



A system of Generalized Operator Quasi-Equilibrium Problems and their solutions

Shivani Sharma¹, Rajesh Kumar Gupta¹, Abdul Raouf²

¹Department of Mathematics, Lovely Professional University, Punjab, India

²Department of Mathematics, Govt. College Mendhar, J&K, India

Email: Shivani.sur81@gmail.com

In our study, the generalized operator quasi-equilibrium problem (GOQEP) has been introduced and investigated in the framework of topological vector space. ϕ -condensing maps have been employed in multiple cases to prove that its solution exists. Additionally, the operator quasi-saddle point problem has been proven by us. For the purpose of obtaining the GOEP existence results, we have utilized the fixed point theorem. For this reason, the maximal element theorem has also been utilized. With the use of this research, we want to consolidate and generalize the conclusions of the comparable equilibrium problems that are already known.

Keywords: Generalized Operator Quasi-Equilibrium Problem (GOQEP), ϕ -Condensing Maps, Fixed Point Theorem.

1. Introduction

In 1994, Blum and Oettli [3] started researching equilibrium problems. The classic variational inequality problem is generalized to provide the equilibrium problem. The equilibrium problem is equivalent to many other problems, eg; the complementarity problem, saddle point problem, and optimization problem. In 1998, Oettli and Schlager [16] extended the equilibrium problem. A family of operator equilibrium problems was developed in 2005 by Kazmi and Raouf [11]. Examples of specific operator equilibrium issues are scalar equilibrium problems and vector equilibrium problems. For the operator equilibrium problems, they offered operator solutions. They created a lemma specifically for this particular class of equilibrium states. Using the lemma and the KKM theorem, a few existence theorems for the operator equilibrium issues were derived. Raouf further introduced general and mixed multi-valued equilibrium issues. Perturbed equilibrium problems having operator solutions were considered by Kazmi and Raouf [18] in 2009. Kim and Raouf [13] looked into a set of generalized operator equilibrium problems in 2017. They proved several existence results for the problems relating to generalized operator equilibrium. In this work, we have constructed

certain existence theorems to deal with a new class of problems known as generalized operator quasi-equilibrium problems (GOQEP).

2. Preliminaries

In our paper, we used the notations that are listed below:

Assume that Z and W are two Hausdorff topological vector spaces. We have considered topology of point-wise convergence and denoted the space of all the linear and continuous operators from Z to W equipped with this topology by $L(Z, W)$. Consider a nonempty convex set $D \subset L(Z, W)$. We represent the family of all subsets of W by $\Pi(W)$ and then consider a multi-valued map $C: D \rightarrow \Pi(W)$ in order to obtain a convex, open and solid cone $C(f)$ for any $f \in D$ such that $0 \notin C(f)$ and $C_0(f) = C(f) \cup \{0\}$,

Consider an ordering cone $C(f)$ in an ordered topological vector space W such that $C(f) \neq W$. Also, consider a partial ordering $\leq_{C(f)}$ defined by $C(f)$ for each $f \in D$ on W

as follows; $h \leq_{C(f)} g$ iff $g - h \in C(f)$, $\forall g, h \in C(f)$.

Let's now assume that $T: D \rightarrow \Pi(D)$ is a non-empty multi-valued map. $\mathcal{F}(D)$ represents the set of multi-valued maps from $D \times D$ to $\Pi(W)$. Let $H \in \mathcal{F}(D)$.

In the current research, we have examined a generalized operator quasi-equilibrium problem (GOQEP) which is as under:

Obtaining $f \in D$ so that

$$f \in \text{cl}_D T(f) \text{ and } H(f, h) \not\subseteq -C(f), \quad \forall h \in T(f), \quad (1)$$

where $\text{cl}_D T(f)$ denotes the closure of $T(f)$ in D .

For appropriate selections of H, D, T, Z, W and C , we note that GOQEP (1) simplifies to the issues raised in ([5, 11–13]). The specifics are left out.

Definition 2.1. Suppose there exists a multi-valued mapping $Q: B \subset L(Z, W) \rightarrow \Pi(W)$

We define its graph as $G(f) = \{(f, w) \in B \times W : f \in B, w \in Q(f)\}$

The inverse Q^{-1} of Q , where Q is a multi-valued map from range of Q to B , is defined as:

$$f \in Q^{-1}(w) \text{ if and only if } w \in Q(f).$$

For each $f \in B$ and an open set K in W containing $Q(f)$, the map Q is an upper semi-continuous map on B if, there is an open neighbourhood U of f in B such that $Q(f) \subseteq K$ for every f in U .

