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# Generalised Γ- Semiideals in Ternary Γ-Semirings

**K. Maha lakshmi[1,2], P. Siva Prasad[3], D. Madhusudana Rao[4]**

[1]*Research Scholar, Department of Mathematics, VFSTR Deemed to be University, Vadlamudi, India, mhlakahmi@gmail.com.*
[2]*Assistant Professor, Department of Mathematics, Vignan Nirulla Engineering College, India, mhlakahmi@gmail.com.*
[3]*Associate Professor, Department of Computer Science & Engineering, School of Computing & Informatics, VFSTR Deemed to University, India, pusapatisivaprasad@gmail.com*
[4]*Professor of Mathematics, Government College For Women(A), Samba Siva Peta Rd, Opp: AC College, India, dmrmaths@gmail.com*

Here in this research article we introduce the concept of generalized semi-ideal in a ternary Γ-semirings.Here we gave several examples to maintain the relationship between the quasi-ideals,bi-ideals, ideals and generalized semi-ideals were introduced. Here a criterion of a commutative ternary Γ-semiring without any zero divisors to a ternary Γ-divison semiring was given.

**Keywords:** Ternary Γ − semiring , Γ−ideals, Quasi Γ−ideals, Generalized Γ-semiideal,

## 1. Introduction

In the year 1971 Lister investigated on ternary rings and their structures. Especially Lister [4] generalized those additive subgroups of rings these are closed under the triple product. Later T.K.Dutta and S.Kar [3] were introduced in the year 2003 the notion of ternary semi rings as a generalization of a ternaryring. In 2005 they were also introduced the notions of left,lateral and right ideals of a ternary semirings and also characterized the regular ternary semirings. After that S.Kar introduced the notions of bi-ideals and quasi-ideals in a ternary semirings.In the same year T.K.Dutta generalized the notion of semi-ideals in a ring has been introduced. Later in the year of 2015 D.Madhusudana rao and sajani lavanya introduced the notion of ternary Γ −semirings. The earlier works of D. Madhusudhana Rao and M. Sajani Lavanya on Ternary Γ-Semirings may be found in [5, 6, 7, 8]. In 2007, T. K. Dutta and M. L. Das [2] introduced and studied right strongly prime Semirings.

In this research article we developed the notion of generalized Γ-semi ideals in a ternary Γ-semiring and study the properties of them. We also introduce an affinity between the Γ-ideals ,bi- Γ-ideals, Γ- semi ideals etc in a ternary Γ-semirings. In that similar way we also study some of the characteristics of a generalized Γ- semi ideals in a ternary Γ-semirings.

## 2. Preliminaries: For the preliminaries we will refer [1],[2] & [3]

Definition 2.1:Let $S_G$ be a commutative semigroup along with a ternary multiplication denoted by [ ] is known as ternary $\Gamma -$ semiring $TS_R$ if

$$\text{i.}\ \left[[p\alpha q\beta r]\gamma s\delta t\right] = [p\alpha[q\beta r\gamma s]\delta t] = [p\alpha q\beta[r\gamma s\delta t]$$

$$\text{ii.}\ [(p+q)\alpha r\beta s] = [p\alpha r\beta s] + [qp\alpha r\beta s]$$

$$\text{iii.}\ [p\alpha(q+r)\beta s] = [p\alpha q\beta s] + [p\alpha r\beta s]$$

$$\text{iv.}\ [p\alpha q\beta(r+s)] = [p\alpha q\beta r] + [p\alpha q\beta s],$$

$$\forall p, q, r, s \in TS_R \text{ and } \alpha, \beta, \gamma, \delta \in \Gamma$$

Throughout of this research article $TS_R$ will denote a ternary $\Gamma -$ semiring unless otherwise stated.

Definition 2.2: Let $TS_R$ be a ternary $\Gamma -$ semiring if there exist an element $0 \in TS_R$ such that $0 + p = p$ and $[0\alpha p\beta q] = [p\alpha q\beta 0] = [p\alpha 0\beta q] = 0, \forall p, q \in TS_R$.then 0 is said to be zero element of $TS_G$.Here in this case we said that $TS_R$. is a ternary $\Gamma -$ semiring with zero element.

Definition 2.3: Let $TS_R$. be a ternary $\Gamma -$ semiring is said to be a commutative ternary $\Gamma -$ semiring if

$$[p\Gamma q\Gamma r] = [q\Gamma r\Gamma a] = [r\Gamma p\Gamma q] = [q\Gamma p\Gamma r] = [r\Gamma q\Gamma p] = [p\Gamma r\Gamma q], \forall p, q, r \in TS_R.$$

Definition 2.4:An additive subsemigroup T of $TS_R$. is called a ternary $\Gamma -$ subsemiring $TSS_R$ of $TS_R$. if $[t_1\alpha t_2\beta t_3] \in T, \forall\ t_1, t_2, t_3 \in T$ and $\alpha, \beta \in \Gamma$.

