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Growth in Mathematical Understanding of a Junior High School Student on the Topic of System of Linear Equations in Two Variables (SPLDV)

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Understanding is crucial for junior high school students to excel in mathematics. However, there's a significant gap in research regarding how a student growth this understanding. One solution lies in leveraging the Pirie-Kieren theory to foster mathematical understanding. This study aims to describe the growth of a student's mathematical understanding of the material on systems of linear equations in two variables (SPLDV). This descriptive-exploratory research examines the growth of a junior high school student's mathematical understanding. The subjects selected were student with a high ability to complete SPLDV assignments. Next, the subject was interviewed based on the SPLDV task. The research results showed that the student could complete tasks up to the seventh layer in the Pirie-Kieren theory, namely at the structuring layer. However, in several layers, the student did fold back. When the MT subject is at the image-making layer, MT folds back to the primitive knowing layer to strengthen the understanding of linear equation in two variables (PLDV) and connect it to SPLDV. Furthermore, when MT is in the image-having layer, MT cannot create a different mathematical model, but after MT folds back to the image-making layer, MT can create a different mathematical model by multiplying the variable coefficients, and the results following the properties of SPLDV which states that two PLDVs can be congruent. Furthermore, when MT is at the property noticing layer, MT folds back to the image-having a layer; MT states that the problem can be solved by utilizing the congruence property of the two equations so that the two equations can be solved by elimination and then substitution. The subject achieves growth in the mathematical understanding of layer structuring by Interpret information in mathematical problems, describe concepts, create topic-specific reports/models, identify similarities and differences in various topic definitions, conceptualize mathematical abstractions, and relate mathematical ideas to problems. This research suggests that its results can help students grow their mathematical understanding.

Keywords: Growth in mathematical understanding; SPLDV; folding back; layer structuring; concept.

1. Introduction

Mathematics is important for students, and every field of science requires it. One of the important ideas in mathematics learning is creating a mathematics learning environment through understanding [1-2]. Understanding is one of the basic skills that junior high school students must master, so understanding is an important part of learning mathematics. In recent years, much interest has been in exploring the nature of mathematical understanding [3-4]. There have been many research studies on the theory of mathematical understanding, but An examination of these studies shows that the literature primarily addresses general characteristics [5-6], NCTM principles and standards of conceptual understanding development [7], types of understanding [8-10]. Based on this, only a few studies have been conducted on the growth of mathematical understanding, even though it is important to know so that teachers can maximize mathematics learning properly. One of the understanding that is widely studied is understanding based on the Pirie-Kieren theory, as carried out by Martin [11], who studied key aspects of the Pirie-Kieren theory, while Gulkilik, Packenham, & Ugurlu [2] researched the characteristics of the growth of students' mathematical understanding. The Pirie–Kieren theory is a medium for students to observe and describe how knowledge processes are organized and reorganized and how students can build their understanding correctly [11].

Research on the growth of students' mathematical understanding has been carried out, such as the Pirie–Kieren theory, which contains eight layers of potential actions to describe the growth of certain understandings for certain topics or concepts. These layers are called primitive knowing, image making, image having, property noticing, formalizing, observing, structuring, and inventing. This growth in understanding is the initial capital for students to understand mathematics subject matter well because understanding is the key to learning mathematics. If students cannot understand mathematical material well, then students will have difficulty solving the mathematical problems given. Students cannot follow the next material if the previous material is not understood well. It is an important discourse for teachers to create meaningful learning for their students. The Pirie-Kieren theory, which consists of eight layers, guides students to understand mathematical material in stages, where the arrangement of these layers illustrates that the growth of understanding does not have to be linear or mono-directional. Additionally, each layer contains all previous layers and is included in all subsequent layers to emphasize the embedded nature of mathematical understanding [11]. Therefore, understanding mathematics is not a static but a dynamic process that is developing and always changing [12].

Previous researchers have studied the growth of mathematical understanding, such as Martin & Towers [13], examining how mathematical understanding grows collectively and how it depends on how a group of students work together collaboratively. Codes, Astudillo, Martin, & Perez [14] conducted research to elucidate the growth of mathematical understanding in students doing assignments in mathematics class with reference to the idea of number series based on the Image-Making, Image-Having and Property-Noticing layers. Patmaniar, Maghfirotun, and Sulaiman [15] studied the knowledge layers of high school students in solving mathematical problems using the folding back method. Several studies that have been conducted have generally examined the growth of students' mathematical understanding of various mathematical topics. However, it has yet to address the topic of Systems of Linear

Equations in Two Variables (SPLDV), even though the topic of SPLDV is very important in students' daily lives.

