thesis, India, 2017.
5. Hardi Ali Shareef, Durga Devi.N, Raja Rajeswari.R and Thangaveli.P: "(1,2)S_P-open sets in bitopological spaces", Journal of Zankoy Sulaimani (2007), 19-2, (part-A), 195-201.
6. Kelly.J.C: "Bitopological spaces", Proc. Londan Math. Soc (3), 13, (1963), pp. 71-89.
7. Lellis Thivagar.M: "Generalization of pairwise α-continuous function, Pure and Applied Mathematics and Sciences, Vol.XXXIII, No.1-2, (1991), pp.55-63.
8. Lellis Thivagar.M and Raja Rajeswari.R: "On Bitopological ultra spaces", Southeast Asian Bullitin of Mathematics (2007), 31(5), pp.993-1008.
9. Levine.N: "Some open sets and Semi continuity in topological spaces", Amer. Math. Monthly, 70 (1963), pp.36-41.