Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

Pitchfork Pendant Equitable Domination In Graphs

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Let G= (V, E) be a finite, simple and undirected graph. An equitable dominating set S in G is called a pendant equitable dominating set if ⟨S⟩ contains atleast one pendant equitable vertex. A subset S of V is a pitchfork pendant equitable dominating set if every vertex v∈S dominates atleast j and atmost k vertices of

V −S, where j and k are non-negative integers.The pitchfork pendant equitable domination number $\gamma_{pfpee}(G)$ is the minimum cardinality of a pitchfork pendant equitable dominating set of G. In this article pitchfork pendant equitable domination when j =1 and k =2 is studied. Some bounds on$\gamma_{pfpee}(G)$related to order, size ,minimum and maximum degree of a graph and some properties are given. Pitchfork pendant equitable domination is determined for some known and new modified graphs.

Keywords: Equitable dominating Set (EDS), Pendant equitable dominating Set(PEDS), Pitchfork Pendant equitable dominating Set (PFPEEDS).

2020 Mathematics Subject Classification: 05C38, 05C69

1 Introduction

Let G = (V, E) be any graph with |V (G)|=n vertices and |E(G)|=m edges. Then n, m are respectively called the order and the size of G. The minimum and maximum of the degree among the vertices of G is denoted by δ(G) and Δ(G) respectively. A graph G is said to be regular if δ(G) = Δ(G). A vertex of degree zero is called an isolated vertex and a vertex of degree one is called a pendant vertex. An edge incident to a pendant vertex is called a pendant edge. The lollipop graph $L_{m,n}$ is obtained by joining a vertex of K_m to P_n by edge. Tadpole graph $T_{m,n}$ is obtained by joining a vertex of C_m to P_n by edge. The daisy graph $D_{m,n}$ is obtained by joining two cycles C_m to C_n by a common node. The graph denoted by $(H_1 \times H_2)$ is the Cartesian product of two graphs H_1 and H_2 with $(H_1 \times H_2)$=V (H1) × V (H_2) (where × denotes the Cartesian product of sets) and two vertices u = (u_1,u_2) and v= (v_1,v_2) in V $(G_1 \times G_2)$ whenever [$u_1=v_1$ and

$(u_2,v_2)\in E(G_2)$] or [$u_2=v_2$ and $(u_1,v_1)\in E(G_1)$]. If each G_1 and G_2 is a path P_m and P_n then we call $(P_m\times P_n)$, a m ×n grid graph for our convenience we refer $(P_m\times P_n)$ by $P_{m,n}$ for graph terminology, we refer to [1],[2],[3].

Definition1.1.[4] A subset S of V(G)is a dominating set of G if each vertex $u\in V-S$is adjacent to a vertex in S. The least cardinality of a dominating set in G is called the domination number of G and is usually denoted by $\gamma(G)$.

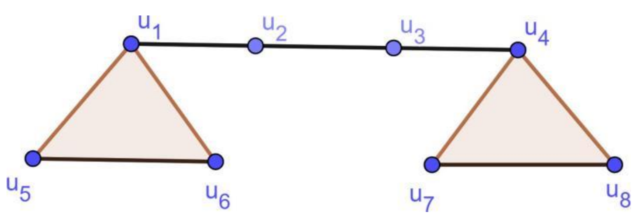
Definition1.2.[5] A dominating set S in G is called a pendant dominating set if $\langle S\rangle$ contains at least one pendant vertex .The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}(G)$.

