Why the most risk-averse take the biggest risks: a quantitative reanalysis of Atkinson's and Litwin's "hoop-the-peg" experiments. Part 1: Simulation and model validation

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Psychologists Atkinson and Litwin designed an experiment based on the hoop-the-peg game to explore the tendency for anxious people to set their aspirations either very low or else, in an apparent contradiction, extremely high. This article derives a model based on risk-aversion to simulate the exercise. This is tested and validated against the probability distribution for selecting a task of given degree of difficulty developed in Part 2 of the paper. The model is able to explain in terms of the mathematically defined parameter, risk-aversion, why the most risk-averse people are prone to taking the biggest risks. Cautious people may not display a risk-aversion that is consistently positive, but can instead exhibit a mixture of positive and negative risk-aversions. What distinguishes the risk-confident from the more cautious is simply that the latter have a higher propensity to demonstrate a positive value of risk-aversion. Interpreting the empirical work of Atkinson and Litwin using the risk-aversion-based model has demonstrated that becoming ever more risk-averse should not be regarded as an unalloyed good. In particular, high levels of risk-aversion can lead to high likelihoods of failure.

Keywords: psychological experiment, psychological measurement

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1. Introduction

Drawing on his personal experience as a former banker, the best-selling author Robert Kelsey[1,2] recalled, "It wasn't that I was unable to take the risks that are part and parcel of an investment banker's role ... I could take ludicrous risks in some of the most volatile trading environments on the planet ... It was my judgement of risk that was undermined by my fears [as was] my ability to sort good fear from bad fear [that] can lead to either paralysis ... or nonsensical leaps." He highlighted "fear of failure" as the root cause of his inconsistent behaviour and directed his readers to the work of John W. Atkinson, the American psychologist who had written in 1957: "The tendency for anxious persons to set either extremely high or very low aspirations has been noted over and over again in the literature on level of aspiration".[3]

Working at the University of Michigan under a grant from the Ford Foundation, Atkinson had set out to explain the behaviour observed with five year-olds playing the game of hoop-the-peg, where greater kudos was on offer the farther back the child stood. Here children who had been found in independent assessments to be highly achievement-motivated took up a throwing position that was a challenging but realistic distance from the peg, somewhere near the centre of the range. But the children with low achievement motivation were found to act differently. Either they stood almost on top of the peg or else, perplexingly, so far back that failure was almost certain. It was the tendency for these more diffident children sometimes to select an extremely difficult if not impossible throwing position that constituted the principal interest of Atkinson's paper. Kelsey identified with this manner of reacting when he remembered the "ludicrously" high risks that he started to take in the later part of his ultimately unrewarding career as an investment banker.

Atkinson's 1957 paper[3] offered a simple mathematical model that came close to explaining the apparently counterintuitive way in which the diffident children sometimes behaved. However, the model possessed the significant limitation that, although diffident children were predicted to regard throwing positions at either end of the range as better than sites in the middle, it predicted that they would have no motivation to take a shot from any location.

The present author improved on this previous attempt at mathematical characterization by devising a model that employed a utility function, with risk-aversion as its governing parameter,[4] to represent the hoop-the-peg game. This gave clear motivations for the strictly risk-averse person to throw both from very close to the target and from very far away. The concept of risk-aversion was, of course, not available to Atkinson at the time he wrote his paper, as it was defined in mathematical terms only in 1964.[5] It has become a well-established

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1 Kelsey, R., Tears for fears, *Square Mile*, issue 73 (2012) 74–76.

2 Kelsey, R. *What's Stopping You? Why Smart People Don't Always Reach Their Potential and How You Can* (2nd edn). London: Capstone (2012).

