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Some Properties of the Cantor Set in the Hyperspace K(X)

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In this scenario, we consider a compact Hausdorff space X and an unceasing map f : X → X. We examine the space K(X), comprising all compact subsets of X with the Hausdorff metric H. A mapping ϕ : K(X) → K(X)) is defined as ϕ(K) = f(K). We explore the relationship between the orbit of ϕ and the orbit of f. Assuming the transitivity of ϕ, we demonstrate the connection between these orbits, shedding light on the dynamics of the system under consideration. We show that X contains a cantor set C with orb(ϕ,C) is dense in K(X). Also we discuss some interesting properties of this Cantor set in K(X).

Keywords: Transitive functions, cantor Set, Hyper Space, dense orbit, strongly invariant.

1. Introduction

In mathematical analysis, particularly in the research of dynamical systems, understanding the behavior of a hyperspace within a compact metric space is crucial. Researchers typically investigate this by examining a dynamical system characterized by a continuous map.

f: X → X, where X represents the compact metric space. This map f describes how points in X evolve under certain transformations or iterations. Additionally, researchers consider the induced map ϕ: K(X) →K(X), where K represents compact subsets of X. This induced map ϕ describes how compact sets within the hyperspace K(X) evolve under the action of f. By analyzing the behavior of these maps, researchers gain insights into the intricate dynamics and geometric properties of the hyperspace within the compact metric space.

In this article, researchers demonstrate a unique phenomenon where a Cantor set C exists within a space X, which is nowhere dense in X but has an orbit that is dense in space K(X). This suggests a surprising lack of correlation between individual and collective confusion. The properties of this Cantor are set within K(X). are also examined, shedding light on the intricate dynamics of confusion within different contexts. This discovery challenges traditional perceptions of confusion, revealing its intricate dynamics at both individual and collective levels. It sheds light on the nuanced relationship between chaos and order within systems, highlighting their interplay. This insight suggests that confusion is not merely disorderly but rather an integral part of organizational functioning, influencing how individuals and groups navigate complexities and maintain equilibrium.

2. Main results

1 Hyperspace and Induced Map

In this mathematical context, X denotes a compact Hausdorff metric space devoid of inaccessible points, characterized by a metric function d. The function f maps points from X to X continuously. Here, H represents the Hausdorff metric on K(X), the collection of all compact subsets of X, determined by the metric d. The function ϕ operates on compact subsets of X, denoted as K, and assigns them to their images under the mapping f. Essentially, $\phi(K) = f(K)$ transforms each compact subset K of X into its corresponding image under the continuous mapping f. This framework refers to the study of dynamical systems and chaos theory, focusing on the behavior of trajectories subjected to continuous mappings on compact spaces. It allows for the analysis of how these trajectories evolve over time, offering valuable insights into the long-term behavior and stability of complex systems. By examining how systems change and interact within defined boundaries, researchers can better understand the intricate dynamics and potential for chaos within these systems. This approach provides a foundation for predicting and interpreting the behavior of dynamic systems across various disciplines, from physics and biology to economics and engineering.

Definition 2.1. The ω-limit of a point x below a dynamical system f is the collection of all limit points of the orbit generated by repeatedly applying f to x. It represents the long-term behavior of the system starting from x and is denoted as $\omega(x, f)$.

Definition 2.2. A subset S of set X is invariant under function f if every element of S, when operated on by f, remains within S. If f(S) equals S, S is strongly invariant under f, implying that every element of S is mapped back to itself by f.

Definition 2.3. If $A = \{A_\mu\}_{\mu\in\daleth}$ is a collection of non-empty subsections of X, then $mesh(A) = \sup\{diam(A_\mu), \mu \in \daleth\}$.

In [2] it is proved that (K(X),H) is compact. In [4] it is proved that $\omega(x, f)$ is closed. We state these results as lemmas.

Lemma 2.4. (K(X),H) is compact.

