Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

Approximation of function in the Generalized Hölder Class using (TC^δ) means for Fourier Series and Conjugate Fourier Series

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In the current work, we explore function of error estimation in the generalized Hölder class $f \in H_k^{(\chi)}(k \geq 1)$ and $\tilde{f} \in H_k^{(\chi)}(k \geq 1)$ classes using (TC^{δ}) means of F. S. as well as C. F. S. A result regarding the generalized Hölder class degree of approximation has been established.

Keywords: Error estimation; Generalized Hölder class; TC^δ means; Fourier series; conjugate Fourier Series.

1. Introduction

A notion is summability evolved from the series summation process and the concept of summability has been correctly shown in a variety of situations, including Fixed point theory, approximation theory, Fourier analysis, numerous more fields. A famous Weierstrass theorem served as the foundation for the development of function approximation theory, and it has become an attractive interdisciplinary topic of study over the previous 130 years. The approximation of functions using generalized Fourier series and conjugate Fourier series dependent on trigonometric polynomials is a closely linked on topic in modern engineering and mathematics study. The virtually summability approach and method of statistical summability are now under investigation in summability theory. Summability is used to approximate the inaccuracy of periodic functions belonging to various H$\ddot{o}$lder classes.

A considerable proportion of work has been done on approximation of function being a part of classes using Hölder class and generalized Hölder class single summability method by the number of researchers such as Dhakal [2,3], Tiwari[7], Mishra et al.[9], Mishra et al. [14] H. K. Nigam and Md Hadish[10] and H. K. Nigam and Supriya rani[12] and Lal and Mishra [13] but nothing appears to have been done in the direction of the current work.

Let $\sum a_k$ be an infinite series, such that s_k=$\sum_{\omega=0}^{n} s_k$ is its k^{th} partial sum. Assume that $T = (a_{k,r})$ is a triangular matrix that is infinite that satisfies a regularity [11], i.e.

$\sum_{r=0}^{k} a_{k,r} = 1$ as $k \to \infty \; \forall \; k \geq 0$ for $r > k$,

$$\sum_{r=0}^{k} \left| a_{k,r} \right| \leq N \qquad (1.1)$$

The sequence-to-sequence transformation

$$t_k^T = \sum_{r=0}^{k} a_{k,r} s_r = \sum_{r=0}^{k} a_{k,k-r} s_r. \qquad (1.2)$$

Specifies the triangular matrix means sequence t_k^T,

which is generated by the coefficient sequence $(a_{k,r})$ and the sequence $\{s_k\}$

If $t_k^T \to s$ as$k \to \infty$, then the infinte series $\sum d_k$ or its sequence $\{s_k\}$ is triangular matrix (T) summable to s [1].

We perform $B_k^0 = s_k = \sum_{\nu=0}^{k} d_\nu = B_k^\delta = B_0^{\delta-1} + B_1^{\delta-1} + \cdots \ldots \ldots \ldots . + B_k^{\delta-1}$

and E_k^δ for the value of B_k^δ when $a_0 = 1$ and $a_k = 0$ for $k > 0$ i.e. when $B_k = 1$.

If $C_k^\delta = \frac{B_k^\delta}{E_k^\delta} = \frac{1}{E_k^\delta} \sum_{r=0}^{k} B_{k-r}^{\delta-1} \to s \; as \; k \to \infty$ (1.3)

Where $B_k^\delta = \sum_{r=0}^{k} \binom{k-r+\delta-1}{\delta-1}$ and $E_k^\delta = \binom{k+\delta}{\delta}$

Then we say that $\sum_{k=0}^{\infty} d_k$ or s is summable to s by C^δ(Cesáro means of orderδ) for the sequence $\{s_k\}$ [5]

The TC^δmean of the sequence $\{s_k\}$ is given by

$$t_k^{TC^\delta} = \sum_{r=0}^{k} a_{k,k-r} C_k^\delta = \sum_{r=0}^{k} a_{k,k-r} \frac{1}{B_k^\delta} \sum_{\nu=0}^{k} B_{k-\nu}^{\delta-1} s_\nu = \sum_{r=0}^{k} a_{k,r} \sum_{\nu=0}^{r} \frac{\binom{k-r+\delta-1}{\delta-1}}{\binom{k+\delta}{\delta}} s_\nu \qquad (1.4)$$

If $t_k^{TC^\delta} \to s \; as \; k \to \infty, \; TC^\delta \to s,$ also $s_k \to s \Rightarrow \; C_k^\delta \to s$ as $k \to \infty$

then C^δ method is regular

$\Rightarrow T\left(C_k^\delta\right) = T_k^{TC^\delta} \to s$ as $k \to \infty$, Given the regularity of the T method.

$\Rightarrow$ The method (TC^δ) is regular.