This research aims to describe the growth in mathematical understanding of a junior high school student on the topic of system of linear equations in two variables (SPLDV)

2. Methodology

This research type is descriptive-exploratory and aims to examine the understanding characteristics of junior high school students. The subjects in this research were class VIII students at Karangjaya State Junior High School, and the subject chosen was student with high ability who could complete the System of Linear Equations in Two Variables (SPLDV) assignment. Furthermore, subject was interviewed based on the SPLDV task. The task in the form of questions used in this research was adapted and modified into eight sub-item questions based on the Pirie-Kieren layer of mathematical understanding growth by paying attention to the process of students' understanding layers. A mathematician, a mathematics education professor, and a mathematics education lecturer validated this task instrument.

In data analysis, subject is thoroughly observed and recorded based on her developing understanding of the task given. When the subject is working on the task, the researcher observes and guides her. Suppose the subject cannot complete each layer of the task given. In that case, the subject is guided to return to the previous layer of understanding (folding back) to strengthen and explore her existing understanding to return to the layer of understanding she is currently going through. The results of the subject's responses are also validated through this technique. Coding is also needed in this research to facilitate analysis such as subject (MT), researcher (P), Primitive knowing (Pk), Image-making (Im), Image-having (Ih), Property noticing (Pn), Formalizing (Fr), Observing (Ob), and Structuring (St). Next, Verification of the information gathered through interviews was done. Results for student answer codes MT (Student) and P (Researcher) are also verified using this procedure. The outcomes of folding back on each student's growth in understanding when finishing task involving mathematics are also summed up.

3. Results and Discussion

A female subject candidate with high mathematics ability was obtained from interviews with a mathematics teacher who teaches class VIII. The outcomes of the MT interview are listed below to obtain further information about the student's mathematical growth.

Folding Back in the process of growing MT students' understanding

Primitive Knowing Layer of MT

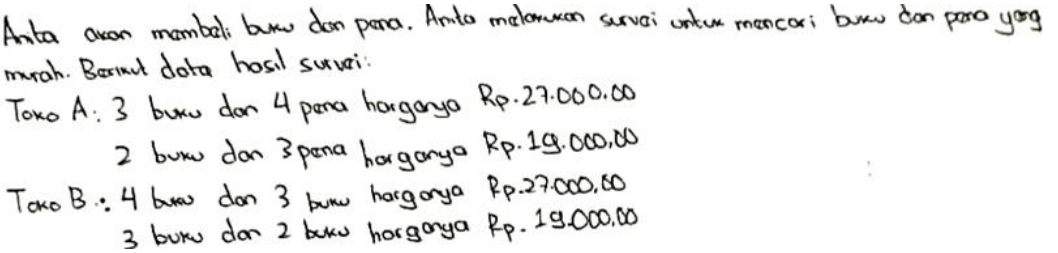
MT understands a mathematical situation and gives comprehensive details, including recognized problem and commonly requested question. An excerpt from an interview by MT on the primitive knowing layer is as follows.

P101 : When presented with a mathematical situation, what would you do?

MT101 : To get the information, I first read the question.

P102 : Write down and tell me the information you get from the mathematical situation.

MT102 :



Anita will buy books and pens. Anita conducted a survey to find cheap books and pens. Form of survey data: At shop A, three books and four pens cost IDR 27,000, while two books and three pens cost IDR 19,000. At shop B, four books and three books cost IDR 27,000, while three books and two books cost IDR 19,000.

P103 : In shop B, why does it say four books and three books cost IDR 27,000 and three books and two books cost IDR 19,000? There is no pen?

MT103 : Oh yes, sir, I was wrong. At shop B, four books and three pens should cost IDR 27,000, while three books and two pens cost IDR 19,000.

P104 : OK, you have to be more careful.

MT104 : Okay sir.

MT observes the given mathematical situation to obtain information without writing down the known answers and being asked on the answer sheet. After the researcher digs up information (P101 & P102), MT uses her knowledge to rewrite the situation in the form of images into written text (MT102). The form of written text MT stated is part of PLDV knowledge. MT has used PLDV knowledge to learn the given SPLDV so that MT can identify the information presented. The understanding task performed by MT is the capacity to explain ideas orally based on prior knowledge [11], [14]. Although the basic information here does not imply lower-layer mathematics, there is no folding back activity in this layer; rather, it serves as a foundation growth of particular mathematical understanding [16].