3 Atkinson, J.W. Motivational determinants of risk-taking behaviour. *Psychological Review* **64** (1957) 359–372.

4 Thomas, P.J. The importance of risk-aversion as a measurable psychological parameter governing risk-taking behaviou. *Proc. Joint IMEKO TC1-TC7-TC13 Symposium on Measurement Across Physical and Behavioural Sciences*, 4–6 September 2013, Genoa; also appears as *Journal of Physics: Conference Series* **459** 012052.

5 Pratt, J.W. Risk aversion in the small and in the large. *Econometrica* **32** (1964) 122–136.

parameter in the fields of economics and decision science since that date. Daniel Bernoulli's and Gabriel Cramer's explanations of the St. Petersburg Paradox,[6,7] can now be linked, as these two great 18th century mathematicians were, effectively, using the same model, except with different values of risk-aversion: 1.0 in the case of Bernoulli and 1/2 in the case of Cramer.

While the improved, risk-aversion-based mathematical model was able to explain in general terms the three modes characterizing the choice of throwing positions, either roughly in the middle of the range or at either end, the lack of detailed empirical data in Atkinson's 1957 paper precluded testing it fully. This deficiency will be remedied in this paper by analysing a further study by Atkinson and Litwin,[8] in which they present the results for 45 male university students taking part in a hoop-the-peg experiment.

A problem remains however, in that their data were not presented in an entirely straightforward way. Rather than give the raw numbers of shots taken from each throwing position, Atkinson and Litwin gave filtered percentages, with no explicit definition of the filter. Some detective work was therefore required to find the most likely form of the smoothing filter and then back-calculate the number of throws taken from each pitching location. This work, which has interest in its own right, is explained in Part 2 of this work.[9] The final output of Part 2 is a set of probability distributions for the choice of throwing position for the subgroups identified by Atkinson and Litwin within their total cohort.

The empirical probability distributions allow the risk-aversion-based model to be tested and validated against experimental data. Part 1, presented here, will describe those experiments and draw conclusions from them.

The layout of this paper is as follows. Following the Introduction, §2 explains the format of the hoop-the-peg exercise devised by Atkinson and Litwin. §3 explains Atkinson's psychological and mathematical insights into motivation. A different approach is employed from that used previously,[4] with the analysis now being referred *ab initio* to the probability of failure rather than the probability of success. The analysis is simplified and results in a different and more easily understood set of formulae. The new approach continues through §4, where a utility model is found for the hoop-the-peg game, and into §5, which explains how it is possible to derive an expression for a risk-based probability density for selecting a task with a given degree of difficulty. §6 uses the model probability densities to produce probability distributions that are compared with the corresponding empirical distributions from Part 2 (the following paper). Sensitivity studies are provided in §7, while §8 contains the discussion. §9 gives the conclusions.

The appendices contain additional mathematical analysis. Appendix A explores the locus of expected utility versus degree of difficulty and finds the zero-crossing points. Appendix B

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6 Bernoulli, D. Specimen theoriae novae de mensura sortis. *Commentarii Academiae Scientiarum Imperialis Petropolitanae* 5 (1738) 175–192. Translation by Dr L. Sommer: Exposition of a new theory on the measurement of risk. *Econometrica* **22** (1954) 23–36.

7 Thomas, P.J. Measuring risk-aversion: The challenge. *Measurement* **79** (2016) 285–301.

8 Atkinson, J.W. and Litwin, G.H. Achievement motive and test anxiety conceived as motive to approach success and motive to avoid failure. *Journal of Abnormal and Social Psychology* **60** (1960) 52–63.

9 Thomas, P. Why the most risk-averse take the biggest risks: a quantitative reanalysis of Atkinson's and Litwin's "hoop-the-peg" experiments. Part 2: Establishing the raw data. *Nanotechnology Perceptions* **17** (2021) 229–250.

derives the probability density for someone with a specified risk-aversion selecting a task of a given degree of difficulty.