Remark-1.1 The TC^δ mean reduces

(i) $\left(H, \frac{1}{k+1}\right) C^\delta$ or HC^δ mean , if $a_{k,r} = \frac{1}{(k-r+1)\log(k+1)}$

(ii) $(N, p, q)C^\delta$ or $N_{p,q}C^\delta$ if $a_{k,r} = \frac{p_k - r q_r}{(k-r+1)\log(k+1)}$, $R_k = \sum_{r=0}^{k} p_r q_{r-k} \neq 0$, where p_k and q_k have their usual meaning

(iii) $(N, p_k)\ C^{\delta}$ or $N_p C^{\delta}$ if $a_{k,r} = \frac{p_{k-r}}{p_k}$, $p_k \neq 0, q_k = 1 \forall\ k$ where the meaning of p_k is as usual

(iv) $(\overline{N}, p_k)\ C^{\delta}$ or $\widetilde{N_p} C^{\delta}$ if $a_{k,r} = \frac{p_r}{p_k}$, $q_k = 1\ \forall\ k$, where q_k has its usual meaning

Remark-1.2 Considering Remark-1.1, C^{δ}, $\delta = 1$ mean also reduces to $HC^1, N_{pq}C^1, N_pC^1, \widetilde{N_p}C^1$ means

Example- Consider the infinite series

$$1 - 4\sum_{\nu=1}^{k}(-3)^{\nu-1} \tag{1.5}$$

The nth Partial sum of (1.5) is given by,

$$s_k = 1 - 4\sum_{\nu=1}^{k}(-3)^{\nu-1} = (-3)^k$$

Let,

$$a_{k,r} = \frac{1}{2^k}\binom{k}{r}(1)^{k-r} \tag{1.6}$$

$$t_k^T = \sum_{r=0}^{n} a_{k,r} = a_{k,0}s_0 + a_{k,1}s_1 + a_{k,2}s_2 + \cdots \ldots\ldots + a_{k,k}s_k$$

$$= \frac{1}{2^k}\left[\binom{k}{0}(1)^k.1 - \binom{k}{1}(1)^{k-1}.3 + (-1)^n\binom{n}{n}\right]$$

$$= \frac{1}{2^k}\left[\binom{k}{0} - \binom{k}{1}(3)^1 + \binom{k}{2}(1)^{k-1}(3)^1 + \binom{k}{3}(1)^{k-1}(3)^2 + \cdots \ldots.(-1)^n\binom{n}{n}(3)^n\right] = \frac{1}{2^k}(1-3)^k$$

$$= \frac{1}{2^k}(-2)^k = (-1)^k$$

Hence,

$$(-1)^k = \begin{cases} 1, & k\ is\ even \\ -1, & k\ is\ odd \end{cases} \tag{1.7}$$

Equation (1.6) can be summable by TC^{δ} but not by T.

The space of the function $L^z[0, 2\pi] = \{f: [0, 2\pi] \to R: \int_0^{2\pi}|f(y)|^z dy < \infty, z \geq 1\}$, is 2π-Periodic and integral, the norm $\|.\|_r$ can be defined as

$\|f\|_z = \{\frac{1}{2\pi}\int_0^{2\pi}|f(y)|^z dy\}^{\frac{1}{z}}, z \geq 1$ as specified in [1] η: $[0, 2\pi] \to R$ be a stochastic function as $\eta(l) > 0$ for $0 < l \leq 2\pi$ and $\lim_{l\to 0^+}\eta(0) = 0$,

Where, $\boldsymbol{H_z^{\eta}} = \left\{\boldsymbol{f \in L^z[0, 2\pi]: sup_{l\neq 0}\frac{\|f(.+l)-f(.)\|_z}{\eta(l)} < \infty, z \geq 1}\right\}$ and

$\|.\|_{\boldsymbol{z}}^{(\boldsymbol{\eta})} = \|\boldsymbol{f}\|_{\boldsymbol{z}}^{(\boldsymbol{\eta})} = \|\boldsymbol{f}\|_{\boldsymbol{z}} + \boldsymbol{sup_{l\neq 0}\frac{\|f(.+l)-f(.)\|_r}{\eta(l)}}, \boldsymbol{z \geq 1}$, clearly $\|.\|_{\boldsymbol{z}}^{(\boldsymbol{\eta})}$ is norm on $\boldsymbol{H_z^{(\eta)}}$.

Note-1: $\eta(l)$ and $\chi(l)$ denoted Zygmund module of continuity of order two such that $\frac{\eta(l)}{\chi(l)}$ to be positive and non-decreasing

$\|f\|_z^{(\chi)} \leq max\left(1, \frac{\eta(2\pi)}{\chi(2\pi)}\right), \|f\|_z^{(\eta)} < \infty$

Thus, $H_z^{(\eta)} \subset H_z^{(\chi)} \subset L^z, z \geq 1$.

Remark-1.3

(i) If $\eta(l) = l^\alpha$ in $H^{(\eta)}$ reduced to the $H^{(\alpha)}$ class

(ii) By taking $\eta(l) = l^\alpha, H_z^{(\eta)}$ reduces to the $H_{\alpha,z}$ class

(iii) If $z \to \infty$ in $H_z^{(\eta)}$, $H_z^{(\eta)}$ class and $H_{\alpha,z}$ class

We represent the k^{th} partial sum of the Fourier series as,

$$s_k(f;y) - f(y) = \frac{1}{2\pi}\int_0^\pi \varphi(y,l)\frac{\sin\,(k+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

The Conjugate Fourier series k^{th} partial sum as,

$$s_k(\tilde{f};y) - \tilde{f}(y) = \frac{1}{2\pi}\int_0^\pi \psi(y,l)\frac{\cos(k+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

The k- order error estimation of function f is provided by

$E_k(f) = min\|f - t_k\|_z$, where t_k is a trigonometric polynomial with degree k.