Image Making Layer of MT

MT stated that the situation given was an SPLDV situation; at first, she thought that SPLDV could only be with one equation; after the researcher dug up information, MT folded back to the primitive knowing layer to strengthen her previous knowledge related to PLDV. MT was asked to recall the linear equation system of two variables. After that, she constructed the information that SPLDV must have two equations. MT can create model the given problem mathematically. MT can develop the idea of PLDV and use it in SPLDV. Excerpt from interview with MT regarding Image Making layer.

P201 : From these problems, what is the material about?

MT201 : Linear equation in two variables (PLDV).

P202 : Before solving the math problem, what did you think about the PLDV material?

MT202 : Equations that have two variables (symbols) or substitutes.

P203 : What is an equation?

MT203 : An equation is a mathematical statement in the form of a variable/symbol/ substitute that states two things that are the same between the left and right segments.

P204 : What is SPLDV?

MT204 : What is it, sir?

P205 : Before SPLDV, do you still remember PLDV?

MT205 : Yes sir, linear equation in two variables.

P206 : Well, Can a linear equation in two variables be solved with just one equation? then a system is needed. What do you remember about the system?

MT206 : As I recall, the system must have two equations, sir.

P207 : Good, so is SPLDV. Try what SPLDV is.

MT207 : SPLDV is two linear equations with two variables, right, sir?

P208 : Yes, but the full definition of SPLDV is a collection of two or more linear equations involving two variables, where the corresponding values of the variables satisfy all the equations simultaneously.

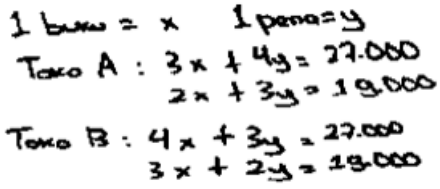
MT208 : Oh yes, sir, SPLDV must have at least two equations.

P209 : What do you know about math models?

MT209 : An equation that contains variables to represent a problem.

P210 : Well, then, what is the mathematical model based on the problem?

MT210 :



Explain your answer?

P211 : I modeled a book with variable x and a pen with variable y, and then I created the mathematical models for store A and store B.

MT211 :

MT stated that the situation given was PLDV; initially, she did not understand what

SPLDV was. After the researcher extracted information (P204), the subject folded back to the primitive knowing layer to understand the meaning of PLDV (MT206); after MT had sufficient understanding of PLDV, the researcher asked MT to relate it to SPLDV (P207). MT already understood the system in SPLDV and that the system must have two equations (MT208). When the researcher asked MT about the mathematical model, MT could answer well (MT209). MT can create a mathematical model from the given situation. MT can develop PLDV ideas and use them in SPLDV. MT's layer of understanding is shown by differentiating previous PLDV knowledge and using PLDV in new ways, such as in SPLDV. It strengthens Pirie Kieren's theory of growth in mathematical understanding, especially in the image-making layer [2], [11].

Image Having Layer of MT

At first, MT could not make a different mathematical model. However, after she folded back to the image-making layer, she realized that she could make a different mathematical model by multiplying the coefficients of the x and y and the result. It follows one of the properties of SPLDV, which states that two PLDV can be congruent. Store A becomes $\boldsymbol{6x + 8y = 27,000}$ and $\boldsymbol{4x + 6y = 19,000}$, while for store B it becomes $\boldsymbol{8x + 6y = 27,000}$ and $\boldsymbol{6x + 4y = 19,000}$. Excerpts of the interview results conducted by MT on the Image Having layer.

P301 : Can it be made using different mathematical models?

MT301: It can't.

P302 : If you want to buy a candy for 500, then two candies cost 1000; if six candies cost 3000. How much does a candy cost if you buy six candies that cost 3000?

MT302: Yes, it is still the same price, a candy is 500 sir, because 3000:6 = 500.

P303 : Do you understand what my example means?

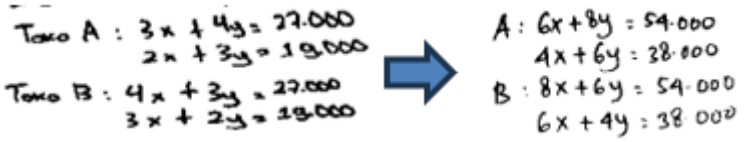
MT303: Yes, sir. That means no matter how many candies you buy, the price is still 500.

