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# S-Pure Rickart Modules

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In this paper we introduce the concept S.pure Rickart module (or s.p.r-module for short) as a generalization of pure Rickart module. Let S semiradical property and let M, N be two modules. We are stating that M is N- s.p.r. module if for every homomorphism f:M→N, ker f is S.pure submodule of M. The main goal of studying S.Pure Rickart modules is to describe the properties of this type of the class modules and to prove some theories and properties their relationships. Also the importance of their applications in constructing algebraic concepts with respect to a special classes of S.

**Keywords:** Pure Rickart module, S. pure Rickart module, pure submodule, S. Pure submodule, Kernel of homomorphism.

## 1. Introduction

Our research includes studying and describing S. Pure Rickart modules and giving some examples and remarks their relationship to some algebraic concepts. As well as their importance in applications of module theory, ring theory, and homological algebra. G. Ahmed in [1], [2] introduced the concepts pure (dual pure) Rickart modules. A module $M$ is called a pure (dual pure) Rickart module if for every homomorphism $f: M \to M$, $\ker f$ (Im f) is a pure submodule of M.

N. Hamad and B. AL-Hashmi in [3] introduced semiradical property, A property $S$ is called a semiradical property if:

1- For each module $M$, there exists a submodule (briefly $S(M)$ ) such that:

a- $A \leq S(M)$ for every submodule $A$ of $M$ such that $S(A) = A$.

b- $S\big(S(M)\big) = S(M)$.

2- If $f: M \to N$ is an epimorphism and $S(M) = M$, then $S(N) = N$.

A semiradical property is said to be radical property if $S\left(\frac{M}{S(M)}\right) = 0$, for each module M.

The following two properties is a radical property.

1- $S = Snr$. For a module M, let $S(M) = Snr(M) = \sum_{\substack{A \leq M \\ J(A)=A}} A$, where $J(A)$ is the Jacobson radical of A, see [4]. It's known that Snr is a radical property, see [3].

2- $S = Sa$. Let M be a module. M is called a semiartinian module (denoted by Sa-module), if for every proper submodule A of M, $Soc(\frac{M}{A}) \neq 0$, see [4]. It's clear that every artinian module is semiartinian. For a module M, let $S(M) = Sa(M) = \sum_{\substack{A \leq M \\ A \text{ is } Sa}} A$, $Sa(M)$ is called the semiartinian submodule of M. It's known that Sa is a radical property, see [3].

While the following three properties are semiradical property, see [3].

1- $S = Z$. For a module M, let $S(M) = Z(M)$ the singular submodule of M. Z is a semiradical property, see [3].

2- $S = Soc$. For a module M, let $S(M) = Soc(M)$, the socle submodule of M. Soc is a semiradical property, see [3].

3- $S = \mathcal{M}$ For a module N, let $S(M) = \mathcal{M}(N) = \sum_{\substack{A \leq N \\ A \text{ is regular}}} A$, , $\mathcal{M}(N)$ is called semi Broun-McCoy radical), see [5]. $\mathcal{M}$ is a semiradical property, see [3].

E. Al-Dhahari and B. Al-Bahrani in [6] introduced the concept S. pure submodule, let N be submodule of M. Then N is s. p. closed submodule (S. pure submodule) of M if and only if $S\left(\frac{M}{N}\right) = 0$.

This observation led us to introduce the concepts S. pure Rickart modules. Let S semiradical property and let M, N be two modules. We are stating that M is N- s. p. r. module if for every homomorphism $f: M \rightarrow N$, $\ker f \leq_{s.p} M$.