We define, $\varphi_y(k) = \varphi(y,k) = f(y+k) + f(y-k) - 2f(y)$,

$\psi_y(k) = \psi(y,k) = f(y+k) - f(y-k)$, $\Delta a_{k,r} = a_{k,r} - a_{k,r+1}$

$$H_k^{TC^\delta}(l) = \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\sum_{\nu=0}^{r}\left(\frac{\begin{matrix}\nu+\delta-1\\ \delta-1\end{matrix}}{\begin{matrix}\delta+r\\ r\end{matrix}}\right)\frac{\sin\,(\nu+\frac{1}{2})l}{sin\frac{l}{2}}$$

$$\tilde{H}_k^{TC^\delta}(l) = \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\sum_{\nu=0}^{r}\left(\frac{\begin{matrix}\nu+\delta-1\\ \delta-1\end{matrix}}{\begin{matrix}\delta+r\\ r\end{matrix}}\right)\frac{\cos\,(\nu+\frac{1}{2})l}{sin\frac{l}{2}}$$

2. Lemmas

Lemma 2.1 Under Condition (1.1) of regularity of matrix $T \equiv (a_{k,r})$ for $0 < l < \frac{1}{k+1}$,

$H_k^{TC^\delta}(l) = O(k+1)$.

Proof: $\mathbf{0 \le l \le \frac{1}{k+1}}$, $\sin\left(\frac{l}{2}\right) \ge \frac{l}{\pi}$, $\mathbf{\sin(kl) \le kl}$ and $\boldsymbol{\delta \ge 1}$, we get

$$H_k^{TC^\delta}(l) = \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\left\{\sum_{\nu=0}^{r}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{sin(\nu+\frac{1}{2})l}{sin\frac{l}{2}}\right\}$$

$$\le \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\left\{\sum_{\nu=0}^{r}\frac{(\nu+\delta-1)!}{(\delta-1)!}\frac{\delta! r!}{(\delta+r)!}\frac{(2\nu+1)\frac{l}{2}}{\frac{l}{\pi}}\right\}$$

$$= \frac{1}{4}\sum_{r=0}^{k} a_{k.k-r}\left\{\frac{r!}{(\delta+1)\dots\dots\dots(\delta+r)\delta!}\sum_{\nu=0}^{r}\frac{(2\nu+1)(\nu+\delta-1)!\delta!}{\nu!}\right\}$$

$$= \frac{1}{4}\sum_{r=0}^{k} a_{k,k-r}\left\{\frac{(r+1)(2r+1)\delta}{(\delta+r)}\right\}$$

$$= \frac{1}{4}\sum_{r=0}^{k} a_{k,k-r}(2r+1)\delta$$

$$= \frac{1}{4}(2k+1)\delta\sum_{r=0}^{k} a_{k,k-r} = O(k+1) \qquad (2.1)$$

Lemma 2.2 Under Condition (1.1) and (1.7) of regularity of matrix $\boldsymbol{T \equiv (a_{k,r})}$ for $\boldsymbol{\frac{1}{k+1} \le l \le \pi}$,

$$H_k^{TC^\delta}(l) = O\left(\frac{1}{l^2(k+1)}\right).$$

Proof: For $\frac{1}{k+1} \le l \le \pi$, using $\sin\left(\frac{l}{2}\right) \ge \frac{l}{\pi}$, $\left|sin^2 kl\right| \le 1$ and $\boldsymbol{\delta \ge 1}$, we get

$$H_k^{TC^\delta}(l) = \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\left\{\sum_{\nu=0}^{r}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{sin(\nu+\frac{1}{2})l}{sin\frac{l}{2}}\right\}$$

$$\left|H_k^{TC^\delta}(l)\right| = \frac{1}{2\pi}\sum_{r=0}^{k} a_{k,k-r}\left|\sum_{\nu=0}^{r}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{sin(\nu+\frac{1}{2})l}{\frac{l}{\pi}}\right|$$

$$\le \frac{1}{2\pi}\times\frac{\pi}{l}\sum_{r=0}^{k} a_{k,k-r}\frac{r!}{(\delta+1)\dots\dots\dots(\delta+r)\delta!}\left|\sum_{\nu=0}^{r}\frac{(\nu+\delta-1)!\delta!}{\nu!}\,sin(\nu+\frac{1}{2})l\right|$$

$$= \frac{1}{2l}\sum_{r=0}^{k} a_{k,k-r}\frac{r!}{(\delta+1)\dots\dots\dots(\delta+r)\delta!}\left|\sum_{\nu=0}^{r}\frac{(\nu+\delta-1)!\delta!}{\nu!}\,\mathrm{Im}\left\{e^{i\left(\nu+\frac{1}{2}\right)l}\right\}\right|$$