P304 : Well, what about the model that has been made? Can it be made in a different form?

MT304: Oh yes sir, that means I can multiply the two equations, right, sir?

P305 : Yes, it can be multiplied by any number. Please give it a try.

MT305: Then this equation can be like this sir.



P306 : The math model for store A becomes $\boldsymbol{6x + 8y = 54,000}$ and $\boldsymbol{4x + 6y = 38,000}$, while for store B it becomes $\boldsymbol{8x + 6y = 54,000}$ and $\boldsymbol{6x + 4y = 38,000}$

MT306: Is the math model you just mentioned equivalent to the old equation?

P307 : What is the equivalent, sir?

MT307: Two equations are equivalent if both equations have the same pair of solution order.

Oh yes sir, the new equation is equivalent to the old equation because if $\boldsymbol{x = 5000}$, $\boldsymbol{y = 3000}$ will have the correct result for the equations $\boldsymbol{3x + 4y = 27000}$ and $\boldsymbol{6x + 8y = 54000}$.

The interview results show that MT has been able to create a mathematical model without being tied to the physical object of SPLDV. At first, MT had difficulty in making different mathematical models. With the help of small illustrations from the researcher, MT can understand the concept of forming a different model than before. MT has understood the concept of equivalence in the system of linear equations of two variables. MT has the concept idea through the activities she did in the previous layer [2], [17].

Property Noticing Layer of MT

MT can solve the problem using a mixed method (elimination and substitution). At first, MT had difficulty solving the given problem by elimination. After MT folded back to the image having layer, MT stated that the problem could be solved by utilizing the equivalence property in two equations so that both equations can be solved by elimination and then substitution, as in the interview result and Figure 1.

P401 : Look at the two equations. What do you know about them?

MT401 : Both equations have two variables, x and y.

P402 : Sip. How do you calculate the value of variables x and y?

MT402 : Em…… [think long enough]

P403 : Take a look at my example.

There are two equations, $\boldsymbol{2a + b = 7}$ and $\boldsymbol{3a + b = 9}$; you want to eliminate the variable $\boldsymbol{a}$, then the first equation is multiplied by 3, and the second equation is multiplied by 2. The equations $\boldsymbol{6a + 3b = 21}$ and $\boldsymbol{6a + 2b = 18}$ are then operated so that $\boldsymbol{b = 3}$ is obtained. Now, how can it be understood?

MT403 : Oh yes, sir, I understand. The two equations have been changed so that the coefficient of variable a is the same in both equations.

P404 : How to do it?

MT404 : Find the kpk of the coefficient of variable an in both pack equations.

P405 : Yes, that is right. How about the equation in this task (pointing to the task)?

MT405 : I multiply first equation by three and second equation by four.

P406 : What do you want to get rid of?

MT406 : Variable y sir.

P407 : Yes, next?

MT407 : The new equation is $\boldsymbol{9x + 12y = 81}$ and $\boldsymbol{8x + 12y = 76}$, and then the value of $\boldsymbol{x = 5}$ or $\boldsymbol{x = 5000}$ is obtained.

P408 : What about the value of y?

MT408 : Oh yes, that means what is next...

P409 : Let us say that the price of a pencil is 1000. Now, if you buy two pencils and a book that cost 5000, how much is the price of a book?

MT409 : Two pencils are 2000, so the price of a book is 5000-2000, which is 3000.

P410 : Well, that is possible; how?

MT410 : I changed the price of the pencil, so I got the book's price.

P411 : Sip, what if there is an equation $\boldsymbol{2p + 2q = 10000}$, with $\boldsymbol{q = 2000}$. What is the value of $\boldsymbol{p}$?

MT411 : Oh then $\boldsymbol{2p + 2(2000) = 10000}$ so $\boldsymbol{2p + 4000 = 10000}$ then $\boldsymbol{2p = 10000 - 4000 = 6000}$ so $\boldsymbol{p = 6000 : 2 = 3000}$.

P412 : That can also be done in other equations.

MT412 : Oh good sir, I take the equation $\boldsymbol{3x + 4y = 27}$ then $\boldsymbol{3(5) + 4y = 27}$, then $\boldsymbol{15 + 4y = 27}$ then $\boldsymbol{4y = 27 - 15}$ so that we get $\boldsymbol{y = 3}$ or $\boldsymbol{y = 3000}$.