Definition 1.3. [7] A subset S of V(G) is called an EDS if for every $v \in (V-S)$ there exists a vertex $u \in S$ such that $uv \in E(G)$ and $|deg(u)-deg(v)| \leq 1$.The minimum cardinality of such an equitable dominating set is called equitable domination number of G and is denoted by $\gamma_e(G)$

If $u \in V$ such that $|deg(u) -deg(v)| \geq 2$ for every $v \in N(u)$ then u is in every equitable dominating set such points are called equitable isolates. I_e denotes the set of all equitable isolates. The equitable neighborhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{v \in V: |v \in N(u),|deg(u)-deg(v)| \leq 1\}$.The maximum and minimum equitable degree of a point in G are denoted by $\Delta_e(G)$ and $\delta_e(G)$ that is $\Delta_e(G) = \max_{u\in V(G)}|N_e(u)|$ and $\delta_e(G) = \min_{u\in V(G)}|N_e(u)|$.The open equitable neighborhood and closed equitable neighborhood of v are denoted by $N_e(v)$ and $N_e[v] = N_e(v)\cup\{v\}$ respectively. If $S \subseteq V$ then $N_e(S) = \cup_{v\in S}N_e(v)$ and $N_e[S] = N_e(S)\cup S$. For a detailed treatment of the pendant domination, equitable domination and pitchfork domination reader may referred to [4], [5], [6],[7].
Let S be an EDS in G. Then S is called a PEDS, if $\langle S\rangle$ contains at least one pendant vertex. The pendant equitable dominating set of minimum cardinality is called the pendant equitable domination number denoted b $\gamma_{pee}(G)$.Any PEDS of cardinality $\gamma_{pee}(G)$ is called a $\gamma_{pee}-$set.[8]

Definition1.4. An equitable dominating set S in G is a PFPEEDS if every vertex v in S dominates maximum two vertices of $V -S$ and $<S>$contains pendant vertex. The set S is a minimum PFPEEDS, if it has no proper pitchfork pendant equitable dominating set. The minimum cardinality over all PFPEEDS in G is called pitchfork pendant equitable domination number of G denoted by $\gamma_{pfpee}(G)$

Example 1.1. Consider a graph as shown in the Fig (a).Then, the set $S =\{ u_1, u_2,u_4\}$ is a minimumγ_{pe}-set of G and the set $S'=\{u_1 ,u_2 ,u_3 ,u_4\}$is a minimum pitchfork pendant equitable dominating set of G.



The new domination parameter is defined for all non-trivial connected graphs of order atleast three. Hence, throughout the paper, we assume that by a graph we mean a connected graph of order at least three.

Proposition1.1. Let G be a graph having maximum degree $\Delta_e(G)\leq 2$, then $\gamma_{pee}(G)=\gamma_{pfpee}(G)$.
Proof. Let S be a minimum PEDS in G with equitable domination number $\gamma_{pee}(G)$. Since each vertex in S is adjacent to one or two vertices of V−S, the S is a $\gamma_{pfpee}-$ set

Observation1.1. Let G be a graph having a pitchfork pendant equitable domination $\gamma_{pfpee}(G)$,then

(i) $|V(G)|\geq 2$
(ii) $\delta_e(G)\geq 1$ and $\Delta_e(G)\geq 1$
(iii) $\gamma_{pfpee}(G)\geq 2$
(iv) $\gamma_{pfpee}(G)=2$ iff $G=P_3$ or P_4 or C_3 or C_4 or $k_{1,2}$

Observation1.2. If P_n and C_n are path and cycle graph then, we have

(1) $\gamma_{pe}(P_n)=\gamma_{pfpee}(P_n)$
(2) $\gamma_{pe}(C_n)=\gamma_{pfpee}(C_n)$

Theorem 1.1. Let G be a graph of size m having a pitchfork pendant equitable domination number $\gamma_{pfpee}(G)$, then

$$\gamma_{pfpee}(G) \leq m \leq \binom{n}{2} + \gamma^2{}_{pfpee}(G) + (2-n)\gamma_{pfpee}(G)$$