Definition 2.5: An element a in $TS_R$ is said to be a regular element if there exist an element x in $TS_G$such that $[a\alpha x\beta a] = a$. $TS_R$ is called regular if all of its elements are to be regular.

Definition 2.6:A ternary $\Gamma -$ semiring $TS_R$. is said to be zero divisor free(ZDF) if for $x, y, z \in TS_R, [x\alpha y\beta z] = 0$ which implies that either $x = 0$ or y= 0 or z= 0.

Definition 2.7: A ternary $\Gamma -$ semiring $TS_R$with $|TS_R| \geq 2$is said to be a ternary division $\Gamma -$ semiring if for each non zero element x in $TS_G$,there exist s non zero element y in $TS_G$ such that $[x\alpha y\beta a] = [y\alpha x\beta a] = [a\alpha x\beta y] = [a\alpha y\beta x] = a, \forall a \in TS_R$.

Definition 2.8: A ternary $\Gamma -$ semiring $TS_R$,A left(right/lateral) ideal I of $TS_R$ is an additive subsemigroup of $TS_R$ such that $[s_1\alpha s_2\beta i] \in I([i\alpha s_1\beta s_2] \in I/([s_1\alpha i\beta s_2] \in I)$ for all $i \in I$ and $\forall s_1, s_2 \in TS_R$ if I is a left,a right and a lateral $\Gamma -$ideal of $TS_R$,then I is said to be an $\Gamma -$ideal of $TS_R$.

Definition 2.9: A ternary $\Gamma -$ semiring $TS_R$ and Q be a additive subsemigroup of $TS_R$is said to be a quasi Γ-ideal of $TS_G$if $[Q\Gamma TS_R\Gamma TS_R] \cap [TS_R\Gamma Q\,\Gamma\, TS_R] + [TS_R\Gamma\ TS_R\Gamma\ Q\ \Gamma\ TS_R\Gamma\ TS_R]\ \cap [TS_R\Gamma\ TS_R\Gamma\ Q] \subseteq Q$.

Definition 2.10: A ternary $\Gamma -$ semiring $TS_R$, B is a ternary gamma subsemiring of $TS_G$ is said to be bi- $\Gamma -$ideal of $TS_R$ if $[B\ \Gamma\ TS_R \Gamma\ B\ \Gamma\ TS_R \Gamma\ B] \subseteq B$.

## 3. Generalised Γ -Semi ideals in ternary $\Gamma -$ semirings

Basically T.K.Dutta was introduced the concept of generalized semi-ideals in semirings. In the year 2011, V.R.Daddvi & Y.S.Pawar defined the same concept in ternary semiring. Now as a generalization we define generalized semi gamma ideals in ternary $\Gamma -$ semirings.

Definition 3.1: Let $TS_R$ be a ternary $\Gamma -$ semiring and a non empty subset A of $TS_R$satisfying the condition $x + y \in A, \forall x, y \in A$ is said to be

i) Generalized left Γ- semi ideal of $TS_R$ if $[[a\alpha a\beta a]a\gamma x] \in A, \forall x \in A, a \in TS_R, \alpha, \beta, \gamma \in \Gamma$

ii) Generalized right Γ- semi ideal of $TS_R$ if$[[x\alpha a\beta a]a\gamma a] \in A, \forall x \in A, a \in TS_R, \alpha, \beta, \gamma \in \Gamma$

iii) Generalized lateral Γ- semi ideal of $TS_R$ if $[[a\alpha a\beta x]a\gamma a] \in A, \forall x \in A, a \in TS_R, \alpha, \beta, \gamma \in \Gamma$

iv) Generalized $\Gamma -$ semi ideal of $TS_R$ if it is a generalized left Γ-semi ideal, a generalized right Γ-semi ideal and a generalized lateral Γ-semi ideal of $TS_R$.

Example 3.2: Let T= $M_2(Z_0^-)$ and $\Gamma = M_2\left(z^+\right)$ then T is $TS_R$ of the set of all 2X2 square matrices over $Z_0^-$. Let $G = \{\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} / a \in Z_0^-\}$ and $\Gamma = \{\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} / b \in Z^+\}$ then G is a $TSS_R$ of T, but not a generalized Γ-semi ideal of T.

Remark 3.3: The concepts of generalized Γ-semi ideal of a ternary $\Gamma -$ subsemiring are independent in $TS_R$. This means that is every ternary $\Gamma -$ subsemiring of $TS_R$ need not be a generalized Γ – semi ideal of $TS_R$ and every generalized Γ– semi ideal of $TS_R$ need not be a ternary subsemiring of $TS_R$. For this consider the following examples.

Example 3.4: Let $TS_R = M_2(Z_0^-)$ be a ternary $\Gamma -$ semiring of the set of all $2 \times 2$ square matrices over $Z_0^-$, the set of all non-positive integers.