P413 : Good job.

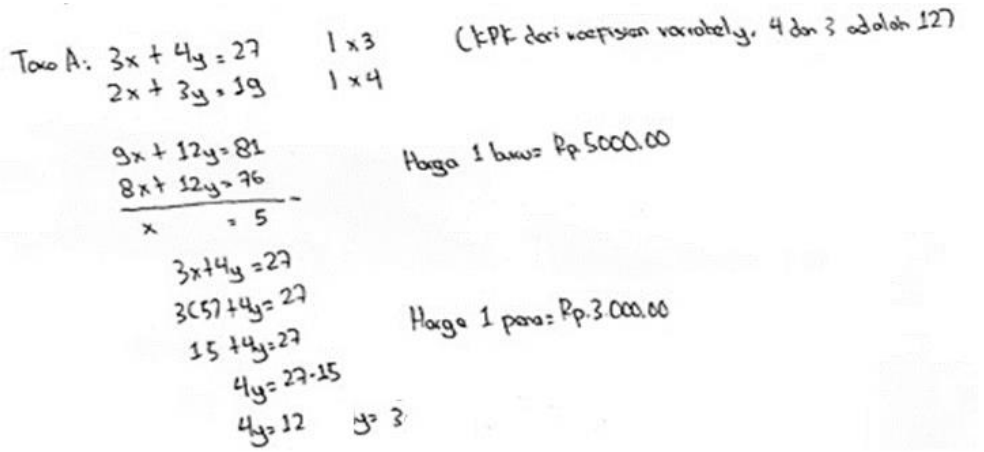


Fig. 1. Solution of SPLDV

At first, MT had difficulty solving the problem given by elimination (MT402). MT folded back to the image having layer, she paid attention to the example illustration given by the researcher (P403). MT tried to repeat and recalculate the example illustrated by the researcher so that MT had sufficient understanding that the two equations were changed so that the coefficient of variable a was the same in both equations (MT403). MT had sufficient

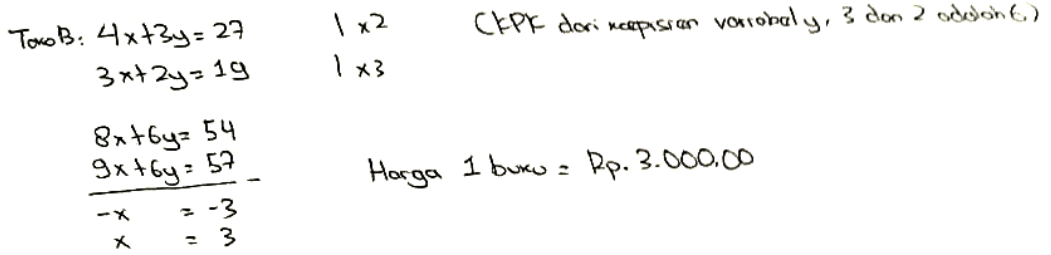
understanding of the equivalence property of two equations, so she returned to the task given (MT405). After she had completed the elimination operation, MT again had difficulty determining the value of the other variable. MT folded back again to the image-having layer by calculating from the example given by the researcher (P409). MT did the calculation $\boldsymbol{2p + 2(2000) = 10000}$ so $\boldsymbol{2p + 4000 = 10000}$ then $\boldsymbol{2p = 10000 - 4000 = 6000}$ so $\boldsymbol{p = 6000 : 2 = 3000}$ (MT411). MT had enough understanding, so she returned to the task given (MT412). MT stated that the problem can be solved by utilizing the equivalence property in the two equations so that the two equations can be solved by elimination and then substitution. In this case, MT can manipulate or combine aspects of the image owned to build relevant properties based on specific contexts. The activities performed by MT are following the theory of mathematical understanding [11].

Formalizing Layer of MT

The interview results with MT on the Formalizing layer are as follows. MT used two different methods to figure out the prices of a book and a pen at Shop B. First, MT used elimination to find the price of the book, where she removed common factors to isolate the unknown. Then, MT apply substitution to find the price of the pen, replacing one variable with another to solve for the unknown. Here are the results of the interview with MT.