Proof.

see[2]

Lemma 2.5. In a compact space X, for every point x, the accumulation points of the orbit of x under a map f are guaranteed to exist, forming a non-empty set that is closed within X. Furthermore, this set remains invariant under the action of f, implying that as the map f iterates over the space X, the set of accumulation points remains consistent and unaltered. This property highlights the stability and predictability of the behavior of orbits under continuous maps within compact spaces, a fundamental concept in dynamical systems theory.

Proof.

see[4]

Lemma 2.6

If $\overline{\mathrm{orb}(\phi, K)} = X$ then for individually A ∈ orb(ϕ,K) and for all x ∈ A, $\overline{\mathrm{orb}(f, x)} = X$

Proof.

Let n ∈ N and x ∈ $f^n(K)$.

We have to demonstrate that $\overline{\mathrm{orb}(f, x)} = X$.

Take y ∈ X and $\epsilon > 0$. Since $\overline{\mathrm{orb}(\phi, K)} = X$ we have $\overline{\mathrm{orb}\left(\phi, f^n(K)\right)} = K(X)$

Let $\{y\} \in \mathrm{orb}(\phi, f^n(K))$, then there exist j ∈ N such that $H\left(f^j(f^n(K), \{y\})\right) < \epsilon$.

Hence $f^j(x) \in f^j\left(f^n(K)\right) \subseteq B_\epsilon(y)$

So $B_\epsilon(y) \cap, \overline{\mathrm{orb}(f, x)} \neq \emptyset$

There fore , $\overline{\mathrm{orb}(f, x)} = X$.

For any natural number n and x in the nth iterate of the compact set K, we aim to show that the orbit of f containing x equals the entire space X. Consider y in X and any positive ε. Since the orbit closure of ϕ over K equals K(X), it follows that the orbit closure of ϕ over the nth iterate of K equals K(X). Therefore, orb(f, x) is dense in X.

Theorem 2.7. If ϕ is transitive, then f is transitive.

Lemma 2.8. If X is weekly mixing, then any power $X \times X \times \cdots \times X$ is ergodic.

Proof.

see[1]

Theorem 2.9. Let $\phi : K(X) \to K(X)$ be transitive. Then there exist a cantor set $C \subseteq X$ such that $\overline{\mathrm{orb}(\phi, C)} = K(X)$.

Proof.

Let δ_0 represent the diameter of the set X. For every positive integer n, we define a series of finite open covers of X, denoted by $\widetilde{V_n} = \{V_{n,1}, V_{n,2}, \ldots, V_{n,t_n}\}$ where each $V_{n,i}$ is a nonempty subset. These covers are constructed such that the size of each cover is smaller than δ_n. In essence, this arrangement ensures that each element of the set X is contained within at least one open set in the cover, and as n increases, the covers become increasingly finer, converging towards the set's diameter.

Step 1

Let us assume W_0 and W_1 to be two non-empty disjoint open sets in X and mesh($\{W_0 \cap W_1\} < \delta_1$. Let $\lambda_1 = \{1,2,\dots,t_1\} \times \{1,2,\dots,t_1\} = \{(a,b) | a,b \in \{1,2,\dots,t_1\}\}$.

Let us take into consideration the following t_1^2+1 collection of open sets

(W_0,W_1) and $\{(V_{1,a},V_{1,b}) : (a,b) \in \lambda_1\}$

Then, by Lemma 1.4 [see 1], two closed subsets of X, C_0 and C_1, having the following characteristics, exist:

- Each int(C_i) is nonempty and which is contained in W_i for i = 0,1.

Hence C_0 and C_1 are disjoint.

- for each A ∈ $\langle C_0,C_1\rangle = \{\boldsymbol{B \in K(X): B \subset (C_0 \cup C_1) and\ B \cap C_i \neq \emptyset}\}$and for each (a,b) ∈ λ_1,there exist n ∈ N such that $f^n(A) \in \langle U_{1,a},U_{1,b}\rangle$.