Lemma 2.1 [2] Let us consider a compact subset B of $L(Z, W)$ and let $S: L(Z, W) \rightarrow \Pi(W)$ be an upper semicontinuous map. Then the set $T(B)$ is a compact set.

Lemma 2.2 [2] Suppose $S: L(Z, W) \rightarrow \Pi(W)$ is a multi-valued map. Then, S is a closed mapping if it is upper semi-continuous on $L(Z, W)$. Assume $\{D_n\}_{n=1}^{\infty}$ to be an increasing sequence of compact non-empty sets. Suppose for all $n \in \mathbb{N}$, $D_n \subseteq D_{n+1}$, $D = \bigcup_{n=1}^{\infty} D_n$, is a subset of $L(Z, W)$. A sequence $\{f_n\}_{n=1}^{\infty}$

contained in D is called an escaping sequence from D with respect to $\{D_n\}_{n=1}^\infty$ if for every $n, \exists$ an

$m \geq M$, such that $f_m \notin D_n$

Consider multi-valued maps $T, S: D \rightarrow \Pi(L(Z, W))$. The multi-valued maps $cl_D T$, coT , $T \cap S: D \rightarrow \Pi(L(Z, W))$ are then defined as

$$(cl_D T)(f) = cl_D T(f), (coT)(f) = coT(f) \text{ and}$$

$$(T \cap S)(f) = T(f) \cap S(f), \forall f$$

$\in D$, where the convex hull of $T(f)$ is represented by $coT(f)$.

Let us assume E to be a Hausdorff topological vector space. Also, take into consideration a lattice L possessing the least element, represented as 0 . Examine the following map $\phi: \Pi(E) \rightarrow L$ meeting the necessary requirements for any sets $X, Y \in \Pi(E)$:

(i) $\phi(X) = 0$ iff X is a pre-compact set

(ii) $\phi(\bar{co}X) = \phi(X)$, where the $\bar{co}X$ represents the closed convex hull of X

(iii) $\phi(X \cup Y) = \max\{\phi(X), \phi(Y)\}$.

Then, ϕ is called a measure of non-compactness

As a result, (iii) would imply that, if $X \subseteq Y$, then $\phi(X) \leq \phi(Y)$. Suppose $J \subseteq E$ and suppose

$\phi: \Pi(E) \rightarrow L$ represents a measure of non-compactness on a set E . Let $S: J \rightarrow \Pi(E)$ be a multi-valued map. It is referred to as a ϕ -condensing map if for $X \subseteq J$ and $\phi(S(X)) \geq \phi(X)$, we have X to be a relatively compact set.

The fact that every multi-valued map that we define on a compact set is unquestionably ϕ -condensing should be highlighted. Additionally, for any given measure of non-compactness, a compact multi-valued map (when $S(J)$ is pre-compact) is certainly ϕ -condensing if E is locally convex. Furthermore, if E is a locally convex set, then a compact multi-valued map (where $S(J)$ is pre-compact) is unquestionably a ϕ -condensing map for any given measure of non-compactness.

S^* is categorically referred to as being ϕ -condensing if

$S: J \rightarrow \Pi(E)$ is a ϕ -condensing map and if

$S^*: J \rightarrow \Pi(E)$ also satisfies the condition that $S^*(b) \subseteq S(b)$ for all $b \in J$

Theorem 2.1 [6] Let E be a topological vector space that is Hausdorff. Suppose that D is a closed, nonempty, convex subset of the space E . Consider multivalued maps $T, S: D \rightarrow \Pi(D)$ so that

$$D = \cup \{int_D T^{-1}(y) : y \in D\} \text{ and } coT(x) \subseteq T(x) \text{ for each } x \in D.$$

Then the map S has a fixed point if S is a ϕ -condensing map.

Theorem 2.2 [14] Let us consider a topological vector space X . Also, suppose that D is a subset of X where $D = \bigcup_{n=1}^\infty D_n$ and $\{D_n\}_{n=1}^\infty$ an ascending sequence of non-empty

subsets of D . Suppose the sets D_n are compact and convex. Let us suppose that the multi-valued map $T: D \rightarrow \Pi(D)$ meets the underlying prerequisites.