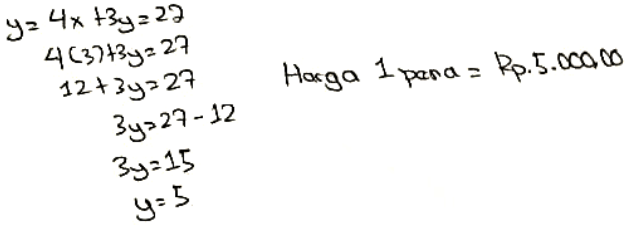
P501 : Earlier, you obtained the price of a book and a pen at Shop A. Now, what about the price of $\boldsymbol{a}$ book at Shop B?

MT501 : By using the same method as before, namely by elimination, the price of $\boldsymbol{a}$ book is IDR 3000.



P 502 : Great, how about the price of $\boldsymbol{a}$ pen at Shop B?

MT502 : By using substitution, the price of $\boldsymbol{a}$ pen is IDR 5000.



MT used the elimination method to obtain a book price and the substitution method to obtain a pen price at store B. This activity shows that MT abstracted an SPLDV concept based on the characteristics of her property noticing understanding [16].

Observing Layer of MT

In this interview, discussed how to find the prices of a book and a pen at two different stores, A and B. MT started by looking up the price of the book at Store A, using a method called elimination to isolate the book's price. Then, she substituted the obtained value into an equation to find the pen's price at Store A. MT repeated the same process to find the prices of the book and pen at Store B. Here are the results of the interview with MT and Figure 2.

P601 : How can you find out how much a pen and a book cost at Stores A and B?

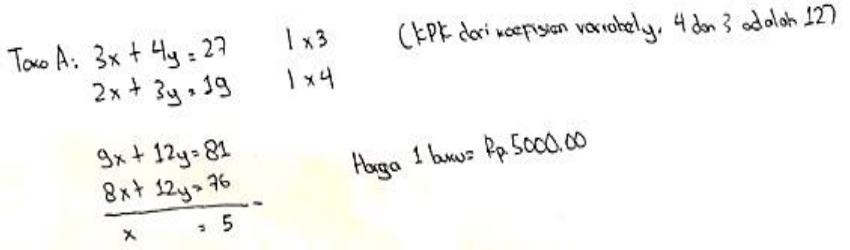
MT601 : I started by checking the cost of a pen and a book at Shop A.

P602 : What do you look for first in store A?

MT602 : I looked up the price of a book first.

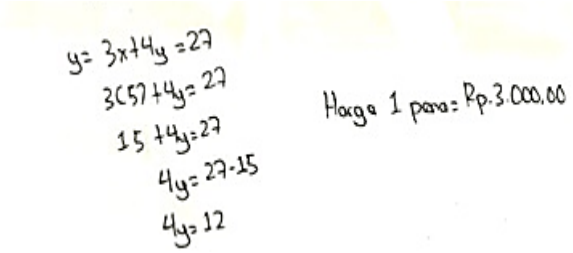
P603 : In what way did you get it?

MT603 : By way of elimination.



P604 : What is your next step?

MT604 : I substituted the value of x obtained into equation (1)