2. The hoop-the-peg experiment and the resulting probability distributions from Part 2

Atkinson and Litwin applied independent tests ahead of the experiment to measure the strength of the desire to achieve ("nAchievement") and the level of anxiety ("Test Anxiety") of each of 45 male students taking part in the hoop-the-peg exercise. As detailed in the source paper,[8] French's Insight Test, where the subject is asked to provide an explanation for the documented behaviour of fictitious characters, was used to measure nAchievement while Test Anxiety scores were obtained from a Mandler-Sarason Test Anxiety Questionnaire. They found they could divide the cohort into four groups:

1. Group 1, high desire to achieve and low level of anxiety: "Hi–Lo", of which there were 13 subjects
2. Group 2, high desire to achieve and high level of anxiety: "Hi–Hi", a class containing 10 students
3. Group 3, low desire to achieve and low level of anxiety: "Lo–Lo", with 9 people in this category
4. Group 4, low desire to achieve and high level of anxiety: "Lo–Hi", which characterized 13 members of the cohort.

A set of lines at discrete distances from the target were marked on the ground: at 1 foot, 2 feet, 3 feet, ..., and 15 feet (1 foot = 0.3048 m), and the students were given the instructions: "Today you are going to play a ring toss game. You will have an opportunity to take 10 shots at the target from any line you wish. You may move after each shot or shoot from the same line. Someone will record your shots and get your code number when you finish. We want to see how good you are at this." Atkinson's and Litwin's interest was in how the subjects would choose their throwing distances.

There was no prize for the person with the highest number of pegs successfully hooped, rather an implicit reward in the kudos accorded to hooping the peg from a difficult distance. This militates against a simple strategy of choosing the distance, perhaps iteratively, to maximize the number of hits. A strategy of taking all one's shots from 1 foot will almost certainly not be favoured by the players (as the results attest). Moreover, a player will not be tempted to settle for any other distance from which he scores a high percentage of hits, as, by definition, this will strike him as too easy and unlikely to earn much credit from the onlookers.

Figure 1 gives the empirical probability distributions derived in Part 2 for selecting a task with a specified degree of difficulty for groups 1 and 2, each of which was characterized by a high desire to achieve. Group 1, Hi–Lo, shows a marked preference for tasks with an intermediate level of difficulty, but retains non-negligible probabilities for choosing both very low and very high degrees of difficulty. An incipient trimodal distribution is observable. Meanwhile, the effect of the higher level of anxiety found in Group 2, Hi–Hi, is to reduce the probability of picking tasks with a medium degree of difficulty, and this leads to a curve with more pronounced trimodal properties. The chances of choosing missions with either a low or a high degree of difficulty have now increased compared with Group 1, Hi–Lo.

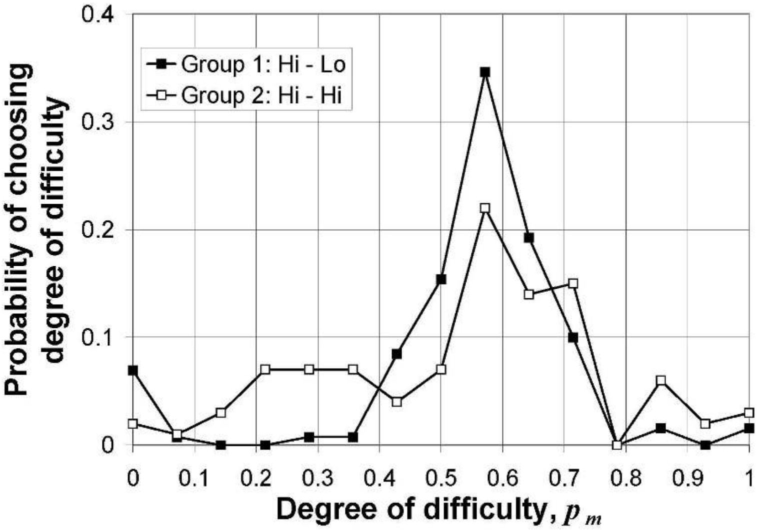

Figure 1. Empirical probability distribution, found in Part 2,[9] for selecting a task of a given degree of difficulty: groups with high desire to achieve.