The work comprises three sections. In Section two, we introduce the concept of s. p. r. module. We illustrate some examples and provides properties. For instance, we prove that, $S(M) = 0$ if and only if M is s. p. r. module see Propo. 2.3. We show that, M be a faithful module. If M is s. p. r. module, then R is s. p. r. ring, see Propo. 2.4. In addition, we prove that M be a f. g faithful multiplication module. Then R is s. p. r. ring if and only if M is s. p. r. module, see Propo. 2.5. We prove that, when S Cohereditary radical property and $M_1, M_2$ be modules. If $M_1$ or $M_2$ are s. p. r. modules, then $M_1 \otimes M_2$ is s. p. r. module, see Propo. 2.6. We show that when S radical property and M be a f. g multiplication module. If M is s. p. r. module, then $E = End(M)$, the endomorphism ring is s. p. r. ring, see Propo. 2.7. We prove that, $M_1$ and $M_2$ be modules such that $S(M_1) = 0$ and M has no non trivial S.pure submodule. If $M_1$ is $M_2$-s. p. r. module, then either $Hom(M_1, M_2) = 0$ or every nonzero homomorphism from $M_1$ to $M_2$ is a monomorphism, see Propo. 2.8. In addition, we show that when $Hom(M_1, M_2) \neq 0$. If $M_1$ is $M_2$-s. p. r. module, then $M_1$ is Quasi Dedekind. In particular if $M_1$ is s. p. r. module, then $M_1$ is Quasi Dedekind, see Coro. 2.9. In addition we give a characterizations for the s. p. r. module, see Theo. 2.10. We prove that when S be a hereditary property and $M_1$ be a module. Then $M_1$ is $M_2$-s. p. r. module if and only if $Hom(M_1, M_2) = 0$, for every module N such that $S(M_2) = M_2$, see Propo. 2.11. We prove that R is s. p. r. ring if and only if every projective (free) module is s. p. r. module, see Propo. 2.12. We prove that $S(R) = 0$ if and only if for each module M is N-s. p. r. module, for every projective (free) module N, see Propo.

2.13.

In section three, we study s. p. r. modules, with respect to a special classes of S. For instance, S be a hereditary property and R be ID. If M be a prime module such that $S(M) \neq M$, then M is s. p. r. module, see Propo. 3.1. In addition, if M is a torsion free (projective, free) module such that $S(M) \neq M$, then M is s. p. r. module, see Coro. 3.2 (3.3). We prove that when R be ID and R not a field. If M is a non-zero torsion free module, then M is $Z(\mathcal{M})$. p. r. module and $Soc(R) = 0$, see Propo. 3.4. In addition If M be a non-zero projective (flat) module, then M is $\mathcal{M}$. p. r. module, Z. p. r. module, Soc. p. r. module and $\mathcal{M}$. p. r. module, see Coro. 3.5. We prove that R be ID and M be a flat module. If M is p. r. module, then M is Z. p. r. module. The converse is true when R is a principle ID, see Propo. 3.6. We show that R be ID and M be a module. If M is Z-regular module. Then M is Z. p. r. module, see Propo. 3.7. In addition if M is Z-regular module, then M is Snr. p. r. module, see Propo. 3.8. We show that M be a non-zero module. If M is Z-regular, then M is not $\mathcal{M}$. p. r. module, see Propo. 3.9. We demonstrate that, M be semisimple projective module. Then M is Z(Snr). p. r. module and not $\mathcal{M}$(Soc). p. r. module, see Propo. 3.10. Recall that K is called a pure submodule of module M, if $K \cap IM = IK$, for every finitely generated ideal I of R, see [7]. Recall that M is called regular module if every submodule of M is pure, see [5]. A ring R is a pure simple if 0 and R are the only pure ideals of R, see [8]. For a module M, the singular submodule of a module M define as $Z(M) = \{m \in M : ann(m) \leq_e R\}$. M is called a singular module if $Z(M) = M$ and M is called a nonsingular module $Z(M) = 0$, see [9]. A submodule N of a module M is called a fully invariant submodule if for every $f \in End(M)$, , $f(N) \subseteq N$, see [10]. A module M is called a Quasi Dedekind module if each $0 \neq f \in End(M)$, is a monomorphisem, see [11]. A module M is called Co-Quasi Dedekind module if for each $0 \neq f \in End(M)$, $Im\, f = M$, see [12]. A module M is called a faithful module if $ann(M) = 0$, where $ann(M) = \{r \in R \mid rx = 0, \forall x \in M\}$, see [13]. Let M be a module, the Jacobson radical of M, $J(M) = \bigcap_{\substack{A \leq M \\ A \text{ is maximal}}} A$. If M has no maximal submodule, then J(M) = M, see [4]. The socle of M, $Soc(M) = \sum_{\substack{A \leq M \\ A \text{ is simple}}} A$, see [9].