- Also, $f^n(A \cap C_0) \subset U_{1,a}$ and $f^n(A \cap C_1) \subset U_{1,b}$.

Let $\mathbb{C}_1 = \langle C_0,C_1\rangle$. Then diam($\mathbb{C}_1$) < δ_1 for each A ∈ $\mathbb{C}_1$, orb(ϕ,A) is δ_1-close to $F_2(X)$,where $F_2(X) = \{A \in K(X) \backslash |A| \leq 2\}$.

For, given any $\{p,q\} \in F_2(X)$, there exist (a,b) ∈ λ_1 such that p ∈ $U_{1,a}$ and q ∈ U1,b.

Since there exist n so that $f^n(A) \in \langle U_{1,a},U_{1,b}\rangle$. We conclude that $H(\{p,q\},f^n(A)) < \delta_1$

Step 2

Let $W_{0,0}$,$W_{1,0}$,$W_{0,1}$.$W_{1,1}$ are 4 non-empty open subsets of X with

$\boldsymbol{W_{0,0} \cap W_{1,0} = \emptyset}$ and $\boldsymbol{W_{0,1} \cap W_{1,1} = \emptyset}$

$\boldsymbol{W_{0,0} \cup W_{1,0} \subset C_0}$ and $\boldsymbol{W_{0,1} \cup W_{1,1} \subset C_1}$ and $\boldsymbol{mesh(\{W_{0,0}, W_{1,0}, W_{0,1}, W_{1,1}\}) < \delta_2}$.

Let $\lambda_2 = \{1,2,\dots t_2\}^4 = \{(a_1,a_2,a_3,a_4) \backslash a_i \in \{1,2,\dots t_2\}\}$. Consider the following $t^4{}_2 + 1$ collection of open sets

$(\boldsymbol{W_{0,0}, W_{1,0}, W_{0,1}, W_{1,1}}),\{(\boldsymbol{U_2, a_1, U_2, a_2, U_2, a_3, U_2, a_4})\}$.

Given four non-empty open subsets of X, $\boldsymbol{W_{0,0}, W_{1,0}, W_{0,1}, and\ W_{1,1}}$, where $\boldsymbol{W_{0,0}}$ and $\boldsymbol{W_{1,0}}$ are disjoint, as are $\boldsymbol{W_{0,1}}$ and $\boldsymbol{W_{1,1}}$. Each pair of subsets is contained within different closed sets, $\boldsymbol{C_0}$and $\boldsymbol{C_1}$, respectively. The mesh of these subsets is less than δ2. Then, λ2 consists of all possible combinations of indices from 1 to t2. A collection of open sets is formed from the given subsets and λ2.

By the same result of Lemma 1.2[see 1] there exist 4 closed sets $C_{0,0}$, $C_{0,1}$,$C_{1,0}$, $C_{1,1}$ with the following properties:

1. Each int($C_{i,j}$) is nonempty and contained in $W_{i,j}$ for $\{i,j\} \in \{0,1\} \times \{0,1\}$

2. For each A ∈ ⟨$C_{0,0}$,$C_{0,1}$,$C_{1,0}$,$C_{1,1}$⟩ and for each (a_1,a_2,a_3,a_4) ∈ λ_2 there exist $\boldsymbol{n} \in \boldsymbol{N}$ such that $f^n(A)$ ∈ ⟨$U_{2,a1}$,$U_{2,a2}$,$U_{2,a3}$,$U_{2,a4}$⟩, subsets of $\mathbb{C}_2$ namely $\mathbb{C}_2^i$ with $\boldsymbol{f^n(A \cap \mathbb{C}_2^i) \subset U_2, a_i}$ for $\boldsymbol{1 \leq i \leq 4}$, where $\mathbb{C}_2 = \langle \boldsymbol{C_{0,0}, C_{0,1}, C_{1,0}, C_{1,1}} \rangle$.