Figure 2 gives the corresponding probability distribution for groups 3 and 4, Lo–Lo and Lo–Hi, in which a low desire to achieve is exhibited. The trimodal nature of the probability distribution is well established in both these groups. In the case of group 4, Lo–Hi, it is striking that the maximum probability at the lowest mode differs little from that of the central mode (which dominates in Figure 1).

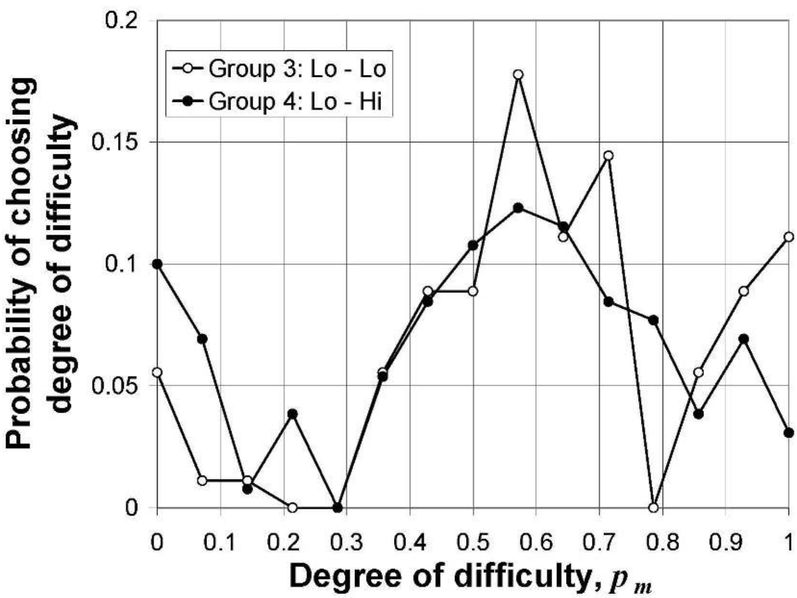

Figure 2. Empirical probability distribution, found in Part 2,[9] for selecting a task of a given degree of difficulty: groups with low desire to achieve .

3. John W. Atkinson's psychological and mathematical insights into motivation

For any task, Atkinson[3] saw the "motivation to succeed", D_s, as the product of three factors: the incentive to succeed, I_s, the "expectancy" or probability of success, p_s, and the "motive to succeed", M_s. Note that, in Atkinson's nomenclature, "motive" has a meaning separate from "motivation".

The motivation to succeed would be counterbalanced by the "motivation to avoid failure", D_m, found by multiplying the incentive to avoid failure, I_m, by the "expectancy" (= probability) of failure, p_m, and the "motive to avoid failure", M_m. (The subscript *m* is derived from "miss" in the sense of "fail to hit the target".) The resulting, net motivation to achieve, D_r, would then be found as the difference between the motivations to succeed and the motivation to avoid failure:

$$D_r = D_s - D_m = M_s p_s I_s - M_m p_m I_m \tag{1}$$

Moreover, Atkinson modeled the incentive to succeed as the complement of the probability of success:

$$I_s = 1 - p_s = p_m \tag{2}$$

reasoning that the more difficult a task, the greater the gain in both personal satisfaction and kudos when it was achieved.

Atkinson accorded a psychological weighting to failure that was higher than that associated simply with lack of success. While the latter might attract, for example, only a null weighting, he took the fact of losing as causing pain over and above this level. His judgment was that the dissatisfaction and embarrassment at failing to execute an easy task were greater than those associated with failure in a difficult assignment. Quantifying the effect, he modeled the incentive to avoid failure in a mission as proportional to its probability of success, with a constant of proportionality equal to unity:

$$I_m = p_s = 1 - p_m \tag{3}$$

Equations (2) and (3) are, of course, complementary. Equation (3) embodies the principle that the greater the task's chance of success, the greater would be the dissatisfaction and shame if the subject failed.