Proof. Let set S be a $\gamma_{pfpee}-$set of a graph G, then
Case1: By the definition of the pitchfork pendant equitable domination, there exist at least one edge from S to V−S, $\gamma_{pfpee}(G) \leq m$ which is the lower bound.
Case 2: To prove the upper bound, suppose that G[S] and G[V −S] are two complete subgraphs to be G have maximum number of edges where the number of edges of S and V−S equal to m_1 and m_2 respectively, then

$$m_1 = \frac{|S||S-1|}{2} = \frac{\gamma_{pfpee}(\gamma_{pfpee}-1)}{2}$$

$$m_2 = \frac{|V-S||V-S-1|}{2} = \frac{(n-\gamma_{pfpee})(n-\gamma_{pfpee}-1)}{2}$$

Now by the definition of pitchfork pendant equitable domination, there exist at most two edges from every vertex of S to V−S, then the number of edges from S to V−S is atmost or equal to $2|D| = 2\gamma_{pfpee} = m_3$, then the number of edges of G equals to

$$m = m_1 + m_2 + m_3$$

$$= \frac{1}{2}\left(\gamma_{pfpee}^2 - \gamma_{pfpee}\right) + \frac{1}{2}\left(n^2 - n\gamma_{pfpee} - n - n\gamma_{pfpee} + \gamma_{pfpee}^2 + \gamma_{pfpee}\right) + 2\gamma_{pfpee}$$

$$= \gamma_{pfpee}^2 - n\gamma_{pfpee} + 2\gamma_{pfpee} + \frac{n^2 - n}{2}$$

which is the upper bound in general.

Theorem1.1. Let G be a graph with pitchfork pendant equitable domination number

$\gamma_{pfpee}(G)$,then. $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{pfpee}(G) \leq n - 1.$

Proof. First we have to prove that lower bound , let S be a γ_{pfpee} set of G and v_i , v_j $\in$ S where $v_i \neq v_j$, then we have two cases,

Case1: If $N_e(v_i) \cap N_e(v_j) \cap (V-D) = \phi$ then every vertex in V−S is dominated by exactly one vertex of S. Since S is a $\gamma_{pfpee}(G)$−set, then every vertex in S dominates at least one vertex of V −S so $\gamma_{pfpee}(G) = \frac{n}{2}$. And every vertex in S dominates at most two vertices of V−S, then we get the result.

Case 2: If $N_e(v_i) \cap N_e(v_j) \cap (V-D) \neq \phi$, then there exist one or more vertice s in V−S which is dominated by the two vertices v_i and v_j of S together , then

$\gamma_{pfpee}(G) \geq \left\lceil \frac{n}{3} \right\rceil$. Therefore we get lower bound, $\left\lceil \frac{n}{3} \right\rceil \leq \gamma_{pfpee}(G)$.

The upper bound proved as follows : since every vertex in S dominates atleast one vertex and atmost two vertices of V−S, then G must contain atleast one vertex in V−S that is dominated by all the other n−1 vertices of G which will be belonging to S. Therefore $\gamma_{pfpee}(G) \leq n - 1$.

Theorem 1.2. Let G be a connected graph with pitchfork pendant equitable domination then , $\gamma_e(G) \leq \gamma_{pee}(G) \leq \gamma_{pfpee}(G)$.

Proof.From the definition of pitchfork pendant equitable domination, every PFPEEDS is a PEEDS and every PEDS is an EDS.

Corollary1.1. Let G be a graph having a pitchfork pendant equitable domination number then:

(I) $\gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\delta_e + 2} \right\rceil$

(II) $\gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\Delta_e + 2} \right\rceil$

$(III)\ \gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\delta_e+\Delta_e+2} \right\rceil$

(IV) $\gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\delta_e^n+2} \right\rceil$

(V) $\gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\Delta_e^n+2} \right\rceil$

$(VI)\ \gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\delta_e\Delta_e+2} \right\rceil$

$(VII)\ \gamma_{pfpee}(G) \geq \left\lceil \frac{n}{\frac{\Delta_e}{\delta_e}+2} \right\rceil$

2 Pitchfork pendant equitable domination of some families of Graphs

Here, the pitchfork pendant equitable domination is determined for several known and modified families of graphs.