(i) For $n = 1, 2, \dots$ and for every $x \in D$, $T^{-1}(x) \cap D_n$ is an open set in D_n

(ii) For every $x \in D$, $x \notin \text{co}T(x)$

(iii) For every escaping sequence $\{x_n\}_{n=1}^\infty$ in D (with respect to $\{D_n\}_{n=1}^\infty$) with $x_n \in D_n$; $n = 1, 2, 3, \dots$, there exists $n \in N$ and $y_n \in D_n$ such that $y_n \in \text{co}T(x_n) \cap D_n$

Then, $x_0 \in D$ exists such that $T(x_0) = \phi$

3. Existence theorems:

We will assume (without loss of generality) W to be a real topological vector space. Also, the family of all the subsets of W shall be denoted by $\Pi(W)$ and suppose $C: D \rightarrow \Pi(W)$ is a multi-valued mapping. For every $f \in D$, $C(f)$ denotes a convex, open and solid cone such that $0 \notin C(f)$. Also, let $C_0(f) = C(f) \cup \{0\}$. The family of multi-valued maps from $D \times D$ to $\Pi(W) / \{\phi\}$ will be referred to as $\mathcal{F}(D)$. Suppose $H \in \mathcal{F}(H)$. For $f \in D$, the existence results for **OQEP (1)** are now presented below.

Theorem 3.1. Let a multivalued map $C: D \rightarrow \Pi(W)$ be defined on D such that, $C(f)$ is a solid, open and convex cone for each $f \in D$ and $0 \notin C(f)$. Here, $D \subset L(Z, W)$ is a convex and compact set which is

non-empty. Let us consider multivalued ϕ – condensing mappings $T, cl_D T: D \rightarrow \Pi(D)$. For each

$f \in D$, let $T(f)$ be non-empty and convex set. Also, for each $h \in D$, let $T^{-1}(h)$ be open in D

and $\mathcal{F} := \{f \in D: f \in T(f)\}$ be closed. Let us suppose that $H, G \in \mathcal{F}$ meet the underlying prerequisites:

(i) For each $f \in D$, $G(f, f) \not\subseteq C(f)$ and $\{h \in D: G(f, h) \subseteq -C(f)\}$ is a convex set

(ii) The set $\{f \in D: H(f, h) \not\subseteq -C(f)\}$ is closed in D , for each $h \in D$;

(iii) For all $f, h \in D$, $H(f, h) \not\subseteq C(f)$ if $G(h, f) \not\subseteq C(f)$

Then we can find $f_0 \in D$ such that

$$f_0 \in cl_D T(f_0) \text{ and } H(f_0, h) \not\subseteq -C(f_0) \forall h \in S(f_0).$$

Proof: Using condition (ii), let a multi-valued map $Q: D \rightarrow \Pi(D)$ be defined as

$$Q(f) = \{h \in D: H(f, h) \subseteq -C(f)\}, \text{ for each } f \in D, \text{ which is a convex set}$$

According to condition (iii), the complement of $Q^{-1}(h)$ in D is defined as:

$$[Q^{-1}(h)]^c = \{f \in D: H(f, h) \not\subseteq -C(f)\}$$

Since it is a closed set in D for every $h \in D$, therefore,

$Q^{-1}(h)$ is an open set in H for all $g \in H$

For any $f \in D$, we define the multivalued maps $N, M: D \rightarrow \Pi(D)$ as:

$$N(f) = \begin{cases} Q(f) \cap T(f) & \text{if } f \in \mathcal{F}, \\ T(f), & \text{if } f \in D \setminus \mathcal{F} \end{cases}$$

and

$$M(f) = \begin{cases} T(f) \cap P(f) & \text{if } f \in \mathcal{F}, \\ T(f), & \text{if } f \in D \setminus \mathcal{F} \end{cases}$$

where $P: D \rightarrow \Pi(D)$ is a multi-valued map defined as

$$P(f) = \{h \in D : G(h, f) \subseteq C(f) \forall f \in D.$$

For every $f \in D$,

$N(f)$ is a convex set for each f because $T(f)$ and $Q(f)$ are also convex.