For a left module M, End(M) that will mean the endomorphism ring of M. The observes $K \leq M$, $K \leq_p M$, $K \leq_\oplus M$, $K \leq_{s.p} M$, f. g and ID mean that K is a submodule, a pure submodule, direct summands s.pure submodule of M, finitely generated and integral domain. In this paper, we mean S is a semiradical property unless otherwise stated. Throughout this article, R is a ring with identity and M is a until left R-module.

## 2- S.pure Rickart Modules

This section introduces the concept of S.pure Rickart module. We illustrate some examples and provides some properties are investigated. We start by the following definition.

Definition 2.1: Let M and N be two modules. We say that M is N-S.pure Rickart module (or s. p. r. module for short) if for every homomorphism $f: M \to N$, $\ker f \leq_{s.p} M$. In particular if M is M-s. p. r. module, then we say that M is s. p. r. module. If $M = R$, then we say R is s. p. r. ring if R is s. p. r. as a module.

Examples 2.2: 1- Let $S = Z$. Consider the modules Z and Q as Z-modules and let $f: Z \to Q$ be a homomorphism, then $\frac{Z}{\ker f} \cong \operatorname{Im} f$, by the first isomorphic theorem. Since Q is nonsingular module, then $\operatorname{Im} f$ is nonsingular. Therefore $S\left(\frac{Z}{\ker f}\right) \cong S(\operatorname{Im} f) = 0$ and hence $\ker f \leq_{s.p} Z$. Thus Z is Q-s. p. r. module.

2- Let $S = \text{Soc}$. Consider the modules $Z_4$ and $Z_2$ as Z-modules. Claim that $Z_4$ is not $Z_2$-s. p. r. module. To show that, let $f: Z_4 \to Z_2$ be a map define by $f(n) = 2n, \forall n \in Z_4$. Clearly that f is a homomorphism. But $\ker f = \{\bar{0}, \bar{2}\}$, so $\frac{Z_4}{\{\bar{0},\bar{2}\}} \cong Z_2$, and hence $S\left(\frac{Z_4}{\{\bar{0},\bar{2}\}}\right) \cong S(Z_2) = Z_2 \neq 0$. Therefore kernel f is not S. pure submodule of $Z_4$. Thus $Z_4$ is not $Z_2$-s. p. r. module.

Proposition 2.3: Let M be a module. The following statements are equivalent:

1- $S(M) = 0$.

2- M is s. p. r. module.

3- Every module N is M-s. p. r. module.

4- For every epimorphism f from any module N to M, $\ker f \leq_{s.p} N$.

Proof: 1⇒2) Let $f: M \to M$ be a homomorphism and let $S(M) = 0$. Since $\frac{M}{\ker f} \cong \operatorname{Im} f$, then $S\left(\frac{M}{\ker f}\right) \cong S(\operatorname{Im} f) = 0$. Hence $\ker f \leq_{s.p} M$. Thus M is s. p. r. module.

2⇒1) Let $1_M: M \to M$ be the identity map. Since M is s. p. r. module, then $\ker 1_M = 0 \leq_{s.p} M$. Therefore $S(M) = 0$.

1⇒3) Let M, N be modules such that $S(M) = 0$. Let $f: N \to M$ be a homomorphism. Since $\frac{N}{\ker f} \cong \operatorname{Im} f$ and $S(M) = 0$, then $S\left(\frac{N}{\ker f}\right) = 0$. So $\ker f \leq_{s.p} N$. Thus N is $M -$ s. p. r-module.