Theorem2.1. Let G be a path or cycle graph with n vertices. Then,

$$\gamma_{pfpee}(G) = \begin{cases} \frac{n}{3}+1 & if\ n \equiv 0(mod3) \\ \left\lceil \frac{n}{3} \right\rceil, & if\ n \equiv 1(mod3) \\ \left\lceil \frac{n}{3} \right\rceil + 1 & if\ n \equiv 2(mod3) \end{cases}$$

Proof. Let G be a path or cycle graph and let V(G) = $\{v_1,v_2,...,v_n\}$. We consider the following possible cases here:

Case1: Suppose n≡0(mod3).Then n=3k, for some integer k>0. Then the set S=$\{v_2,v_{3i}|1\leq i\leq k\}$ will be a pitchfork pendant equitable dominating set of G and each vertex in G dominates atmost two vertices of G. Hence, $\gamma_{pfpee}(G)\leq|S|$.

i.e., $\gamma_{pfpee}(G)=\frac{n}{3}+1$. On the other hand , we have $\gamma_e(G)=\frac{n}{3}$ and any least dominating set of G contains only vertices of degree zero.

Thus $\gamma_{pfpee}(G) \geq \frac{n}{3} + 1$. Therefore, $\gamma_{pfpee}(G) = \frac{n}{3} + 1$.

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Case 2: Suppose n ≡ 1(mod 3). Then it is easy to check that any γ_e −set in G contains a pendant vertex and each vertex in γ_e set dominates atmost two vertices inG. Hence any γ_e −set in G itself a pitchfork pendant equitable dominating set in G.

Therefore , $\gamma_{pfpee}(G) = \gamma_e(G) = \left\lceil \frac{n}{3} \right\rceil$.

Case3: Proof of this case is analogous to Case 1.

Observation2.1. For a path graph P_n and cycle graph C_n, we have

(1) $\gamma_{pee}(P_n) = \gamma_{pfpee}(P_n)$

(2) $\gamma_{pee}(C_n) = \gamma_{pfpee}(C_n)$

Theorem2.2. Let G be the tadpole graph $T_{m,n}$ where(m≥4) and (n≥3). Then,

$$\gamma_{pfpee}(T_{m,n}) = \begin{cases} \frac{m}{3} + \left\lceil \frac{n-1}{3} \right\rceil + 1 & if\ n \equiv 0(mod3) \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil, & if\ n \equiv 1(mod3) \\ \left\lceil \frac{m}{3} \right\rceil + \left\lceil \frac{n-1}{3} \right\rceil + 1 & if\ n \equiv 2(mod3) \end{cases}$$

Proof. The tadpole graph contains a cycle C_m joined by a bridge to a path graph P_n then it contains m+n number of vertices and edges .The vertices of C_m can be labeled as $\{u_j$; j=1,2,...,,m$\}$ and also the vertices of P_n as $\{v_i$; i = 1,2,...,,n$\}$ such that the vertex $u_1 \in C_m$ is adjacent with the vertex $v_n \in P_n$ and the vertex u_1 dominates n^{th} vertex of P_n and deg(u_1) =3. Let the pitchfork pendant equitable dominating $S=S_1 \cup S_2$ where S_1 is a pendant equitable dominating set of C_m and S_2 is the equitable dominating set of P_{n-1}. According to m we have three cases.

Case1: If m = 3k, then let $S_1=\{v_1, v_{3i} \backslash 1 \leq i \leq k\}$ and $S_2=\{u_{3i-1}$,i= 1,2,...,$\left\lceil \frac{n-1}{3} \right\rceil\}$.Therefore the set S=|$S_1$|+|$S_2$|= $\frac{m}{3} + \left\lceil \frac{n-1}{3} \right\rceil + 1$

Case2: If m=3k+1 ,then any equitable dominating set of C_m contains a pendant Equitable vertex and dominates atmost two vertices and the vertex u_n is dominated

By the vertex $v_1 \in C_m$.Therefore , the set

$$S = \gamma_{pee}(C_m) + \gamma_{pee}P(n-1) = \left\lceil\frac{m}{3}\right\rceil + \left\lceil\frac{n-1}{3}\right\rceil$$

Is a minimum PFPEEDS of $T_{m,n}$
Case3: Proof of this case is analogous to Case 1.