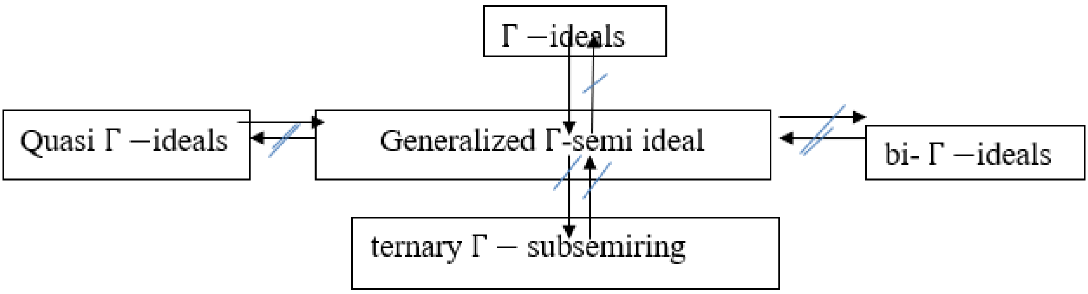
And let $T = \{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} / x \in Z_0^-\}$. Tis a ternary $\Gamma -$ subsemiring of $TS_R$. but it is not a generalized Γ-semi ideal of $TS_R$.

Example 3.5: Let $TS_R = \{\dots\dots - 2i, -i, 0, i, 2i \dots.\}$ and $TS_R = \Gamma$, then $TS_R$ be a ternary $\Gamma -$ semiring with respect to addition and complex triple multiplication. Let $A = \{\dots\dots - 8i, -4i, 0, 4i, 8i \dots\}$, A is a generalized Γ-semi ideal of $TS_R$,but not a ternary subsemiring of $TS_R$.

Remark 3.6:Every $\Gamma -$ideal of $TS_R$ is a generalized Γ-semi ideal of $TS_R$.but the converse need not be true.

Example 3.7:Every quasi $\Gamma-$ideal need not be a generalized Γ-semi ideal of $TS_R$ and every generalized Γ-semi ideal need not be a quasi $\Gamma-$ideal $TS_R$.(in example 3.4) T is a quasi $\Gamma-$ideal of $TS_R$, but it is not a generalized Γ-semi ideal of $TS_R$.In Example 3.5 A is a generalized Γ-semi ideal of $TS_R$ but it is not a quasi $\Gamma-$ideal of $TS_R$.

Remark 3.8:Every quasi $\Gamma-$ideal is a bi-$\Gamma-$ideal in $TS_R$[6]. Hence the bi-$\Gamma-$ideal and generalized Γ-semi ideal in $TS_R$ are independent concepts.The flow-chart of the relation between $\Gamma-$ideals, bi-$\Gamma-$ideals, quasi $\Gamma-$ideals, ternary $\Gamma-$ subsemiring and generalized Γ-semi ideal in a ternary $\Gamma-$ semiring is given below




## 4. Properties of Generalized Γ-semi ideal

Let $\{G_\alpha\}_{\alpha\in\Delta}$ be the arbitrary collection of generalized $\Gamma-$semi ideals of a ternary $\Gamma-$ semiring $TS_R$ then the arbitrary intersection of these generalized $\Gamma-$semi ideals is again generalized $\Gamma-$semi ideal of $TS_R$. But the union of two generalized $\Gamma-$semi ideals of $TS_R$ may not be a generalized $\Gamma-$semi ideal of $TS_R$. This we establish in following example.

Let $TS_R = \{\dots\dots\dots\dots -2i, -i, 0, i, 2i \dots\dots\}$ be a ternary $\Gamma-$ semiring with respect to the operations addition and complex triple multiplication. Then $P = \{\dots\dots\dots\dots -16i, -8i, 0, 8i, 16i, \dots\dots\}$ and $Q = \{\dots\dots - 15i, -5i, 0, 5i, 15i, \dots\dots\}$ are two generalized $\Gamma-$semi ideal of $TS_R$ but the union $P \cup Q$ is not a generalized $\Gamma-$semi ideal of $TS_R$.

Theorem 4.1:Let $TS_R$ be a ternary $\Gamma-$semiring and let A be a generalized $\Gamma-$semi ideal of $TS_R$ and let T be a ternary $\Gamma-$ subsemiring of $TS_R$.If $A \cap T \neq \emptyset$,then $A \cap T$ is a generalized $\Gamma-$semi ideal of $TS_R$.

Proof: Assume that $TS_R$ be a ternary $\Gamma-$semiring and let $x, y \in A \cap T$.then $x + y \in A \cap T$.For any $a \in T$ and $x \in A \cap T$ we have $[[a\Gamma a\Gamma a]\Gamma a\Gamma x] \in A \cap T, [[x\Gamma a\Gamma a]\Gamma a\Gamma a] \in A \cap T, [[a\Gamma a\Gamma x]\Gamma a\Gamma a] \in A \cap T$.Hence $A \cap T$ is a generalized $\Gamma-$semi ideal of $TS_R$.