Note that diam($\mathbb{C}_2$) < δ_2 and $\mathbb{C}_2 \subset \mathbb{C}_1$ let A ∈ $\mathbb{C}_2$ and {p_1,p_2,p_3,p_4} ∈ F_4(X), then there exist (a_1,a_2,a_3,a_4) ∈ λ_2 such that $p_i \in U_{2,ai}$

Since there exist n so that $f^n(A)$ ∈ ⟨$U_{2,a1}$,$U_{2,a2}$,$U_{2,a3}$,$U_{2,a4}$⟩.

We conclude that H({p_1,a_2,a_3,a_4},f^n(A)) < δ_2.

So for every A ∈$\mathbb{C}_2$,orb(ϕ ,A) is δ_2-close to F_4(X) .

Step 3

Let's say that $\mathbb{C}_r$ has previously been defined and has the following attributes:

$\mathbb{C}_r$ is defined as a collection of closed sets of X, represented as ⟨$C_{0,0,0,\ldots}$,...,$C_{1,1,\ldots}$⟩, where each closed set is indexed by a binary sequence :{$C_{j1,j2,\ldots,jr}$\(j_1,j_2,...,j_r) ∈ $\{0,1\}^r$}. This indexing scheme corresponds to 2^r possible combinations, denoting the presence or absence of each closed set in $\mathbb{C}_r$. Each combination delineates the composition of $\mathbb{C}_r$ and its constituent closed sets.

- $(\boldsymbol{j_1, j_2, \ldots, j_r}) \in \{\boldsymbol{0,1}\}^{\boldsymbol{r}}$ and $\boldsymbol{int}(\boldsymbol{C_{j_1 j_2 \ldots j_r}})$ is non empty,$\boldsymbol{C_{j_1 j_2 \ldots j_r} \subset C_{j_2 \ldots j_r}}$ and $\boldsymbol{diam(C_{j_1 j_2 \ldots j_r}) < \delta_r}$
- diam($\mathbb{C}_r$) is less than δ_r and $\mathbb{C}_r$ contained in $\mathbb{C}_{r-1}$
- For each pair $(\boldsymbol{j_1, j_2, \ldots j_r}) \neq (\boldsymbol{l_1, l_2, \ldots, l_r})$ in $\{0,1\}^r$;

$\boldsymbol{C_{j_1 j_2 \ldots j_r}}$ and $\boldsymbol{C_{l_1 l_2 \ldots l_r}}$ are disjoint.

- For each A ∈ $\mathbb{C}_r$ and each $(\boldsymbol{t_1, t_2, \ldots, t_{2^r}})$in $\lambda_r = \{\boldsymbol{t_1, t_2, \ldots t_r}\}^{\boldsymbol{2^r}}$ there exist $\boldsymbol{n} \in \boldsymbol{N}$,$\boldsymbol{f^{n(A)}}$ ∈ $\langle \boldsymbol{U_r, t_1, U_r, t_2, \ldots, U_r, t_{2^r}} \rangle$, subsets of $\mathbb{C}_r$ namely $\mathbb{C}_r^i$ with $\boldsymbol{f^n(A \cap \mathbb{C}_r^i) \subset U_r, t_i}$, for $\boldsymbol{1 \leq i \leq 2^r}$.

Then for if $\boldsymbol{A} \in \mathbb{C}_r$, then $\boldsymbol{orb(\phi, A)}$ is $\boldsymbol{\delta_r}$ close to $\boldsymbol{F_{2^r}}$(X).

Similarly, for $\mathbb{C}_{r+1}$.

Step 4

So we get a declining order of compact subsets of $\boldsymbol{X}$, $\{\mathbb{C}_r\}_1{}^{\infty}$

Let $\{\boldsymbol{C_{l_1 l_2 \ldots l_r}} : (\boldsymbol{l_1, l_2, \ldots, l_r}) \in \{\boldsymbol{0,1}\}^{\boldsymbol{r}}\}$ be the $\boldsymbol{2^r}$ compact subset of X that define $\mathbb{C}_r$.