Theorem2.3. For the lollipop graph $L_{m,n}$; m=3,4, n≥2 we have,

$$\gamma_{pfpee}(L_{m,n}) = \gamma_{pee}(K_m) + \left\lceil\frac{n}{3}\right\rceil$$

Proof. All the vertices of K_m can be labeled as $\{v_i\,;\, i=1,2,...,m\}$ and the vertices of P_n as $\{u_j\,;\, j=1,2,...,n\}$, where the vertex v_1 is adjacent with a vertex u_n.

The set S= $S' \cup \gamma_e(P_n)$ will be a minimum PFPEEDS of $L_{m,n}$ where S' is the PEDS of K_m.

Therefore $\gamma_{pfpee}(L_{m,n}) = |S| = \gamma_{pee}(K_m) + \left\lceil\frac{n}{3}\right\rceil$

Proposition2.1. If $G \cong L_{m,n}$, m > 4 , n > 3,

then $\gamma_{pfpee}(L_{m,n}) = \gamma_{pee}(P_n) + \left\lceil\frac{m}{3}\right\rceil$ 3
Theorem2.4. Let G be a wheel graph W_n where n≥6 , then

$$\gamma_{pfpee}(W_n) = 3 + \left\lceil\frac{n-5}{3}\right\rceil$$

Proof. Let W_n be a wheel graph and let us label the vertices of W_n as : $\{v_1,v_2,,...,v_{n+1}\}$ where $\deg(v_i)=3$ for all i=1,2,...,n and $\deg(v_{n+1})=n$. The set $S'=\{v_1,v_2,v_{n+1}\}$ is a PEDS of the graph W_n. Here the vertex v_1 dominates two vertices v_2 and v_{n-1} and the vertex v_{n+1} is equitable isolate.The set$S=\{v_1,v_2,v_{n+1}\} \cup \{v_4,v_5,...,v_{n-4},v_{n-3}\}$ will be a PFPEEDS of W_n.

Therefore, $\gamma_{pfpee}(W_n) = |S| = 3 + \left\lceil\frac{n-5}{3}\right\rceil$

Theorem2.5. Let $D_{m,n}$ be the daisy graph ,then $\gamma_{pfpee}(D_{m,n}) = \gamma_{pee}(C_m) + \left\lceil \frac{n-2}{3} \right\rceil$

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Proof. The daisy graph contains two cycles C_m and C_n with common vertex. Let us label the vertices of C_m as $\{u_i;i=1,2,...,m\}$ so that the vertices of C_n as $\{v_j;j=1,2,...,n\}$. The set $S = S_1 \cup S_2$ is a γ_{pfpee}-set of daisy graph where S_1 is the PEDS of C_m and S_2 is the EDS of C_{n-2}. Therefore $\gamma_{pfpee}(D_{m,n}) = |S| = \gamma_{pee}(C_m) + \left\lceil \frac{n-2}{3} \right\rceil$

Theorem2.6. Let $G \cong P_{2,n}$ be a grid graph where $n \geq 2$, $\gamma_{pfpee}(P_{2,n}) = \left\lceil \frac{2n}{3} \right\rceil$

Proof. If n=2, then its easy to find out the pitchfork pendant equitable domination number of grid graph $P_{2,2}$. If $n \geq 3$, then a minimum PFPEEDS of $P_{2,n}(n \geq 3)$ is presented as follows.

Let $n = 3q$. Here, we split the set of columns of $P_{2,n}$ into blocks B_i, where $B_i \cong P_{2,3}$ For i=1,...,q. The vertices • enclosed within the round symbol in each of the blocks in the figures represent the vertices to be included for a minimal PFPEEDS D.