$$= \frac{1}{2l}\times\frac{\pi}{l}\sum_{r=0}^{k} a_{k,k-r}\frac{r!}{(\delta+1)\dots\dots\dots(\delta+r)\delta!}\times\frac{(\delta+1)\dots\dots\dots(\delta+r-1)\delta!}{r!}\;\mathrm{Im}\left\{\frac{e^{i(k+1)l}-1}{2i\,sin(\frac{l}{2})}\right\}$$

$$= \frac{\pi}{2l^2}\sum_{r=0}^{k} a_{k,k-r}\frac{\delta}{(\delta+r)}\times sin^2(k+1)\frac{l}{2}$$

$$= \frac{\pi}{2l^2}\sum_{r=0}^{k} a_{k,k-r}\frac{\delta}{(\delta+r)}$$

$$= \frac{\pi}{2l^2} \sum_{r=0}^{k} \frac{a_{k,k-r}}{(k-r+1)}$$

$$= \frac{\pi}{2l^2} O\left(\frac{1}{(k+1)}\right)$$

$$= O\left(\frac{1}{l^2(k+1)}\right) \tag{2.2}$$

Lemma 2.3 Under Condition (1.1) of regularity for matrix $T \equiv (a_{k,r})$, for $0< l < \frac{1}{k+1}$, $\widetilde{H}_k^{TC^\delta}(l) = O\left(\frac{1}{k}\right)$.

Proof: For $0 < l \le \frac{1}{k+1}$, using, $\sin\left(\frac{l}{2}\right) \ge \frac{l}{\pi}$ and $|cosrl| \le 1$, we get

$$\widetilde{H}_k^{TC^\delta}(l) = \frac{1}{2\pi} \sum_{r=0}^{k} a_k a_{k-r} \left\{ \sum_{\nu=0}^{r} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \frac{cos(\nu+\frac{1}{2})l}{sin\left(\frac{l}{2}\right)} \right\}$$

$$\left|\widetilde{H}_k^{TC^\delta}(l)\right| \le \frac{1}{2\pi} \sum_{r=0}^{k} a_{k,k-r} \left| \sum_{\nu=0}^{r} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \frac{cos(\nu+\frac{1}{2})l}{\frac{l}{\pi}} \right|$$

$$\left|\widetilde{H}_k^{TC^\delta}(l)\right| = \frac{1}{2\pi} \times \frac{\pi}{l} \sum_{r=0}^{k} a_{k,k-r} \sum_{\nu=0}^{r} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \left| cos(\nu+\frac{1}{2})l \right|$$

$$\left|\widetilde{H}_k^{TC^\delta}(l)\right| = \frac{1}{2l} \sum_{r=0}^{k} a_{k,k-r} \frac{r!}{(\delta+1)\ldots\ldots\ldots(\delta+r)\delta!} \sum_{\nu=0}^{r} \frac{(\nu+\delta-1)!\,\delta!}{\nu!}$$

$$\left|\widetilde{H}_k^{TC^\delta}(l)\right| = \frac{1}{2l} \sum_{r=0}^{k} a_{k,k-r} \frac{r!}{(\delta+1)\ldots\ldots\ldots(\delta+r)\delta!} \sum_{\nu=0}^{r} \frac{(\delta+1)\ldots\ldots\ldots(\delta+r-1)\delta!}{\nu!}$$

$$\left|\widetilde{H}_k^{TC^\delta}(l)\right|$$

$$= \frac{1}{2l} \sum_{r=0}^{k} a_{k,k-r} \frac{r!}{(\delta+1)\ldots\ldots\ldots(\delta+r)\delta!} \left[\frac{(r+1)\delta(\delta+1)\ldots\ldots\ldots(\delta+r-1)}{r!}\right]$$

$$= \frac{1}{2l} \sum_{r=0}^{k} a_{k,k-r} \frac{\delta(r+1)}{(\delta+r)}$$

$= O\left(\frac{1}{k}\right)$ since $\sum_{r=0}^{k} a_{k,k-r} = 1$

$$= O\left(\frac{1}{k}\right) \tag{2.3}$$

Lemma 3.4 Under Condition (1.1) and (1.7) of regularity of matrix $\boldsymbol{T \equiv (a_{k,r})}$, for $\boldsymbol{\frac{1}{k+1} \leq l \leq \pi}$, $\boldsymbol{\widetilde{H}_k^{TC^\delta}(l) = O\left(\frac{1}{l^2(k+1)}\right)}$.