∞

Let $\boldsymbol{C} = \bigcap_{\boldsymbol{r=1}}^{\infty} (\cup \{\boldsymbol{C_{l_1, l_2, \ldots l_r}}, (\boldsymbol{l_1, l_2, \ldots, l_r}) \in \{\boldsymbol{0,1}\}^{\boldsymbol{r}}\})$.

Then C is a cantor set in X .

for all $\boldsymbol{r}, \boldsymbol{C} \in \mathbb{C}_{\boldsymbol{r}}$, so $\overline{\boldsymbol{orb}(\boldsymbol{\phi}, \boldsymbol{C})} = \boldsymbol{K}(\boldsymbol{X})$.

We establish a descending sequence of compact subsets in X. Let $\{\boldsymbol{C}_{\boldsymbol{l_1 l_2 \ldots l_r}}: (\boldsymbol{l_1}, \boldsymbol{l_2}, \ldots, \boldsymbol{l_r}) \in \{\boldsymbol{0}, \boldsymbol{1}\}^{\boldsymbol{r}}\}$ denote the $\boldsymbol{2^r}$ compact subsets defining $\mathbb{C}_{\boldsymbol{r}}$. By repeating this process infinitely, we obtain the Cantor set C in X. For any r and C in $\mathbb{C}_{\boldsymbol{r}}$, it follows that orb(ϕ,C) dense in K(X). This implies that the orbit of ϕ acting on C covers the entirety of X. Therefore, C represents a Cantor set that exhausts the space X, providing a comprehensive characterization of its structure through the Cantor-like construction.

Theorem 2.10. Let ϕ : K(X) → K(X) be transitive. Then there exists a cantor set C ⊆ X such that $\overline{\boldsymbol{orb}(\boldsymbol{f}, \boldsymbol{x})} = \boldsymbol{X}$, for every $x \in f^n(C)$ and for all n ∈ N.

Proof. clear from Theorem 2.

Theorem 2.11. Let ϕ : K(X) → K(X) be transitive. Then there exist a cantor set C ⊆ X such that $\overline{\boldsymbol{orb}(\boldsymbol{f}, \boldsymbol{x})} = \boldsymbol{X}$ for each x ∈ C

Proof. clear from lemma 2.6 and theorem 2.9 .

Theorem 2.12. if orb(ϕ,K) is dense in K(X), then for each A ∈ K(X) and for each x ∈ K, x is not a intermittent point of f.

Proof.

for each x ∈ K, $\overline{\boldsymbol{orb}(\boldsymbol{f}, \boldsymbol{x})} = \boldsymbol{X}$. ie, each point of K admits dense orbit. Hence the Theorem

For every point x belonging to set K, the orbit of f at x covers the entire space X, implying that the orbit is dense. Therefore, the theorem holds true because it confirms that every point in K has a dense orbit under the function f.

Theorem 2.13. Let ϕ : K(X) → K(X) be transitive. The there exist a Cantor set C ⊆ X such that for each x ∈ C, x is not a periodic point of f.

Proof. Clear from Theorem 2.11 and 2.12 .

Theorem 2.14. Let X be a continum, k ∈ X and $A_1, A_2, \ldots, A_k \in K(X)$. Given j ∈ {1,2,3,...,k},let $\Lambda_j = \{A \in K(X) : A_j \subseteq A\}$ and $\Lambda = A_1 \cup A_2 \cup \cdots \cup A_k$. Then Λ is not solid in K(X).

Proof.

see[3]

Theorem 2.15. Let X be a continum. Assume that ϕ : K(X) → K(X) be transitive. Let C be the Cantor set in K(X) with orb(ϕ,C) is dense in K(X). Then for every m ∈ N ∪ {0}, the set $f^m(C)$ has an empty interior in X .

Proof.

The proof is grounded on the process of contradiction. So assume that there exists an s in N such that $int_X(f^s(C))$ is nonempty.

Now take $V = f^s(C)$.

Note that orb(ϕ,C) is dense in K(X).