Theorem 4.2: Let $TS_R$ be a ternary $\Gamma-$semiring and if Pand Q are any two generalized $\Gamma-$semi ideals of $TS_R$then $P + Q = \{x + y / x \in P, y \in Q\}$ is a generalized $\Gamma-$semi ideal of $TS_R$.

Proof: Assume that $TS_R$ be a ternary $\Gamma-$semiring also let Pand Q are any two generalized $\Gamma-$semi ideals of $TS_R$,then define $P + Q = \{x + y, x \in P, y \in Q\}$. Let $p, q \in P + Q$ ,hence $p = a + b, q = c + d$ for any $a, c \in P$ and $b, d \in Q$. Then $p + q = (a + b) + (c + d) = (a + c) + (b + d) \in P + Q$.and let $s \in TS_R$ and $p \in P + Q$ ,hence $p = a + b$

for some a ∈ P and b ∈ Q. Now $S \in TS_R$,a∈ P and P is a generalized Γ −semi ideal

$$\Rightarrow [[s\Gamma s\Gamma s]\ \Gamma s\Gamma a] \in P,\ [s\ \Gamma[[s\Gamma s\Gamma s]\Gamma s] \in P, [a\ \Gamma s\Gamma[s\Gamma s\Gamma s] \in P$$

therefore $\left[[s\Gamma s\Gamma s]\Gamma s\Gamma p\right] = \left[[s\Gamma s\Gamma s]\Gamma s\Gamma(a+b)\right] = \left[[s\Gamma s\Gamma s]\Gamma s\Gamma a\right] + \left[[s\Gamma s\Gamma s]\Gamma s\Gamma b\right] \in P + Q$.

In this similar way we will prove that $[[s\Gamma s\Gamma p]\ \Gamma s\Gamma s] = \left[[s\Gamma s\Gamma(a+b)]\Gamma s\Gamma s\right] = [s\Gamma s\Gamma a]\Gamma s\Gamma s] + [s\Gamma s\Gamma b]\Gamma s\Gamma s] \in P + Q$. and $[\ [p\Gamma s\Gamma s]\ \Gamma s\Gamma s] = [\ [(a+b)\Gamma s\Gamma s]\ \Gamma s\Gamma s] = [[a\Gamma s\Gamma s]\ \Gamma s\Gamma s] + [[b\Gamma s\Gamma s]\ \Gamma s\Gamma s] \in P + Q$

Therefore $P+Q$ is a generalized $\Gamma$ −semi ideal of $TS_R$.

Theorem 4.3:Let $TS_R$ be a ternary $\Gamma$ −semiring with zero. And let $P$and $Q$ are any two generalized $\Gamma$ −semi ideals of $TS_R$ containing zero.Then $P+Q$ is the smallest generalized $\Gamma$ −semi ideal of $TS_R$ containing both $P$and $Q$.

Proof: By the above Theorem 4.2 $P+Q$ is a generalized $\Gamma$ −semi ideal of $TS_R$. Since $\mathbf{0} \in P, \mathbf{0} \in Q$ we will get $\mathbf{0} \in P+Q$ and for any $p \in P, p = p + \mathbf{0} \in P + Q$. Hence $P \subseteq P + Q$,similarly $Q \subseteq P+Q$. Assume that I be any other generalized $\Gamma$ −semi ideal containing both $P$and $Q$.Let $p \in P+Q$ then $p = a + b$,for some $a \in P$ and $b \in Q$.Hence $p = a + b \in I$,therfore $P + Q \subseteq I$,thus $P+Q$ is the smallest generalized $\Gamma$ −semi ideal containing both $P$and $Q$.

Note: If $P, Q$ *and* $R$ are the three subsets of $TS_R$then by [P $\Gamma Q \Gamma R$] it means the set of all finite sums of the form $\sum[p_i \Gamma q_i \Gamma r_i]$,where $p_i \in P, q_i \in Q, r_i \in R$.

Theorem 4.4: Let $TS_R$ be a ternary $\Gamma$ −semiring and $P$ be a generalized left $\Gamma$ −semi ideal of $TS_R$.Then $[P\Gamma Q\Gamma R]$ is a generalized left $\Gamma$ −semi ideal for any non empty subsets $Q$ and $R$ of $TS_R$.

Proof: Let $TS_R$ be a ternary $\Gamma$ −semiring.And for $x, y \in [P\Gamma Q\Gamma R]$ ,let $x = \sum_{i=1}^{n}[p_i\Gamma q_i\Gamma r_i]$ and $y = \sum_{j=1}^{m}\left[p_j\Gamma q_j\Gamma r_j\right]$. Unanimously $x+y$ is a finite sum of the form $\sum[p_i\Gamma q_i\Gamma r_i]$.So that $x + y \in [P\Gamma Q\Gamma R]$.For any $s \in TS_R$ then $[[s\Gamma s\Gamma s]\ \Gamma s\Gamma x] = [[s\Gamma s\Gamma s]\ \Gamma s\Gamma \sum_{i=1}^{n}[p_i\Gamma q_i\Gamma r_i]]$

$$= \sum_{i=1}^{n}\left[[s\Gamma s\Gamma s]\Gamma s\Gamma[p_i\Gamma q_i\Gamma r_i]\right]$$

$$= \sum_{i=1}^{n}\left[\left[[s\Gamma s\Gamma s]\Gamma s\Gamma p_j\right]\Gamma q_i\Gamma r_i\right] \in [P\Gamma Q\Gamma R]$$

Since $P$ is generalized left $\Gamma$ −semi ideal of $TS_R$. $[P\Gamma Q\Gamma R]$ is a generalized left $\Gamma$ −semi ideal of $TS_R$.