Proof: For $\boldsymbol{\frac{1}{k+1} \leq l \leq \pi}$, $\sin\left(\frac{l}{2}\right) \geq \frac{l}{\pi}$, using Abel's lemma, we get

$$\boldsymbol{\widetilde{H}_k^{TC^\delta}(l) \leq \frac{1}{2\pi} \times \frac{\pi}{l}\left|\sum_{r=0}^{k-1} a_{k,r} \frac{sin(k-r+1)\left(\frac{l}{2}\right)cos^{k-r}\left(\frac{l}{2}\right)}{\sin\left(\frac{l}{2}\right)}\right|}$$

$$\boldsymbol{\leq \frac{1}{2l}\left|\sum_{r=0}^{k-1}(a_{k,r} - a_{k,n+1})\sum_{v=0}^{k} sin(k-v+1)\left(\frac{l}{2}\right)cos^{k-v}\left(\frac{l}{2}\right) + a_{k,k}\sum_{r=0}^{k} in(k-r+1)\left(\frac{l}{2}\right)cos^{k-r}\left(\frac{l}{2}\right)\right|}$$

$$\boldsymbol{= \frac{\pi}{l^2}\left[\sum_{r=0}^{k-1}\left|\Delta a_{k,r}\right| + a_{k,k}\right]\max_{0\leq k\leq r} sin(2k-r+1)\left(\frac{l}{2}\right)sin(k+1)\left(\frac{l}{2}\right)}$$

$$\boldsymbol{= \frac{\pi}{l^2}\left[\sum_{r=0}^{k-1}\left|\Delta a_{k,r}\right| + a_{k,k}\right]}$$

$$\boldsymbol{= \frac{\pi}{l^2}\left[O\left(\frac{1}{k+1}\right) + O\left(\frac{1}{k+1}\right)\right]}$$

$$\boldsymbol{= O\left(\frac{1}{(k+1)l^2}\right)} \tag{2.4}$$

Lemma 3.5 [13] Suppose $\boldsymbol{f \in H_z^{(\eta)}}$ for $\boldsymbol{0 < l \leq \pi}$,

(i) $\boldsymbol{\|\varphi(.,l)\|_z = O(\eta(l))}$;

(ii) $\boldsymbol{\|\varphi(.+u,l) + \varphi(.,l)\|_z = \begin{cases} O(\eta(l)), \\ O(\eta(|u|)); \end{cases}}$

(iii) If $\boldsymbol{\eta(l)}$ and $\boldsymbol{\chi(l)}$ are defined Note-1, then $\boldsymbol{\|\varphi(.+u,l) - \varphi(.,l)\|_z = O\left(\chi(|u|)\frac{\eta(l)}{\chi(l)}\right)}$

Lemma 3.6 [13] Let $\boldsymbol{\tilde{f} \in H_z^{(\eta)}}$ for $\boldsymbol{0 < l \leq \pi}$,

(i) $\boldsymbol{\|\psi(.,l)\|_z = O(\eta(l))}$;

(ii) $\boldsymbol{\|\psi(.+u,l) + \psi(.,l)\|_z = \begin{cases} O(\eta(l)), \\ O(\eta(|u|)); \end{cases}}$

(iii) If $\boldsymbol{\eta(l)}$ and $\boldsymbol{\chi(l)}$ are defined Note-1, then $\boldsymbol{\|\psi(.+u,l) - \psi(.,l)\|_z = O\left(\chi(|u|)\frac{\eta(l)}{\chi(l)}\right)}$.

3. Main Theorem

Theorem- 3.1 If $\boldsymbol{f \epsilon H_z^{(\chi)}}$ class, $\boldsymbol{z \geq 1}$, and $\frac{\eta(l)}{\chi(l)}$ are non-decreasing and positive then the error estimation of $\boldsymbol{f}$ by Fourier series $\boldsymbol{TC^{\delta}}$ method is provided by

$$\left\| t_k^{TC^{\delta}} - f \right\|_z^{(\chi)} = O\left(\frac{1}{k+1} \int_{\frac{1}{k+1}}^{\pi} \frac{\eta(l)}{l^2 \chi(l)} dl\right)$$

Where $\boldsymbol{\eta(l)}$ and $\boldsymbol{\chi(l)}$ are a defined in Note-1, provided.

Proof :Let $s_k(f,y) - f(y) = \frac{1}{2\pi} \int_0^{\pi} \varphi(y,l) \frac{\sin\,(k+\frac{1}{2})l}{\sin\frac{l}{2}} dl, k = 0, 1, 2, 3, 4, \ldots\ldots$

Then, $\sum_{r=0}^{k} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} [s_k(f,y) - f(y)] = \frac{1}{2\pi} \int_0^{\pi} \varphi(y,l) \sum_{r=0}^{k} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \frac{\sin\,(k+\frac{1}{2})l}{\sin\frac{l}{2}} dl$

Now denoting $\boldsymbol{TC^{\delta}}$ transform of $\boldsymbol{s_k(f,y)}$ by $\boldsymbol{t_k^{TC^{\delta}}}$

$$t_k^{TC^{\delta}}(y) - f(y) = \sum_{\nu=0}^{k} a_{k,k-r} \{C_{\nu}^{\delta} - f(y)\}$$

$$t_k^{TC^{\delta}}(y) - f(y) = \frac{1}{2\pi} \int_0^{\pi} \varphi(y,l) \sum_{\nu=0}^{k} a_{k,k-r} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \frac{\sin(\nu+\frac{1}{2})l}{\sin\frac{l}{2}} dl \qquad (3.1)$$

$$t_k^{TC^{\delta}}(y) - f(y) = \frac{1}{2\pi} \int_0^{\pi} \varphi(y,l) \sum_{\nu=0}^{k} \frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}} \frac{\sin(\nu+\frac{1}{2})l}{\sin\frac{l}{2}} dl$$