Let $x \in int_X(V)$ and let $\epsilon > 0$ such that $B_X(x,\epsilon) \subseteq V$.

Since {x} $\in \overline{orb(\phi, C)} = K(X)$., we can find r ∈ N such that $H(f^r(V),\{x\}) < \epsilon$.

Then $f^r(V) \subseteq B_X(x,\epsilon) \subseteq V$.

So we have $f^r(V) \subseteq V$

$\Longrightarrow f^{r(n+1)}(V) \subseteq f^{rn}(V)$, for each n ∈ N .

Hence $\{f^{rn}(V)\}_n$ is declining order in K(X) so it converges to W $=\bigcap_{r=1}^{\infty} f^{rn}(V)$

This implies that $\omega(\phi^r,V) = \{W\}$ and since $\omega(\phi^r,V)$ is strongly invariant under ϕ^r .

We have

$\{f^r(W)\} = \phi^r(W) = \phi^r((\omega(V,\phi^r))$

$= \omega(V,\phi^r) = \{W\}$

That is ,$f^r(W) = W$

From this inequality we have if $t \geq r$ and $t \equiv j(\text{mod} r)$ for

$j = \{0,1,2,...,r-1\}$,$f^j(W) \subseteq f^t(C)$

this implies $orb(\phi,f^t(C)) \subseteq \Lambda$ where $\Lambda = \bigcup_{j=0}^{r-1}\{F \in K(X): f^j(W) \subseteq F\}$

Since $orb(\phi,f^t(C))$ is solid in K(X), we have Λ is solid in K(X), a contradiction to Theorem 2.14

This proof employs the concept of contradiction, assuming the existence of a set within a specific context. By demonstrating a series of logical deductions, it establishes the convergence of a sequence within a certain space. The convergence implies the existence of a stable point under repeated iterations of a function. However, the subsequent analysis reveals a contradiction by showing that this stable point violates a fundamental theorem. This contradiction highlights the intricacies of mathematical reasoning and underscores the importance of rigorous proof techniques in establishing mathematical truths.

Theorem 2.16. Let X be a continum and $\phi : K(X) \to K(X)$ be transitive.

Let C be the Cantor set in K(X) with orb(ϕ,C) is dense in K(X). Let V ∈ K(X) and j ∈ N such that $V \subseteq f^j(C)$. Then

1. $V \not\subseteq f^{j+k}(C)$ for some k ∈ N
2. $V \not\subseteq f(V)$ and $f(V) \not\subseteq V$

Proof.

Assume on the conflicting that $V \subseteq f^{j+k}(C)$ for each k ∈ N.

Let $\Lambda = \{W \in K(X) : V \subseteq W\}$.

Then orb(ϕ,f^j(C))is dense in K(X), we have Λ is dense in K(X).

This is a contradiction to Theorem 2.14

This proves (1)

Now we prove (2), if we assume f(V) ⊆ V , then we have f^{n+1}(V) ⊆ f^n(V) for every n ∈ N and this leads that $\{f^n(V)\}_n$ is a declining order K(X) and it converges, in the Hausdorff metric, to

$F = \bigcap_{n=1}^{\infty} f^n(V)$. Note that F ∈ K(X) and that j ∈ N is such that F ⊆ f^j(C), so by (1), there is an k in N such that F⊈ f^{j+k}(C)

Since V ⊆ f^j(C), we have f^n(V) ⊆ f^{j+n}(C) for each n ∈ N.

Thus F ⊆ f^{j+n}(C) for every n ∈ N.

This contradiction displays that f(V) ⊈ V .

Now assume that V ⊆ f(V) .

Then, f^n(V) ⊆ f^{n+1}(V), for each n ∈ N .

Hence V ⊆ f^j(C), we have f^n(V) ⊆ f^{j+n}(C) for each n ∈ N.

This implies, V ⊆ f^{j+n}(C) for each n ∈ N

This contradicts (1),hence we have V ⊈ f(V)

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