Corollary 4.5: Let $TS_R$ be a ternary $\Gamma$ −semiring and $P$ be a generalized right $\Gamma$ −semi ideal of $TS_R$.Then $[P\Gamma Q\Gamma R]$ is a generalized right $\Gamma$ −semi ideal for any non empty subsets $Q$ and $R$ of $TS_R$.

Proof: Similar to the proof of theorem 4.4

Corollary 4.6: Let $TS_R$ be a ternary $\Gamma$ −semiring and $P$ be a generalized lateral $\Gamma$ −semi ideal of $TS_R$.Then $[P\Gamma Q\Gamma R]$ is a generalized lateral $\Gamma$ −semi ideal for any non empty subsets $Q$ and $R$ of $TS_R$.

Proof: Proof: Similar to the proof of theorem 4.4

Corollary 4.7: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring and $\boldsymbol{P}$ be a generalized $\boldsymbol{\Gamma}$ −semi ideal, then $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized $\boldsymbol{\Gamma}$ −semi ideal.

Proof: By the theorem 4.4 $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized left $\boldsymbol{\Gamma}$ −semi ideal ,By theorem 4.5 $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized right $\boldsymbol{\Gamma}$ −semi ideal, By theorem 4.6 $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized lateral $\boldsymbol{\Gamma}$ −semi ideal.Hence $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized $\boldsymbol{\Gamma}$ −semi ideal.

Definition 4.8: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring and P is a subset of $\boldsymbol{TS_R}$ then P is said to be duo generalized $\boldsymbol{\Gamma}$ −semi ideal if P is a left generalized $\boldsymbol{\Gamma}$ −semi ideal then P is also a right generalized $\boldsymbol{\Gamma}$ −semi ideal.

Theorem 4.9: : Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring and $\boldsymbol{P}$ be a generalized $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.Then $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a duo generalized $\boldsymbol{\Gamma}$ −semi ideal for any non empty subsets $\boldsymbol{Q}$ and $\boldsymbol{R}$ of $\boldsymbol{TS_R}$.

Proof: By the theorem 4.4 $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized left $\boldsymbol{\Gamma}$ −semi ideal ,By theorem 4.5 $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized right $\boldsymbol{\Gamma}$ −semi ideal.Therefore $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a duo generalized $\boldsymbol{\Gamma}$ −semi ideal

Theorem 4.10: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring.And let $\boldsymbol{P}$ be a generalized left(right) $\boldsymbol{\Gamma}$ −semi ideal and $\boldsymbol{Q}$ be a bi $\boldsymbol{\Gamma}-$ ideal of $\boldsymbol{TS_R}$.Then $[\boldsymbol{P\Gamma Q\Gamma Q}]$($[\boldsymbol{Q\Gamma Q\Gamma P}]$) is a generalized left(right) $\boldsymbol{\Gamma}$ −semi ideal as well as bi $\boldsymbol{\Gamma}-$ ideal of $\boldsymbol{TS_R}$.

Proof: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring. $\boldsymbol{x,y,z} \in [\boldsymbol{P\Gamma Q\Gamma Q}]$,and let

$$\boldsymbol{x} = \sum_{i=1}^{n}[\boldsymbol{p_i\Gamma q_i\Gamma r_i}],$$

$$\boldsymbol{y} = \sum_{i=n+1}^{m}[\boldsymbol{p_i\Gamma q_i\Gamma r_i}],$$

$$\boldsymbol{z} = \sum_{i=m+1}^{p}[\boldsymbol{p_j\Gamma q_j\Gamma r_j}] \text{ for all } \boldsymbol{p_i} \in \boldsymbol{P}, \boldsymbol{q_i}, \boldsymbol{r_i} \in \boldsymbol{Q}.$$

Thus $\boldsymbol{x+y}$ is a finite sum of the form $\sum[\boldsymbol{p_i\Gamma q_i\Gamma r_i}]$.