Let, $\boldsymbol{T_k(y) = t_k^{TC^{\delta}}(y) - f(y) = \int_0^{\pi} \varphi(y,l) H_k^{TC^{\delta}}(l)\, dl}$

Then, $\boldsymbol{T_k(y+u) - T_k(y) = \int_0^{\pi} \{\varphi(y+u,l) - \varphi(y,l)\}\, H_k^{TC^{\delta}}(l) dl}$

Using generalized Minikowski's inequality [4], we obtain

$$\|T_k(.+u) + T_k(.)\|_z \leq \int_0^{\pi} \|\varphi(.+u,l) - \varphi(.\,l)\|_z\, H_k^{TC^{\delta}}(l) dl$$

$$= \left(\int_0^{\frac{1}{k+1}} + \int_{\frac{1}{k+1}}^{\pi} \|\varphi(.+u,l) - \varphi(.\,l)\|_z H_k^{TC^{\delta}}(l) dl\right)$$

$$= I_1 + I_2 \qquad (3.2)$$

Using lemma 2.1 & lemma 2.5(iii), we get

$$I_1 = \int_0^{\frac{1}{k+1}} \|\varphi(.+u,l) - \varphi(.\,l)\|_z H_k^{TC^{\delta}}(l)\, dl$$

$$= O\left(\int_0^{\frac{1}{k+1}} \chi(|u|)\frac{\eta(l)}{\chi(l)}(k+1)dl\right)$$

$$= O\left((k+1)\chi(|u|)\int_0^{\frac{1}{k+1}} \frac{\eta(l)}{\chi(l)}dl\right)$$

$$= O\left((k+1)\chi(|u|)\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\int_0^{\frac{1}{k+1}} dl\right)$$

$$= O\left(\chi(|u|)\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\right) \qquad (3.3)$$

Using lemma 2.2 and 2.5 (ii)

$$I_2 = \int_{\frac{1}{k+1}}^{\pi} \|\varphi(.+u,l)-\varphi(.\,l)\|_z H_k^{TC^\delta}(l)\,dl$$

$$= O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\chi(|u|)\frac{\eta(l)}{l^2\chi(l)}dl\right) \qquad (3.4)$$

From (3.2), (3.3) and (3.4), we have

$$sup_{u\neq 0}\frac{\|T_k(.+u)-T_k(.)\|_z}{\chi(|u|)} = O\left(\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\right)+O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \qquad (3.5)$$

Using, Lemma 3.1 and 3.2, we get

$$\|T_k(.)\|_z = \left\|t_k^{TC^\delta}-f\right\|_z$$

$$\leq \int_0^{\frac{1}{k+1}}\|\varphi(.\,l)\|_z H_k^{TC^\delta}(l)dl + \int_{\frac{1}{k+1}}^{\pi}\|\varphi(.\,l)\|_z H_k^{TC^\delta}(l)dl$$

$$= O\left((k+1)\int_0^{\frac{1}{k+1}}\eta(l)dl\right) + O\left(\frac{1}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2}dl\right)$$

$$= O\left(\eta\left(\frac{1}{k+1}\right)\right) + O\left(\frac{1}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2}dl\right) \qquad (3.6)$$

We know that,

$$\|T_k(.)\|_z^{\chi} = \|T_k(.)\|_z + sup_{u\neq 0}\frac{\|T_k(.+u)-T_k(.)\|_z}{\chi(|u|)} \qquad (3.7)$$

Now, by (3.5) and (3.6) in (3.7), we have

$$\|T_k(.)\|_z^{\chi} = O\left(\eta\left(\frac{1}{k+1}\right)\right) + O\left(\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2}dl\right) + O\left(\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\right) + O\left(\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.8)$$

Due to the monotonicity of the function $\chi(l)$,

$\eta(l) = \frac{\eta(l)}{\chi(l)}\chi(l) \leq \chi(\pi)\frac{\eta(l)}{\chi(l)}$for $0 < l \leq \pi$, we get

Again, due to the monotonicity of the function $\chi(l)$,

$$\|T_k(.)\|_z^{(\chi)} = O\left(\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\right) + O\left(\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.9)$$

Since η and χ are continuity moduli, $\frac{\eta(l)}{\chi(l)}$ is positive and non-decreasing, hence,

$$\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl \geq \frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\left(\frac{1}{k+1}\right)\int_{\frac{1}{k+1}}^{\pi}\frac{1}{l^2}dl \geq \frac{\eta\left(\frac{1}{k+1}\right)}{2\chi\left(\frac{1}{k+1}\right)}$$

Then

$$\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)} = O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.10)$$

From (3.9) and (3.10)

$$\|T_k(.)\|_z^{(\chi)} = O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right)$$

Thus,

$$\left\|t_k^{TC^\delta} - f\right\|_z^{(\chi)} = O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.11)$$

We now have the proof of Theorem 3.1 finished. □

Theorem-3.2 Given that $\tilde{f}\epsilon\, H_z^{(\chi)}$ class, $z \geq 1$, and $\frac{\eta(l)}{\chi(l)}$ is positive and non-decreasing then the Conjugate Fourier series error estimation f using the TC^δ method

$$\left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_z^{(\chi)} = O\left(\frac{1 + log(k+1)}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right)$$