Then, in the same way, I find the price of a book and a pen at store B

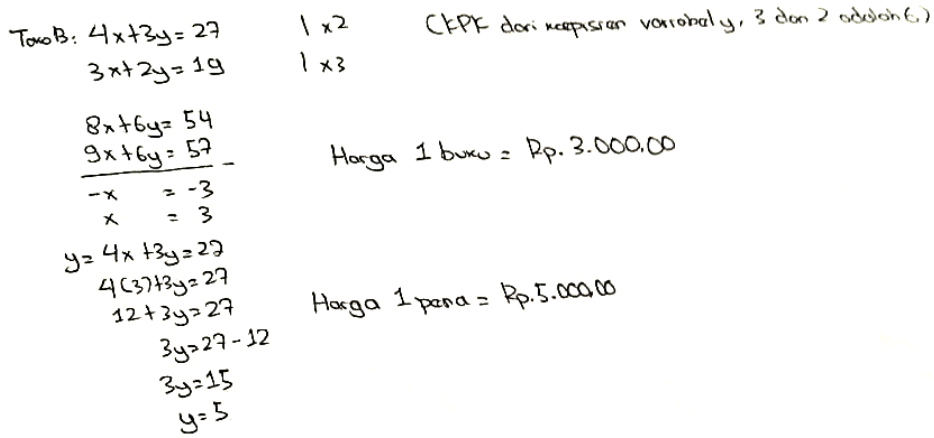


Fig. 2. SPLDV Solution with Mixed Method

At store A, MT found the price of a book (x) using the elimination method in equation 1 and equation 2. After obtaining the price of x, MT substituted the price of x into equation one and obtained the price of a pen (y). In the same way, MT obtained the price of books (x) and pens (y) at store B. In this process, MT thought about the concept of elimination in one of the variables by finding the KPK of the variable's coefficient to be eliminated. After obtaining the value of one of the variables, MT substituted it into one of the equations. In this case, MT coordinated and used formal activities to solve the problem. She reflects and coordinates formal activities as SPLDV theorems. MT is able to identify patterns in algorithms or theorems and provide formal explanations regarding mathematical topics [2]. In this layer, MT did not fold back.

Structuring Layer of MT

Based on interview, Store B is found to offer the cheapest prices for books, while Store A has the lowest prices for pens. When asked about the cost of buying two books and two pens at the cheapest prices, it's calculated that purchasing two books from Store B and two pens from Store A would be the most economical option. This illustrates a straightforward comparison between the prices of different items at each store, leading to a decision based on affordability. The following is an interview with MT and Figure 3.

P701 : Based on your previous answer, which store sells books at the cheapest price?

MT701 : Shop B

P702 : Why is that?

MT702 : Because Store A sells ***a*** book for IDR 5,000, while Store B only sells it for IDR 3,000.

P703 : Now, which store sells pens at the cheapest price?

MT703 : Of course, Shop A

P704 : What is your reason?

MT704 : Yes, because the price of $\boldsymbol{a}$ pen in Shop A is IDR 3,000, while Shop B sells it for IDR 5,000.

P705 : So, how much would it cost to buy two books and two pens at the cheapest price?

MT705 : To choose the cheapest price from shops A and B, I compared the prices first and then took the cheapest price of the goods in those shops. I bought two books at shop B, $\boldsymbol{2xIDR\ 3,000 = IDR\ 6,000}$, and two pens at shop A, $\boldsymbol{2xIDR\ 3,000 = IDR\ 6,000}$. The total money I have to spend to buy two books and two pens is $\boldsymbol{IDR\ 6,000 + IDR\ 6,000 = IDR\ 12,000}$.

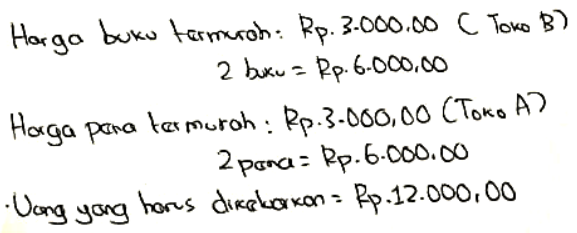


Fig. 3. The price paid for purchasing two books and two pens

MT can compare the prices of two stores that sell the same item. MT can determine which store has the cheaper item so that she can decide to buy it. MT bought two books at shop B, which is $\boldsymbol{2xIDR\ 3,000 = IDR\ 6,000}$, and bought two pens at shop A, which is $\boldsymbol{2xIDR\ 3,000 = IDR\ 6,000}$. The total money she had to spend to buy two books and two pens was $\boldsymbol{IDR\ 6,000 + IDR\ 6,000 = IDR\ 12,000}$. This activity shows that MT makes logical formal observations and verifies ideas before developing them. MT can connect theorems in SPLDV material with comparison material to solve math problems based on logical arguments [18]. In this layer, students do not fold back.

Based on the activities above, the activity of MT's folding back into the mathematical understanding layer is shown in figure 4.

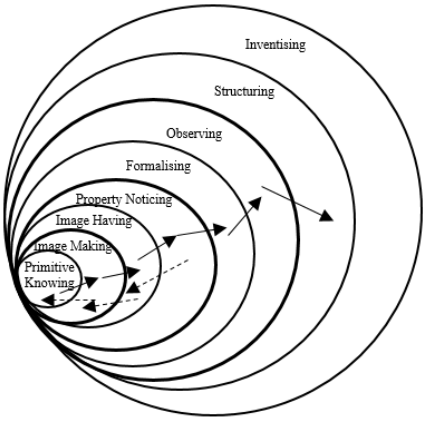


Fig. 4. Folding back in Mathematical Understanding of MT

Figure 4 shows MT's folding back activity on the mathematical understanding layer. This