3⇒4) Clear.

4⇒1) Let M be a module. By [ [4], Coro. 4.4.4, p.89], there exist a free module F and an epimorphism $f: F \to M$. By our assumption $\ker f \leq_{s.p} F$ and hence $S\left(\frac{F}{\ker f}\right) = 0$. But f epimorphism, so $\frac{F}{\ker f} \cong M$. Thus $S(M) = 0$.

Proposition 2.4: Let M be a faithful module. If M is s. p. r. module, then R is s. p. r. ring.

Proof: Let $f: R \to R$ be a homomorphism. Since M is s. p. r. module, then $S(M) = 0$, by Propo. 2.6. But $S(R)M \subseteq S(M)$, by [ [3], Propo. 19-3, p.61 ], therefore $S(R)M = 0$. But M be a faithful module, therefore $S(R) = 0$. Thus R is s. p. r. ring, by Propo. 2.3.

Proposition 2.5: Let M be a f. g faithful multiplication module. Then R is s. p. r. ring if and only if M is s. p. r. module.

Proof: Suppose that R is s. p. r. ring. Then $S(R) = 0$, by Propo. 2.6. Since M be a f. g faithful multiplication module, then $S(R)M = S(M)$, by [ [3], Propo. 24-3, p.63 ]. Therefore $S(M) = 0$. Thus M is s. p. r. module, by Propo. 2.3.

For the converse, is clear by Propo. 2.4.

Proposition 2.6: Let S Cohereditary radical property and $M_1, N$ be modules. If $M_1$ or $M_2$ are $s.p.r.$ modules, then $M_1 \otimes M_2$ is $s.p.r.$ module.

Proof: Assume that $M_1$ is $s.p.r.$ modules. We want to show that $M_1 \otimes M_2$ is $s.p.r.$ module. Then $S(M_1) = 0$, by Propo. 2.6. By [ [3], Coro. 46-3, p.74] $S(M_1 \otimes M_2) = S(M_1) \otimes M_2 = 0 \otimes M_2 = 0$. Thus $M_1 \otimes M_2$ is $s.p.r.$ module, by Propo. 2.3.

Proposition 2.7: Let S radical property and M be a $f.g$ multiplication module. If M is $s.p.r.$ module, then $E = End(M)$, the endomorphism ring is $s.p.r.$ ring.

Proof: Suppose that M is $s.p.r.$ module. Since M be a $f.g$ multiplication faithful module as a ring R, then M be a $f.g$ multiplication faithful module as a ring E, by [14]. Therefore $S(E)M = S(M)$, by [ [3], Propo. 24-3, p.63 ]. But M is $s.p.r.$ module, so $S(M) = 0$, by Propo. 2.3 and hence $S(E)M = 0$. Since M be a faithful module, then $S(E) = 0$. Thus E is $s.p.r.$ ring, by Propo. 2.3.

Proposition 2.8: Let $M_1$ and $M_2$ be two modules such that $S(M_1) = 0$ and $M_1$ has no non trivial S. pure submodule. If $M_1$ is $M_2$- $s.p.r.$ module, then either

1- $Hom(M_1, M_2) = 0$ or

2- Every nonzero homomorphism from $M_1$ to $M_2$ is a monomorphism.

Proof: Suppose that $Hom(M_1, M_2) \neq 0$. Let $f: M_1 \to M_2$ be a nonzero homomorphism. Since $M_1$ is $M_2$-$s.p.r.$ module, then $\ker f \leq_{s.p} M_1$. But $M_1$ has no non trivial S. pure submodule, therefore $\ker f = \{0\}$. Hence f is a monomorphism.