Hence $\boldsymbol{x+y} \in [\boldsymbol{P\Gamma Q\Gamma Q}]$. And let $\boldsymbol{s} \in \boldsymbol{TS_R}, \boldsymbol{x} = \sum_{i=1}^{n}[\boldsymbol{p_i\Gamma q_i\Gamma r_i}] \in [\boldsymbol{P\Gamma Q\Gamma Q}]$, then $[[\mathrm{s}\boldsymbol{\Gamma s\Gamma s}]\ \boldsymbol{\Gamma s\Gamma x}] = \left[[\boldsymbol{s\Gamma s\Gamma s}]\boldsymbol{\Gamma s\Gamma} \sum_{i=1}^{n}[\boldsymbol{p_i\Gamma q_i\Gamma r_i}]\right]$ $=\sum_{i=1}^{n}\left[\left[[\boldsymbol{s\Gamma s\Gamma s}]\boldsymbol{\Gamma s\Gamma p_j}\right]\boldsymbol{\Gamma q_i\Gamma r_i}\right] \in [\boldsymbol{P\Gamma Q\Gamma R}]$.Hence $[\boldsymbol{P\Gamma Q\Gamma R}]$ is a generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.Now $[\boldsymbol{P\Gamma Q\Gamma Q}][\boldsymbol{P\Gamma Q\Gamma Q}][\boldsymbol{P\Gamma Q\Gamma Q}] = [\boldsymbol{P\Gamma}[[\boldsymbol{Q\Gamma}[\boldsymbol{Q\Gamma P\Gamma Q}]\boldsymbol{\Gamma Q}]\boldsymbol{\Gamma P\Gamma Q}]\boldsymbol{\Gamma Q}]$

$$\subseteq [\boldsymbol{P\Gamma}[\boldsymbol{Q\Gamma TS_R\Gamma Q\Gamma TS_R\Gamma Q}]\boldsymbol{\Gamma Q}]$$

$\subseteq [\boldsymbol{P\Gamma Q\Gamma Q}]$. (Since $[\mathrm{Q}\,\boldsymbol{\Gamma P\Gamma Q}] \subseteq \boldsymbol{TS_R}$and $\boldsymbol{Q}$ is a bi $\boldsymbol{\Gamma}-$ ideal). This shows that $[\boldsymbol{P\Gamma Q\Gamma Q}]$ is ternary subsemiring of $\boldsymbol{TS_R}$. Again

$$[\boldsymbol{P\Gamma Q\Gamma Q}]\boldsymbol{\Gamma TS_G\Gamma}[\boldsymbol{P\Gamma Q\Gamma Q}]\boldsymbol{\Gamma TS_G\Gamma}[\boldsymbol{P\Gamma Q\Gamma Q}]$$
$$= [\boldsymbol{P\Gamma}[\boldsymbol{Q\Gamma}[\boldsymbol{Q\Gamma TS_R\Gamma P}]\boldsymbol{\Gamma Q\Gamma}[\boldsymbol{Q\Gamma TS_R\Gamma P}]\boldsymbol{\Gamma Q}]\boldsymbol{\Gamma Q}]$$
$$\subseteq [\boldsymbol{P\Gamma}[\boldsymbol{Q\Gamma TS_R\Gamma Q\Gamma TS_R\Gamma Q}]\boldsymbol{\Gamma Q}]$$
$$\subseteq [\boldsymbol{P\Gamma Q\Gamma Q}]$$

Since $\boldsymbol{Q}$ is a bi $\boldsymbol{\Gamma}-$ ideal.Hence $[\boldsymbol{P\Gamma Q\Gamma Q}]$is a bi $\boldsymbol{\Gamma}-$ ideal of $\boldsymbol{TS_R}$.

Theorem 4.11: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring.And let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two ternary $\boldsymbol{\Gamma}$ −subsemirings of $\boldsymbol{TS_R}$such that $\boldsymbol{A^3 = A}$ and $\boldsymbol{A}$ is a left $\boldsymbol{\Gamma}$ −ideal of $\boldsymbol{B}$.and $\boldsymbol{B}$ is a generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.Then $\boldsymbol{A}$ is generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.
Proof: Let $\boldsymbol{TS_R}$be a ternary $\boldsymbol{\Gamma}$ −semiring and let $\boldsymbol{p \in A}$, therefore $\boldsymbol{p = [p_1 \Gamma p_2 \Gamma p_3]}$ where $\boldsymbol{p_1, p_2, p_3 \in A}$, Now for any $\boldsymbol{s \in TS_R}$ then

$$\boldsymbol{\big[[s\Gamma s\Gamma s]\Gamma s\Gamma p\big] = \big[[s\Gamma s\Gamma s]\Gamma s\Gamma[p_1\Gamma p_2\Gamma p_3]\big]}$$

$$\boldsymbol{= [[[s\Gamma s\Gamma s]\Gamma s\Gamma p_1]\Gamma p_2\Gamma p_3 \in [B\Gamma p_2\Gamma p_3] \subseteq A}$$

(Since $\boldsymbol{A}$ is a left $\boldsymbol{\Gamma}$ −ideal of $\boldsymbol{B}, \boldsymbol{p_1 \in A \subset B}, \boldsymbol{B}$is generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$) so that $\boldsymbol{A}$ is generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.