Where $\eta(l)$ and $\chi(l)$ are a defined in Note-1 provided

$\sum_{r=0}^{k-1}\left|\Delta a_{k,r}\right| = O\left(\frac{1}{k+1}\right)$ and $(k+1)a_{k,k} = O(1)$

Proof: Given $s_k(f,y)$, the integral representation is given by

$$s_k(\tilde{f},y)-\tilde{f}(y)=\frac{1}{2\pi}\int_0^\pi \psi(y,l)\frac{\cos\,(k+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

Then,

$$\sum_{r=0}^{k}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\left[s_k(\tilde{f},y)-\tilde{f}(y)\right]=\frac{1}{2\pi}\int_0^\pi \psi(y,l)\sum_{r=0}^{k}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{\cos\,(k+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

$$C_k^\delta(y)-\tilde{f}(y)=\frac{1}{2\pi}\int_0^\pi \psi(y,l)\sum_{r=0}^{k}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{\cos\,(k+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

Now,

$$t_k^{TC^\delta}(y)-\tilde{f}(y)=\textstyle\sum_{r=0}^{k}a_{k,k-r}\{C_k^\delta-f(y)\}$$

$$t_k^{TC^\delta}(y)-\tilde{f}(y)=\frac{1}{2\pi}\int_0^\pi \psi(y,l)\sum_{r=0}^{k}a_{k,k-r}\sum_{\nu=0}^{r}\frac{\binom{\nu+\delta-1}{\delta-1}}{\binom{\delta+r}{r}}\frac{\cos\,(\nu+\frac{1}{2})l}{sin\frac{l}{2}}dl$$

$$t_k^{TC^\delta}(y)-\tilde{f}(y)=\int_0^\pi \psi(y,l)H_k^{TC^\delta}(l)\,dl$$

Let, $\quad \tilde{T}_k(y)=t_k^{TC^\delta}(y)-\tilde{f}(y)=\int_0^\pi \psi(y,l)H_k^{TC^\delta}(l)\,dl$

$$\tilde{T}_k(y+u)-\tilde{T}_k(y)=\int_0^\pi\{\psi(y+u,l)-\psi(y,u)\}\,H_k^{TC^\delta}(l)dl$$

Using, the GMI, we get

$$\left\|\tilde{T}_k(.+u)-\tilde{T}_k(.)\right\|_z\le\int_0^\pi\|\psi(.+u,l)-\psi(.,u)\|_z\,H_k^{TC^\delta}(l)dl$$

$$\begin{aligned}\left\|\tilde{T}_k(.+u)-\tilde{T}_k(.)\right\|_z &\le\int_0^{\frac{1}{k+1}}\|\psi(.+u,l)-\psi(.,u)\|_z\,H_k^{TC^\delta}(l)dl\\ &+\int_{\frac{1}{k+1}}^{\pi}\|\psi(.+u,l)-\psi(.,u)\|_z\,H_k^{TC^\delta}(l)dl\end{aligned}$$

$$\left\|\tilde{T}_k(.+u)-\tilde{T}_k(.)\right\|_z=I_3+I_4 \qquad (3.12)$$

Using, Lemma 2.3 and 2.6(iii), we get

$$\boldsymbol{I_3 = \int_0^{\frac{1}{k+1}} \|\psi(.+u,l) - \psi(.,u)\|_z H_k^{TC^\delta}(l)dl}$$

$$\boldsymbol{I_3 = O\left(\chi(|u|)\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\int_0^{\frac{1}{k+1}}\frac{1}{l}dl\right) = O\left(\chi(|u|)\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}log(k+1)\right)} \qquad (3.13)$$

Again, using Lemma 2.4 and 2.6(iii), we have

$$\boldsymbol{I_4 = \int_{\frac{1}{k+1}}^{\pi} \|\psi(.+u,l) - \psi(.,u)\|_z H_k^{TC^\delta}(l)dl}$$

$$= \frac{\mathbf{1}}{k+1}\int_{\frac{1}{k+1}}^{\pi}\chi(|u|)\frac{\eta(l)}{\chi(l)l^2}dl$$

$$I_4 = O\left(\chi(|u|)\frac{\eta(l)}{\chi(l)}\int_0^{\frac{1}{k+1}}\frac{1}{l^2}dl\right) \qquad (3.14)$$

Using, (3.12),(3.13) and (3.14), we have

$$sup_{u\neq 0}\frac{\|T_k(.+u)-T_k(.)\|_z}{\chi(|u|)} = O\left(\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}log(k+1)\right) + O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \qquad (3.15)$$

Apply GMI, again using Lemmas 2.3, Lemma 2.4 and Lemma 2.6(i), we obtain

$$\left\|\tilde{T}_k(.)\right\|_z = \left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_z$$

By the monotonic of $\chi(l)$, we have $\frac{\eta(l)}{\chi(l)}\chi(l)$, $0 \le l \le \pi$, we get

$$\left\|\tilde{T}_k(.)\right\|_z^\chi = O\left(\frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}log(k+1)\right) + O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \qquad (3.16)$$