Corollary 2.9: Let $M_1$ and $M_2$ be modules such that $S(M_1) = 0$ and $M_1$ has no non trivial S. pure submodule such that $Hom(M_1, M_2) \neq 0$. If $M_1$ is $M_2$-$s.p.r.$ module, then $M_1$ is Quasi Dedekind. In particular if $M_1$ is $s.p.r.$ module, then $M_1$ is Quasi Dedekind.

Proof: Assume that $Hom(M_1, M_2) \neq 0$. Hence there is a monomorphism $f: M_1 \to M_2$, by Propo. 2.8. Suppose $M_1$ is not Quasi Dedekind. Therefore there exists homomorphism $f_1: M_1 \to M_1$ such that $\ker f_1 \neq 0$. But f is a monomorphism, so $\ker fof_1 = \ker f_1 \neq 0$. Since $M_1$ is $M_2$-$s.p.r.$ module by our assumption, then $\ker fof_1 = \ker f_1 \leq_{s.p} M_1$. But $M_1$ has no non trivial S. pure submodule, therefore $\ker f_1 = M_1$. So $f_1 = 0$, which is contradiction. Thus $M_1$ is Quasi Dedekind.

The following theorems are characterizations for the $s.p.r.$ module.

Theorem 2.10: Let $M_1$ be a module. Then the following statements are equivalent:

1- $M_1$ is $s.p.r$-module.

2- For every $M_2 \leq M_1$, every $K_1 \leq_{\oplus} M_1$ is $M_2$ - $s.p.r.$ module.

3- For every pair $K_1, L_1 \leq_{\oplus} M_1$ and every $f \in Hom(M_1, L_1)$, the kernel of the restricted map $ker\, f|_K \leq_{s.p} K_1$.

Proof: 1⇒2) Let $M_2 \leq M_1$ and $M_1 = K_1 \oplus K_2$, for some submodule $K_2$ of $M_1$.To show that $K$ is $M_2$-$s.p.r.$ module. Let $f: K_1 \to M_2$ be a homomorphism. Let $f_1: M_1 \to M_1$ be a map defined by

$$f_1(x) = \begin{cases} f(x), & if\ x \in K_1 \\ 0\,, & if\ x \in K_2 \end{cases}$$

Clearly that $f_1$ is a homomorphism. Since $M_1$ is $s.p.r$-module, then $ker\, f_1 \leq_{s.p} M_1$. But

$$ker\, f_1 = \{m + m_1 \in M_1;\ f_1(m + m_1) = 0, m \in K_1, m_1 \in K_2\}$$
$$= \{m + m_1 \in M;\ f(m) = 0, m \in K_1, m_1 \in K_2\}$$
$$= ker\, f \oplus K_2.$$

Therefore $ker\, f \oplus K_2 \leq_{s.p} M_1$. Hence $S\left(\frac{M_1}{ker\, f \oplus K_2}\right) = 0$. But $\frac{M_1}{ker\, f \oplus K_2} = \frac{K_1 \oplus K_2}{ker\, f \oplus K_2} \cong \frac{K_1}{ker\, f}$, so $S\left(\frac{K_1}{ker\, f}\right) = 0$. So $ker\, f \leq_{s.p} K_1$. Thus $K_1$ is $M_2$-$s.p.r.$ module.

2⇒3) Let $K_1, L \leq_{\oplus} M_1$ and let $f: M_1 \to L_1$ be a homomorphism. Now consider the map $f|_{K_1}: K_1 \to L_1$. But $K_1$ is $L_1$- $s.p.r.$ module, therefore $ker\, f|_{K_1} \leq_{s.p} K_1$.

3⇒1) Let $f: M_1 \to M_1$ be a homomorphism. Since $f|_{K_1}: K_1 \to L_1$ and $K_1$ is $L_1$- $s.p.r.$ module, then $ker\, f \leq_{s.p} K_1$. Take $K_1 = L_1 = M_1$. Thus $M_1$ is $s.p.r$-module.