Theorem 4.12: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring.If H is a is a generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$ and $\boldsymbol{P_1, P_2}$ be any two ternary $\boldsymbol{\Gamma}$ − subsemirings of $\boldsymbol{TS_R}$ then$\boldsymbol{[H\,\Gamma P_1 \Gamma P_2]}$ is a generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.

Proof: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring. For any $\boldsymbol{x, y \in [H\Gamma P_1\Gamma P_2]}$, $\boldsymbol{x = \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$ , $\boldsymbol{y = \sum_{i=n+1}^{m}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$ for any $\boldsymbol{h_i \in H, p_1 \in P_1, p_2 \in P_2}$ Therefore $\boldsymbol{x + y}$ is a finite sum of the form $\sum\boldsymbol{\big[h_i\Gamma p_{1_i}\Gamma p_{2_I}\big]}$, will imply $\boldsymbol{x + y \in [H\Gamma P_1\Gamma P_2]}$, and let $\boldsymbol{x = \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}] \in [H\Gamma P_1\Gamma P_2]}$ and let $\boldsymbol{s \in TS_R}$ then $\boldsymbol{[s\Gamma s\Gamma s]\Gamma s\Gamma x] = [[s\Gamma s\Gamma s]\Gamma s\Gamma \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$

$\boldsymbol{= \sum_{i=1}^{n}[[\,[s\Gamma s\Gamma s]\Gamma s h_i]\Gamma p_{1_i}\Gamma p_{2_i}] \in [H\Gamma P_1\Gamma P_2]}$

Hence $\boldsymbol{[H\Gamma P_1\Gamma P_2]}$ is generalized left $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.

Theorem 4.13: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring.If H is a is a generalized right $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$ and $\boldsymbol{P_1, P_2}$ be any two ternary $\boldsymbol{\Gamma}$ − subsemirings of $\boldsymbol{TS_R}$ then$\boldsymbol{[H\,\Gamma P_1\Gamma P_2]}$ is a generalized right $\boldsymbol{\Gamma}$ −semi ideal of $\boldsymbol{TS_R}$.

Proof: Let $\boldsymbol{TS_R}$ be a ternary $\boldsymbol{\Gamma}$ −semiring. For any $\boldsymbol{x, y \in [H\Gamma P_1\Gamma P_2]}$, $\boldsymbol{x = \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$ , $\boldsymbol{y = \sum_{i=n+1}^{m}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$ for any $\boldsymbol{h_i \in H, p_1 \in P_1, p_2 \in P_2}$ Therefore $\boldsymbol{x + y}$ is a finite sum of the form $\sum\boldsymbol{\big[h_i\Gamma p_{1_i}\Gamma p_{2_I}\big]}$, will imply $\boldsymbol{x + y \in [H\Gamma P_1\Gamma P_2]}$, and let $\boldsymbol{x = \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}] \in [H\Gamma P_1\Gamma P_2]}$ and let $\boldsymbol{s \in TS_R}$ then $\boldsymbol{[s\Gamma s\Gamma s]\Gamma s\Gamma x] = [[s\Gamma s\Gamma s]\Gamma s\Gamma \sum_{i=1}^{n}[h_i\,\Gamma p_{1_i}\Gamma p_{2_i}]}$

$\boldsymbol{= \sum_{i=1}^{n}[[\,[s\Gamma s\Gamma s]\Gamma s h_i]\Gamma s\Gamma p_{1_i}\Gamma p_{2_i}] \in [H\Gamma P_1\Gamma P_2]}$

Hence $\boldsymbol{[H\Gamma P_1\Gamma P_2]}$ is generalized right $\boldsymbol{\Gamma}$ −semi ideal of $TS_R$.

Note: A necessary and sufficient condition for a commutative ternary Γ −semiring $TS_R$ without any divisors of zero to be a ternary division Γ −semiring which is given the below theorems

Theorem 4.14: Let $TS_R$ be a commutative ternary Γ −semiring without any divisors of zero which is a ternary division Γ −semiring if and only if for any generalized Γ −semi ideal of $A, a \in TS_R \backslash A$(the complement of A in$TS_R$ ) and $0 \neq x \in TS_R$ which gives

$[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \in TS_R \backslash A$.