So, by condition (iv), $N(f) \subseteq M(f)$ for every $f \in D$

Since $T^{-1}(h)$ and $Q^{-1}(h)$ are open in D , for each $f \in D$,

$$\text{consequently, } N^{-1}(h) = (T^{-1}(h) \cap Q^{-1}(h)) \cup ((D \setminus \mathcal{F}) \cap T^{-1}(h))$$

As is clear from Lemma 2.3 in [4] and since $D \setminus \mathcal{F}$ is an open set in D , hence, $N^{-1}(h)$ is also open in D . Let us suppose that the set $T(f) \cap Q(f) \neq \phi$, for each $f \in D$

Therefore, for each $f \in D$, $N(f) \neq \phi$ and consequently

$$D = \bigcup_{h \in D} N^{-1}(h) = \bigcup_{h \in D} \text{int}_D N^{-1}(h)$$

Hence, M is ϕ -condensing since T is ϕ -condensing and

$M(f) \subseteq T(f)$ for each $f \in D$.

Consequently, Theorem 2.1 shows the existence of $f_o \in D$ such that $f_o \in M(f)$.

Additionally, based on the definition and meaning of $\mathcal{F}$ and M , we have $\{f \in D : f \in M(f)\} \subseteq \mathcal{F}$. This implies that $f_o \in \mathcal{F}$ and $f_o \in T(f_o) \cap P(f_o)$ and, in particular

$G(f_o, f_o) \subseteq C(f_o)$, which contradicts (i). So, there lies

$$f_o \in \text{cal } H \text{ such that } T(f_o) \cap Q(f_o) = \phi$$

That is,

$$f_o \in T(f_o) \text{ and } H(f_o, h) \not\subseteq -C(f_o) \text{ for every } h \in T(f_o).$$

The proof is therefore complete.

By including $G(f, h) = -H(h, f)$ in the Theorem 3.1, the following conclusion is generated:

Corollary 3.1 Let us suppose that D is a real Hausdorff topological vector space. Also, D is a compact, convex and non – empty subset of $L(Z,W)$. Consider a multi-valued map $C: D \rightarrow \Pi(D)$. Suppose $C(f)$ is a convex, solid and open cone, for every $f \in D$. Also, $0 \notin C(f)$. Consider ϕ – condensing multi-valued maps $T, cl_D T: D \rightarrow \Pi(D)$. Suppose $T(f)$ is a non-empty and convex set for each $f \in D$. Also, for each $h \in D$, $T^{-1}(h)$ is an open set in D and

$$\mathcal{F} = \{f \in D: f \in T(f)\} \text{ is a closed set.}$$

Suppose $H \in \mathcal{F}$ meets the following criteria:

For each $f \in D$, $H(f, f) \not\subseteq C(f)$; $\{h \in D: H(f, h) \subseteq -C(f)\}$ is a convex set and $\{f \in D: H(f, h) \not\subseteq C(f)\}$

is a closed set in D , for each $h \in D$.

Then we can find $f_0 \in D$ such that

$$f_0 \in cl_D T(f_0) \text{ and } H(f_0, h) \not\subseteq -C(f_0) \forall h \in T(f_0).$$

When T is not ϕ – condensing necessarily, we can derive the following conclusion:

Theorem 3.2 Let $\{D_n\}_{n=1}^\infty$ be an increasing sequence of convex, compact and non-empty subsets of D and $D = \cup_{n=1}^\infty D_n$. Let $D \subset L(Z, W)$ be non – empty. Consider a multi – valued map $C: D \rightarrow \Pi(D)$. Suppose for each $f \in D$, $C(f)$ is a convex, open and solid cone. Suppose $0 \notin C(f)$. Consider an ordered topological vector space $(W, C(f))$. Let $H \in \mathcal{F}(D)$. Consider multi-valued maps $T, cl_D T: D \rightarrow \Pi(D)$ such that for each $f \in D$, $T(D)$ is non-empty and convex set. Also, $T^{-1}(h)$ is compactly open in D for each $h \in D$ and $\mathcal{F} = \{f \in D: f \in T(f)\}$ is a closed set. Let $H \in \mathcal{F}$ meet the underlying criteria:

- (i) For each $f \in D$, $H(f, f) \not\subseteq -C(f)$;
- (ii) For each $f \in D$, the set $\{h \in D: H(f, h) \subseteq -C(f)\}$ is convex and
- (iii) $\{f \in D: H(f, h) \not\subseteq -C(f)\}$ is compactly closed in D for each $h \in D$.
- (iv) Also, there exists $m \in N$ and $h_m \in D_m \cap T(f_m)$ such that for every $f_m \in T(f_m)$, $H(f_m, h_m) \subseteq -C(f_m)$, where $\{f_n\}_{n=1}^\infty$ is a sequence in D , $f_n \in D_n$, $n \in N$ which is an escaping sequence from D relative to $\{D_n\}_{n=1}^\infty$

Then we can find $f_0 \in D$ where

$$f_0 \in cl_D T(f_0) \text{ and } H(f_0, h) \not\subseteq -C(f_0), \text{ for each } h \in T(f_0).$$

Proof: Using condition (ii), let us define a multi-valued map $Q: H \rightarrow \Pi(H)$ as:

$$Q(f) = \{h \in D: H(f, h) \subseteq -C(f)\}, \text{ for each } f \in D.$$

Q is convex. Using condition (iii), the complement of $Q^{-1}(h)$ in H is defined as:

$$[Q^{-1}(h)]^c = \{f \in D: H(f, h) \not\subseteq -C(f)\}$$

This is a closed set in D for all $h \in D$. Thus, $Q^{-1}(h)$ is open in D for each $h \in D$.

Let a multi-valued map $N : D \rightarrow \Pi(D)$ be defined for each $f \in D$ by

$$N(f) = \begin{cases} T(f) \cap Q(f) & \text{if } f \in \mathcal{F}, \\ T(f), & \text{if } f \in D \setminus \mathcal{F} \end{cases}$$

$N(f)$ is convex for each $f \in D$. Suppose now that there lies $f_0 \in D$ such that

$f_0 \in N(f_0)$. $f_0 \in T(f_0) \cap Q(f_0)$, if $f_0 \in \mathcal{F}$. Consequently, $H(f_0, f_0) \subseteq -C(f_0)$,

which is a contradiction of (i). And, $N(f_0) = T(f_0)$, if $f_0 \notin \mathcal{F}$. As such, $f_0 \in T(f_0)$,

which is again a contradiction.

Thus, $f \notin N(f) = coN(f)$, for all $f \in D$

On applying (iii), $Q^{-1}(h)$ is a compactly open set in D for each $h \in D$. By utilising the justification in proof of Theorem 3.1,

we, thus, conclude that $N^{-1}(h)$ is a compactly open set in D , for each $h \in D$. By Theorem 2.2, Condition (iv) implies Condition (iii). As a result, according to Theorem 2.2, there shall exist $f_0 \in D$ such that $N(f_0)$ is empty. Because $T(f)$ is nonempty for each $f \in D$, so, we have $f_0 \in T(f_0)$ and $T(f_0) \cap Q(f_0) = \emptyset$. Thus, $f_0 \in D$ such that

$$f_0 \in cl_D T(f_0) \text{ and } H(f_0, h) \not\subseteq -C(f_0), \text{ for all } h \in T(f_0)$$

The proof is therefore complete

4. Application

We have applied Corollary 3.1 to prove operator quasi-saddle point problem in the present section.

Definition 4.1. For $i=1,2$, suppose $L(Z_i, W)$, $i = 1, 2$ are real vector spaces that are topological. Consider non-empty and convex subsets D_i of $L(Z_i, W)$, where $i = 1, 2$. Also, consider a multi-valued map $T: D = D_1 \times D_2 \rightarrow \Pi(D)$ which is defined as $T(f_1, f_2) = T_1(f_1) \times T_2(f_2)$ for all $f_1 \in D_1$ and

$f_2 \in D_2$. Here, $T_i: D_i \rightarrow \Pi(D_i) \setminus \{\emptyset\}$, $i = 1, 2$ are multi-valued maps. Also, $\phi: D \rightarrow W$ is a function with operator values and W and C are exactly the same as taken above. Then, we call

$$\bar{f} = (\bar{f}_1, \bar{f}_2) \in H_1 \times H_2 \text{ as:}$$

(i) a regular C – saddle point of the operator ϕ if

$$\text{for all } (h_1, h_2) \in D_1 \times D_2, \phi(h_1, \bar{f}_2) - \phi(\bar{f}_1, h_2) \notin -C;$$