Proposition 2.11: Let $S$ be a hereditary property and $M_1$ be a module. Then $M_1$ is $M_2$-$s.p.r.$ module if and only if $Hom(M_1, M_2) = 0$, for every module $M_2$ such that $S(M_2) = M_2$.

Proof: Let $f: M_1 \to M_2$ be a homomorphism. Then $ker\, f \leq_{s.p} M_1$ and hence $S\left(\frac{M_1}{ker\, f}\right) = 0$. Since $\frac{M_1}{ker\, f} \cong Im\, f$ by the first isomorphisem theorem, then $S(Im\, f) = 0$. But $S(M_2) = M_2$ and $S$ hereditary, therefore $S(Im\, f) = Im\, f$ and hence $Im\, f = 0$. Thus $Hom(M_1, M_2) = 0$.

The converse is clear.

Proposition 2.12: Let $R$ be a ring and $M$ be a module. The following statements are equivalent:

1- $R$ is $s.p.r.$ ring.

2- Every projective module is $s.p.r.$ module.

3- Every free module is $s.p.r.$ module.

Proof: 1⇒2) Let $M$ be a projective module and let $f: M \to M$ be a homomorphism. Since $R$ is $s.p.r.$ ring, then $S(R) = 0$, by Propo.2.3. But $M$ is a projective module, therefore $S(R)M = S(M)$, by [ [3], Propo. 23-3, p.62 ]. and hence $S(M) = 0$. Thus $M$ is $s.p.r.$ module, by Propo.2.3.

2⇒3) Clearly since every free module is projective module.

3⇒1) Since $R$ as $R$-module is free, then $R$ is $s.p.r.$ module by our assumption. Thus $R$ is $s.p.r.$ ring.

Proposition 2.13: Let $R$ be a ring. The following statements are equivalent:

1- $S(R) = 0$

2- Each module $M$ is $N$- $s.p.r.$ module, for every projective module $N$.

3- Each module $M$ is $N$- $s.p.r.$ module, for every free module $N$.

Proof: 1⇒2) Let $M$ be a module and $N$ be a projective module. Let $f: M \to N$ be a homomorphism. Since $N$ be projective, then $S(R)N = S(N)$, by [ [3], Propo. 23-3, p.62 ]. But $S(R) = 0$, by our assumption, therefore $S(N) = 0$. Since $\frac{M}{ker\, f} \cong Im\, f$ by the first isomorphism Theorem and $Im\, f \leq N$, then $S\left(\frac{M}{ker\, f}\right) \cong S(Im\, f) = 0$. Therefore $ker\, f \ \leq_{s.p} M$. Thus $M$ is $N$- $s.p.r.$ module.

2⇒3) Clear since every free module is projective module.

3⇒1) Let $R$ be a ring. To show that $S(R) = 0$. Since $R$ as $R$-module is free module, then $R$ is $s.p.r.$ module. Thus $S(R) = 0$, by Propo. 2.3.

3- $Z(Soc, Snr, \mathcal{M}).p.r.$ modules

This section introduces the study $s.p.r$-modules, with respect to a special classes of $S$.

Proposition 3.1: Let $S$ be a hereditary property and $R$ be $ID$. If $M_1$ is a prime module such that $S(M_1) \neq M_1$, then $M_1$ is $s.p.r.$ module.

Proof: Let $M_1$ be a prime module such that $S(M_1) \neq M_1$, then $S(M_1) = 0$. To show that, assume that $S(M_1) \neq 0$. Since $S(M_1) = \sum_{\substack{K \leq M_1 \\ K\ has\ S}} N$, by [9], then there exists a submodule $K$ of $M_1$ such that $K \neq 0$ and $K$ has $S$. Hence there exists $0 \neq x \in K$ such that $Rx \leq K$. But $S$ hereditary property, therefore $Rx$ has $S$. Let $y \in M_1$, claim that $Ry$ has $S$. Let $f_x: R \to Rx$ be a map defined by $f_x(r) = rx, \forall r \in R$. Clearly that $f_x$ is an epimorphism and $ker\, f_x = ann(x)$. Hence $\frac{R}{ann(x)} \cong Rx$ by the first isomorphism Theorem. But $Rx$ has $S$, therefore $\frac{R}{ann(x)}$ has $S$. Since$M_1$be a prime, then $ann(x) = ann(y)$. Therefore $\frac{R}{ann(x)} = \frac{R}{ann(y)} \cong Ry$ and hence $Ry$ has $S$. So $S(M_1) = M_1$ be a contradiction. Hence $S(M_1) = 0$. Thus $M_1$ is $s.p.r.$ module, by Propo. 2.3.