activity demonstrates how students, faced with challenges they are unable to solve right away, go back to more profound layers of understanding. Three folding back steps are shown in this study. To reinforce prior knowledge about PLDV, MT must revert to basic knowledge in the initial stage of the image-making layer. In the second phase in the image-having layer, MT returns to image-making to strengthen its understanding of the congruency property in SPLDV, and then in the third phase of the property noticing layer, MT folds back to image-having to construct an understanding of the elimination method that utilizes the congruency property of two PLDV. The result of folding back is that student can develop inadequate and incomplete understanding while at a certain layer and reorganize their concept understanding. Students can also reach a certain layer by generating and creating a new picture on the previous layer, if the existing construction is insufficient to solve the problem. In this case, student already have thinking skills that include critical, logical, reflective, metacognitive, and creative thinking [19].

Data analysis in this study is on the seven layers of Pirie-Kieren's mathematical understanding growth. An explanation of the characteristics of MT students in the growth of mathematical understanding can be seen in Table 1.

Table 1 Growth of students' mathematical understanding MT

Layer of growth mathematical understanding	Description of mathematical understanding growth characteristics
Primitive Knowing	❖ MT delineates data derived from mathematical situation. ❖ State concepts connected to mathematical situations.
Image-Making	❖ Describe a concept based on prior knowledge to develop a specific image.
Image-Having	❖ Folding back to primitive knowing layer. ❖ Have an overview of congruence properties used in SPLDV
Property Noticing	❖ Folding back to the image-making layer. ❖ Manipulate aspects of existing images to build relevant context-specific properties.
Formalising	❖ Folding back to the image-having layer. ❖ MT abstracts an SPLDV concept based on the characteristics of the property noticing understanding that they have.
Observing	❖ MT can coordinate and carry out official activities to address problems. ❖ MT reflects and coordinates formal activity as SPLDV theorem.
Structuring	❖ MT can relate to making logical connections between concepts.

Table 1 shows the growth layer of MT student's mathematical understanding. Mathematical understanding is a dynamic growth in thinking according to the initial stage of learning [3], [20]. In this study, the Pirie-Kieren theory was utilized to determine the activities associated with the student's activities. This idea offers a framework for looking into how students think. The task that has been given require subject to critically evaluate information, make inferences, and make generalizations [21]. If student understands a mathematical concept, she may apply that concept to solve a variety of mathematical problems, as well as problems outside of

mathematics [7]. The findings significantly contribute to mathematical problem-solving by expanding on folding back and putting forth a more comprehensive framework for source-based classification, form, and outcome, and illustrating its impact on student's mathematical understanding.

4. Conclusion

This research explores the growth characteristics of a student's mathematical understanding. The student can imagine concepts through mental and physical actions by utilizing prior information. Based on the research results, it is found that a student can complete task up to the seventh layer in Pirie-Kieren's theory, namely at the structuring layer even though in some layers, a student's experience folding back to develop and explore prior knowledge in order to construct a mathematical understanding that will be achieved at the layer of understanding she is going through. When the MT subject is in the image-making layer, MT folds back to the primitive knowing layer to strengthen the understanding of PLDV and connect it to SPLDV. Furthermore, when MT is in the image-having layer, MT cannot create a different mathematical model, but after MT folding back to the image-making layer, MT can create a different mathematical model by multiplying the variable coefficients, and the results following the SPLDV property which states that two PLDV can be congruent. Furthermore, when MT is in the property noticing layer, MT folds back to the image-having layer; MT states that the problem can be solved by utilizing the congruence property in the two equations to solve the two equations by elimination and then substitution.

Furthermore, the results show that student achieve mathematical understanding by providing explanations for data derived from mathematical problems, outlining ideas, creating reports on specific subjects, pointing out discrepancies and similarities between topic definitions, creating concepts of linking mathematical concepts to problems and mathematical abstraction. However, this student's learning outcomes has not reached the inventing layer. Consequently, more research must be conducted utilizing a qualitative study covering a range of themes and grade levels. The way that the current study characterizes students' thinking understanding is innovative. The findings of this study have some implications for how students' understanding will continue to improve at higher layer of understanding. From the outcomes and implications, practical factors are deduced when creating activities to solve mathematical SPLDV difficulties. However, this study is limited to observational data, leading to a small-scale investigation of a subject with high mathematical ability in a public school.

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