(ii) a regular C – quasi-saddle point of the operator ϕ , if

$$\text{for all } (h_1, h_2) \in T(\bar{f}), \phi(h_1, \bar{f}_2) - \phi(\bar{f}_1, h_2) \notin -C, \text{ where } \bar{f} \in T(\bar{f});$$

Definition 4.2. Let us suppose that E is a real Hausdorff topological vector space. Also suppose

D is a non-empty, closed and convex subset of E and also suppose K is a real valued topological vector space. Suppose C is a cone that is pointed, proper, convex and closed and $\text{int}C \neq \emptyset$. If

- (i) for all elements $\alpha \in K$, $U(\alpha) = \{a \in D: \phi(x) - \alpha \notin -\text{int}C\}$ is convex set, the function $\phi: D \rightarrow K$ is known as C – upper semi-continuous mapping on D
- (ii) $-\phi$ is a C – upper semi-continuous function on D , then ϕ is called C -lower semi-continuous mapping on D
- (iii) for all $\alpha \in K$, $\{x \in D: \phi(x) - \alpha \in -C\}$ is a convex set in D , ϕ is called C -quasi-convex
- (iv) ϕ is C -quasi-convex then the set $\{a \in D: \phi(x) \in -\text{int}C\}$ is also convex.

Theorem 4.1. Consider Hausdorff topological vector spaces Z_i and W , for $i = 1 \& 2$. Let the spaces of all the continuous linear operators from $Z_i, i = 1, 2$ to W , with point-wise convergence topology be denoted by $L(Z_i, W)$. Let $K_i \in L(Z_i, W)$, $i = 1, 2$, be sets that are convex, closed and non-empty. Let us consider multivalued maps $T_i: D_i \rightarrow \Pi(D_i)$, $i = 1, 2$. Let us define multi-valued and

ϕ -condensing map $T: D = D_1 \times D_2 \rightarrow \Pi(D)$ as $T(f_1, f_2) = T_1(f_1) \times T_2(f_2)$ for all $f_1 \in D_1$ and $f_2 \in D_2$ so that for each function $f \in D$, $T(f)$ is a nonempty set which is also convex. Also, it is open in D for each $h \in D$.

Let a multi-valued map $C: D \rightarrow \Pi(W)$ be defined in such a way that for each $f \in D$, $C(f)$ is an open, convex and solid cone. Also, $0 \notin C(f)$. Let $\mathcal{F} = \{f = (f_1, f_2) \in D_1 \times D_2: f \in T(f)\}$ be a closed set. Let $\phi: D \rightarrow W$ be an operator-valued function such that:

- (i) For each $f_2 \in D_2$, the mapping $h_1 \rightarrow \phi(h_1, f_2)$ is C – lower semi-continuous and C – quasi-convex on D
- (ii) For each $f_1 \in D_1$, the mapping $h_2 \rightarrow \phi(f_1, h_2)$ is C – upper semi-continuous and C – quasi-concave on H

Then, for the function ϕ , we get a regular C – quasi-saddle point.

Proof: On considering H to be a single-valued mapping, i.e. $H: D \times D \rightarrow W$ (in corollary 3.1) and defining

$$H(f, h) = \phi(h_1, f_2) - \phi(f_1, h_2)$$

for each $f = (f_1, f_2)$, $h = (h_1, h_2) \in D_1 \times D_2$, the desired outcome is attained from Corollary 3.1.

Remark 4.1. If $T(f) = D$, for every $f \in D$ and $H(f, h) = \phi(f) - \phi(h)$ for every $f, h \in D$, where

$\phi: D \rightarrow W$ in Corollary 3.1, reduces to a problem of finding $f \in D$ so that $\phi(f) - \phi(h) \notin C(f)$, for all $h \in D$. Then, we obtain the existence result for operator minimization problem.

5. Conclusion :

In this study, we have studied and explored the problem of operator quasi-equilibrium which is a generalization of the operator equilibrium problem. This type of equilibrium problems has been discovered recently. We made use of C-quasi convex and C-quasi-concave functions and escaping sequence to prove existence results for it. The framework of topological vector spaces was employed by us. The problem of operator quasi-saddle point was obtained as an immediate application of corollary 3.1.

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