Corollary 3.2: Let $S$ be a hereditary property and $R$ be $ID$. If $M$ is a torsion free module such that $S(M) \neq M$, then $M$ is $s.p.r.$ module.

Proof: Suppose that $M$ is a torsion free module. Then $M$ be a prime and faithful module, by [15]. Thus $M$ is $s.p.r.$ module, by Propo.3.1.

Corollary 3.3: Let $R$ be ID. If M be a projective (free) module such that $\mathrm{S(M)} \neq \mathrm{M}$, then M is $\mathrm{Z}(\mathrm{Soc}, \mathcal{M}).\mathrm{p.r.}$ module.

Proof: Suppose that M be a projective (free) module such that $\mathrm{S(M)} \neq \mathrm{M}$. Then M is torsion free, by [ [16], Propo. 3.49, p.134]. Since $\mathrm{S} = \mathrm{Z}(\mathrm{Soc}, \mathcal{M})$ is hereditary, then M is $\mathrm{s.p.r.}$ module, by Propo.3.1.

Proposition 3.4: Let R be ID and R not a field. If M is a non-zero torsion free module, then M is $\mathrm{Z}(\mathcal{M}).\mathrm{p.r.}$ module and $\mathrm{Soc(R)} = 0$.

Proof: Since R be ID not a field and $0 \neq \mathrm{M}$ torsion free, then M is not Z-regular, by [ [17], Propo. 1.2.4, p.17]. Claim that $\mathcal{M}(\mathrm{M}) = 0$. To show that, let $\mathrm{K} \leq \mathrm{M}$. Since M be torsion free, then N be torsion free and hence N is not Z-regular. But $\mathcal{M}(\mathrm{M}) = \sum_{\substack{\mathrm{K}\leq\mathrm{M} \\ \mathrm{K\ is\ regular}}} \mathrm{K}$, therefore $\mathcal{M}(\mathrm{M}) = 0$. Since R be ID and M be torsion free, then $\mathrm{T(M)} = \mathrm{Z(M)} = 0$. Thus M is $\mathrm{Z}(\mathcal{M})$. p. r. module, by Propo.2.3. Now to show $\mathrm{Soc(R)} = 0$. Since R be ID, then R be a prime as R-module. Hence $\mathrm{Soc(R)} = 0$, by [ [16], Coro. 2.3.25, p.83].

Corollary 3.5: Let R be ID and R not a field. If M be a non zero projective (flat) module, then M is $\mathcal{M}$. p. r. module, Z. p. r. module, Soc. p. r. module and $\mathcal{M}$. p. r. module.

Proof: Clearly by Propo. 3.4 and Propo. 2.3.

Proposition 3.6: Let R be ID and M be a flat module. If M is p. r. module, then M is Z. p. r. module. The converse is true when R is a principle integral domain.

Proof: Let $\mathrm{f}: \mathrm{M} \to \mathrm{M}$ be a homomorphism. But M is p. r. module and M be a flat, therefore $\ker \mathrm{f} \leq_{\mathrm{p}} \mathrm{M}$ and flat, by [ [16], Propo. 3.67, p.147]. Hence $\frac{\mathrm{M}}{\ker \mathrm{f}}$ is flat, by [ [16], Propo. 3.60, p.139]. But R be ID, therefore $\mathrm{T}\left(\frac{\mathrm{M}}{\ker \mathrm{f}}\right) \cong \mathrm{Z}\left(\frac{\mathrm{M}}{\ker \mathrm{f}}\right) = 0$. So $\ker \mathrm{f} \leq_{\mathrm{Z.p}} \mathrm{M}$. Thus M is Z. p. r-module.