Using, the fact that $\frac{\eta(l)}{\chi(l)}$ is non-decreasing and positive, we get

$$\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl \ge \frac{\eta\left(\frac{1}{k+1}\right)}{\chi\left(\frac{1}{k+1}\right)}\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{1}{l^2}dl \ge \frac{\eta(\frac{1}{k+1})}{2\chi(\frac{1}{k+1})}$$

Then,

$$\frac{\eta(\frac{1}{k+1})}{\chi(\frac{1}{k+1})} = O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \qquad (3.17)$$

From (3.16) and (3.17), we get

$$\left\|\tilde{T}_k(.)\right\|_z^{(\chi)} = O\left(\frac{log(k+1)}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) + O\left(\frac{1}{k+1}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.18)$$

$$\left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_z^{(\chi)} = O\left(\frac{1+log(k+1)}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right) \quad (3.19)$$

The completes proof of Theorem 3.2. finished. □

4. Corollaries

Corollary 4.1 let $0 \leq \beta < \alpha \leq 1$ and $\tilde{f} \in H_z^{(\chi)}, z \geq 1$ then

$$\left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_{(\beta),z}^{(\chi)} = \begin{cases} O\left(\frac{1+log(k+1)^{\beta-\alpha}}{(k+1)}\right) if\ 0 \leq \beta < \alpha < 1 \\ \left(\frac{1+\log(k+1)}{(k+1)}\right) if\ \beta = 0, \alpha = 1 \end{cases}$$

Proof: In Theorem 3.2, put $\eta(l) = l^\alpha, \chi(l) = l^\beta, 0 \leq \beta < \alpha \leq 1$.

$$\left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_{(\beta),z}^{(\chi)} = \begin{cases} O\left(\frac{1+log(k+1)}{(k+1)}\int_{\frac{1}{k+1}}^{\pi} l^{\alpha-\beta-2}dl\right) if\ 0 \leq \beta < \alpha < 1 \\ \left(\frac{1+\log(k+1)}{(k+1)}\int_{\frac{1}{k+1}}^{\pi} dl\right) if\ \beta = 0, \alpha = 1 \end{cases}$$

Corollary 4.2 A function $f \in H_k^{(\chi)}$ can have its error estimation determine by $\left(H, \frac{1}{k+1}\right)(C,1)$ mean of F.S. if $a_{k,r} = \frac{1}{(k-r+1)\log(k+1)}$, and TC^δ means is reduced to means

$$\left\|t_k^{TC^\delta} - f\right\|_z^{(\chi)} = O\left(\frac{1}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right)$$

Corollary 4.3 If $a_{k,r} = \frac{p_k - rp_r}{(k-r+1)\log(k+1)}$, then TC^δ means reduced to $(N,p,q)C^\delta$means and error estimation of a function $f \in H_k^{(\chi)}$ by $(N,p,q)C^\delta$means of F.S. is

$$\left\|t_k^{TC^\delta} - f\right\|_z^{(\chi)} = O\left(\frac{1}{(k+1)}\int_{\frac{1}{k+1}}^{\pi}\frac{\eta(l)}{l^2\chi(l)}dl\right)$$

Corollary 4.4 If $a_{k,r} = \frac{1}{(k-r+1)\log(k+1)}$, then TC^δ means reduced to $\left(H, \frac{1}{k+1}\right)(C,1)$ the function of mean and error estimation function $\tilde{f} \in H_k^{(\chi)}$ by $\left(H, \frac{1}{k+1}\right)(C,1)$means of C.F.S. are

$$\left\|\tilde{t}_k^{TC^\delta} - \tilde{f}\right\|_z^{(\chi)} = O\left(\frac{1 + log(k+1)}{(k+1)} \int_{\frac{1}{k+1}}^{\pi} \frac{\eta(l)}{l^2\chi(l)} dl\right)$$

Remark 4

i. In our Theorem 3.1, if $z \to \infty$, then $H_k^{(\chi)}$ class. Also putting $\eta(l) = l^\alpha$ and $\chi(l) = l^\beta$ in this theorem 3.1, The $H^{(\chi)}$ class is reduced to H_α class, For $\beta = 0$, the H_α class reduced to $Lip\alpha$ class.

ii. In this theorem 3.1 with putting $\eta(l) = l^\alpha$ and $\chi(l) = l^\beta$ in $H_k^{(\chi)}$class, $H_k^{(\chi)}$class reduced to $H_{\alpha,k}$; then with putting $\beta = 0$ in $H_{\alpha,k}$ class, $H_{\alpha,k}$ class reduced to $Lip(\alpha, k)$ class.

5. Conclusion

In the current study, we have found the ideal error approximation for a 2π periodic function f within the generalized Hölder class $H_k^{(\chi)}, k \geq 1$ by using TC^δ (matrix-cesaro) means of its F.S. as well as the C.F.S.

Declarations

I hereby certify that the submission is completely my own work, written wholly in my own words, and that all sources used in researching it are fully recognized and all quotations properly idented.

Conflicts of Interest

The authors declare that they have no competing interests.

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