Let $P_i = \{X_{1,3i-1}, X_{2,3i-1}\}, i = 1,2,...,q$.

Let $D = \cup_{i=1}^{q} P_i$ Therefore $|D| = 2\left\lfloor \frac{n}{3} \right\rfloor$

Case1: $n \equiv 1 \pmod 3$

Consider the set $D_1 = D \cup \{X_{2,n}\}$ (Figure 2.2(a)). This set is a equitable dominating set and induced subgraph of D_1 contains a pendant equitable vertex and each vertex in D_1 dominates maximum two vertices in G. Therefore , the set D_1 is a minimal

PFPEEDS of $P_{2,n}$. Hence , $|D_1| = 2\left\lfloor \frac{n}{3} \right\rfloor + 1 = \left\lceil \frac{2n}{3} \right\rceil$

Case2: $n \equiv 2 \pmod 3$

Here the set $D_2 = D \cup \{X_{1,n}, X_{2,n}\}$ (Figure2.2(b)) is a minimal PFPEEDS of $P_{2,n}$.

Hence , $|D_2| = 2\left\lfloor \frac{n}{3} \right\rfloor + 2 = \left\lceil \frac{2n}{3} \right\rceil$

Case3: $n \equiv 0 \pmod 3$

In this case ,the set $D_3 = D \cup \{X_{1,n-1}, X_{2,n-1}\}$ (Figure2.2(c)) is a minimal PFPEEDS of $P_{2,n}$ and $|D_3| = \left\lceil \frac{2n}{3} \right\rceil$

From all the cases , $\gamma_{pfpee}(P_{2,n}) = \left\lceil \frac{2n}{3} \right\rceil$

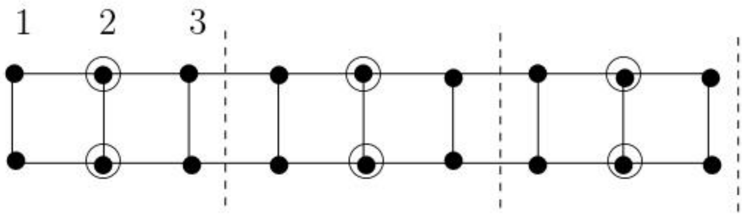

Figure 2.1

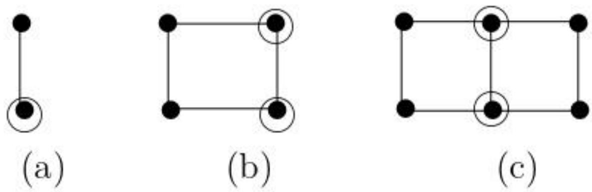

Figure 2.2

References

[1] BondyJ.A.and MurtyU.S.R,(1984),Graph theory with application , Elsevier science Publishing Co, sixth printing.

[2] Haynes T.W., Hedetniemi S.T., and Slater P.J.,(1998) Fundamentals of Domination in Graphs, Marcel Dekker, New york.

[3] HararyF.,(1969),GraphTheory, Addison-Wesley, ReadingMass.

[4] Al-Harere M. N. and Abdlhusein M. A., Pitchfork domination in graphs,(2020), Discrete Math. Algorithm Appl. 12(2) , 2050025.

[5] Nayaka S.R, Puttaswamy and Purushothama S,(2020) ,Pendant Domination in Graphs. Journal of Combinatorial Mathematics and Combinatorial Computing, 112, pp. 219-230.

[6] Purushothama.S , Puttaswamy and NayakaS.R, Pitchfork Pendant Domination in Graphs. elsevier publication , 100, pp. 139-142.

[7] A.Anitha,S.Arumugam and Mustapha chellali,(1998) Equitable domination in graphs, Networks, 32(3), pp 199-206.

[8] Deepak B.P, Puttaswamy , Sivakumar S. and Nayaka S.R Pendant Equitable dom- ination in graphs, (communicated)