Proof: Assume that $TS_R$ be a commutative ternary $\Gamma$ −semiring without any divisors of zero which is a ternary division $\Gamma$ −semiring. And let A be a generalized $\Gamma$ −semi ideal of $TS_R$. Select an element $a \in TS_R \backslash A$ and $0 \neq x \in TS_R$. Hence $\exists 0 \neq y \in TS_R$ such that $[x\Gamma y\Gamma z] = [y\Gamma x\Gamma z] = [z\Gamma x\Gamma y] = [z\Gamma y\Gamma x] = z$, for all $z \in TS_R$Therefore $[x\Gamma y\Gamma a] = [y\Gamma x\Gamma a] = [a\Gamma x\Gamma y] = [a\Gamma y\Gamma x] = a$, This proves that $[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \in TS_R \backslash A$.Assume that $[x\Gamma x\Gamma x]\Gamma x\Gamma a] = x^4 a \in A$. Therefore $a = [[y\Gamma x\Gamma y]^4 \Gamma a\Gamma x^4] \in A$.(Since $TS_R$ is a commutative ternary $\Gamma$ −semiring, A is a generalized $\Gamma$ −semi ideal of $TS_R$) Which is a contradiction Therefore $[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \in TS_R \backslash A$. Conversely suppose that for any generalized $\Gamma$ −semi ideal A, $a \in TS_R \backslash A$ and $0 \neq x \in TS_R$ implies that $[[x\Gamma x\Gamma x]\Gamma x\Gamma a] \in TS_R \backslash A$. Now we prove that $TS_R$ is a ternary division $\Gamma$ −semiring.It is enough we show that for any $0 \neq x \in TS_R, \exists 0 \neq y \in TS_R$ such that $[x\Gamma y\Gamma TS_R] = TS_R$.If possible suppose that $[x\Gamma y\Gamma TS_R] \neq TS_R$ and $b \in TS_R \backslash A$,then $[x\Gamma x\Gamma x]\Gamma x\Gamma b] = [x^3 \Gamma x\Gamma b] = [x\Gamma x^3 \Gamma b] = [x\Gamma y\Gamma b] \in [x\Gamma y\Gamma TS_R]$,where $y = x^3 (\neq 0) \in TS_R$.Which is a contradiction since $[x^3 \Gamma x\Gamma b] \in TS_R \backslash A$.Hence $[x\Gamma y\Gamma TS_R] = TS_R$.Therefore $TS_R$is a ternary division $\Gamma$ −semiring. Note: Let $TS_R$ be a commutative ternary $\Gamma$ −semiring and A is generalized $\Gamma$ −semi ideal of $TS_R$. Let $\beta(A)$ denote the set of all those elements afor which there exist a non zero element $s \in TS_R$ such that $[[s\Gamma s\Gamma s]\Gamma a] \in A$. It is then clear that $A \subseteq \beta(A)$.

Theorem 4.15: A Commutative ternary $\Gamma$ −semiring $TS_R$ without any divisors of zero.And if A is generalized $\Gamma$ −semi ideal of $TS_R$ then $\beta(A)$ is also a generalized $\Gamma$ −semi ideal of $TS_R$.

Proof: Let $TS_R$ be Commutative ternary $\Gamma$ −semiring and A is generalized $\Gamma$ −semi ideal $TS_R$. Let $\beta(A)$ denote the set of all those elements a. Let $x, y \in \beta(A)$ so there exist a non zero element $a, b \in TS_R$ such that $c = \left[[a\Gamma a\Gamma a]\Gamma a\Gamma x\right] \in A, d = [[b\Gamma b\Gamma b]\Gamma b\Gamma y] \in A$ Now

$\pi = [[a\Gamma a\Gamma a]\Gamma a\Gamma [b\Gamma b\Gamma b]\Gamma b\Gamma (x + y)]$

$= [ [a\Gamma a\Gamma a]\Gamma a\Gamma [b\Gamma b\Gamma b] \Gamma b\Gamma x] + [ [a\Gamma a\Gamma a]\Gamma a\Gamma [b\Gamma b\Gamma b] \Gamma b\Gamma y]$

$= [[b\Gamma b\Gamma b] \Gamma b\Gamma [[a\Gamma a\Gamma a]\Gamma a\Gamma x]] + [[a\Gamma a\Gamma a]\Gamma a\Gamma [b\Gamma b\Gamma b]\Gamma b\Gamma y]$

$=\left[[b\Gamma b\Gamma b]\Gamma b\Gamma c\right] + [[a\Gamma a\Gamma a]\Gamma x\Gamma d \in A$, For any$0 \neq z \in TS_R, [[z\Gamma z\Gamma z]\Gamma z\Gamma \pi] \in A$(since A is generalized $\Gamma$ −semi ideal of $TS_G$) therefore $[[[a\Gamma b\Gamma z][a\Gamma b\Gamma z][a\Gamma b\Gamma z]][[a\Gamma b\Gamma z](x + y) \in A$.Hence $(x + y) \in \beta(A)$ . For any $r \in \beta(A)$ ,$[[a\Gamma a\Gamma a]\Gamma a\Gamma x] \in A$ and let $0 \neq z \in TS_R$ Hence $\left[[a\Gamma a\Gamma a]\Gamma a\Gamma r\right][z\Gamma z\Gamma z]\Gamma z\Gamma x]] = [[z\Gamma z\Gamma z]\Gamma z\Gamma [[a\Gamma a\Gamma a]\Gamma a\Gamma r]] \in A$. Therefore $[[z\Gamma z\Gamma z]\Gamma z\Gamma a] \in \beta(A)$ for all $z \in TS_R$.Hence $\beta(A)$ generalized $\Gamma$ −semi ideal of $TS_R$.

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