For the converse, let $\mathrm{f}: \mathrm{M} \to \mathrm{M}$ be a homomorphism. Since M is Z. p. r. module, then $\ker \mathrm{f} \leq_{\mathrm{Z.p}} \mathrm{M}$ and hence $\mathrm{Z}\left(\frac{\mathrm{M}}{\ker \mathrm{f}}\right) = 0$. But R be principle integral domain, therefore $\mathrm{T}\left(\frac{\mathrm{M}}{\ker \mathrm{f}}\right) \cong \mathrm{Z}\left(\frac{\mathrm{M}}{\ker \mathrm{f}}\right) = 0$. Hence $\frac{\mathrm{M}}{\ker \mathrm{f}}$ is flat, by [ [16], Coro. 3.51, p.134]. Therefore $\ker \mathrm{f} \leq_{\mathrm{p}} \mathrm{M}$. Thus M is p. r. module.

Proposition 3.7: Let R be ID and M be a module. If M is Z-regular module. Then M is Z. p. r. module.

Proof: Assume that M is Z-regular. Then $\mathrm{J(M)} = 0$, by [ [18],Propo.6-3, p.60]. But J(M) is fully invariant submodule of M, therefore $\mathrm{Z(M)} \leq \mathrm{J(M)}$, by [ [16], Propo.2.1.6, p.54]. So $\mathrm{Z(M)} = 0$. Thus M is Z. p. r. module, by Propo.2.3.

Proposition 3.8: Let $\mathrm{M}_1$ be a module. If $\mathrm{M}_1$ is Z-regular module, then $\mathrm{M}_1$ is Snr. p. r. module.

Proof: Assume that $\mathrm{M}_1$ is Z-regular. Then every submodule of $\mathrm{M}_1$ is regular, by [ [14], Remark 2-1, p.28]. Then $\mathrm{J}(\mathrm{M}_1) = 0$, by [ [18],Propo.6-3, p.60]. Hence $\mathrm{J(A)} = 0, \mathrm{A} \leq \mathrm{M}_1$, by [ [4], Coro. 9.1.5, p.215]. So $\mathrm{Snr}(\mathrm{M}_1) = \sum_{\substack{\mathrm{A}\leq\mathrm{M}_1 \\ \mathrm{J(A)=A}}} \mathrm{A} = 0$. Thus $\mathrm{M}_1$ is Snr. p. r. module, by Propo. 2.3.

Proposition 3.9: Let $\mathrm{M}_1$ be a non-zero module. If $\mathrm{M}_1$ is Z-regular, then $\mathrm{M}_1$ is not $\mathcal{M}$. p. r. module.

Proof: Assume that $\mathrm{M}_1$ is Z-regular module. Then every submodule of $\mathrm{M}_1$is regular, by [ [14], Remark 2-1, p.28]. Hence $\mathrm{M}_1$ is the only $\mathcal{M}$. p-submodule of $\mathrm{M}_1$. Thus M is not $\mathcal{M}$. p. r-module.

Proposition 3.10: Let M be a semisimple projective module. Then M is Z(Snr). p. r. module and not $\mathcal{M}$(Soc). p. r. module.

Proof: Since M be semisimple projective module, then M is Z-regular, by [ [17], Remark 1.2.3, p.17]. Therefore M is Z(Snr). p. r. module, by Propo.3.7 and Propo.3.8. So M is not

$\mathcal{M}.\mathrm{p.r.}$ module, by Propo. 3.9. Since M be semisimple, then $\mathrm{Soc(M) = M}$. Thus M is not $\mathrm{Soc.p.r.}$ module.

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