Atkinson's model[3] brought undoubted psychological insights but a number of deficiencies remained. While the equations he developed suggested that the anxious individual would tend to prefer tasks where success was either completely certain or completely impossible over those where there was a reasonable prospect of success, that person would theoretically perform neither, since the net motivation produced by the mathematics would be zero, whereas, in reality, the participants actually took throws from both ends of the range. To cover this, Atkinson invoked other influences, "e.g. social pressures", as necessary to persuade the individual to take on either the easiest or the hardest available task. But such auxiliary effects were not included in the mathematical model.

Moreover, Atkinson's model was unable to distinguish between degrees of anxiousness beyond just three states of mind, which might be characterized as risk-averse, risk-neutral and risk-confident. Yet common experience suggests that people vary in their levels of anxiety in a more gradual way than this ternary system implies, indicating that a fuller description might be needed. (The sensitivity studies in §7 will show that Atkinson's idea has merit; this paper's analysis will convert what was an assumption into a more precise statement that can be inferred from the mathematics to be approximately true.)

4. Modelling the hoop-the-peg game using utility theory

The reward associated with success in a task may be described by a utility function for which the argument is not money (as in the common economic case) but the incentive for success, I_s. The utility of success is then modeled using a power utility function,[7] with risk-aversion, ε, as a parameter:

$$u_s = a_s I_s^{1-\varepsilon} = a_s p_m^{1-\varepsilon} \tag{4}$$

where a_s is a constant, reflecting the fact that the utility function is correct to a positive multiplier.[10] In future discussion, the probability of failure, p_m, will be referred to synonymously as "the degree of difficulty" of the task. The utility function of equation (4) becomes identical to Atkinson's incentive for success, $u_s = I_s$, when $\varepsilon = 0$ and $a_s = 1$, in much the same way that a utility function applied to money becomes either the money or some multiple of it at the risk-neutral condition, $\varepsilon = 0$.

Meanwhile the penalty associated with failure in a task is described by a disutility function, the argument of which is the incentive to avoid failure. Although utility functions are usually assumed concave for a risk-averse person, reflecting increasing satiation with good things, disutility functions, introduced as a concept by Frank Ramsey,[11] may be taken to be convex to accommodate an accelerating growth in dissatisfaction as bad things accumulate and potentially threaten to overwhelm a person's normal coping mechanisms. The implication for a risk-averse person is that the disutility of owing one's creditors £20,000 would be more than 10 times greater than being indebted to the sum of £2,000. On the other hand, a risk-confident person would see the larger debt as bringing less than 10 times as much disutility as owing the smaller amount, which would imply a concave utility function. Daniel Kahneman suggests disutility functions can be concave in this way.[12] The premiss that people will get ever more inured to bad things seems to underlie his claim that "The pain of losing \$900 is more than 90% of the pain of losing \$1000" (so that the extra 11% loss brings less than 10% extra pain). He accepts this as evidence of a risk-seeking (or risk-confident) stance, but proposes that this risk-seeking viewpoint can coexist with a cautious attitude. Kahneman may, in fact, be reporting on behaviour similar to that highlighted in the hoop-the-peg exercise, where risk-averse people have been observed to act in a way that is, *prima facie*, counterintuitive. (Full details of Atkinson's and Litwin's results are given in Part 2 of this paper.)

The power disutility function used here possesses the desired property of convexity when risk-aversion is positive, indicating caution, and concavity, signifying risk confidence, when that parameter is negative. It has the same form as the power utility function, except that the risk-aversion, ε, is preceded by a plus rather than a minus sign. Making the further assumption that the individual's utility and disutility functions will use the same value of risk-aversion, the

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[10] von Neumann, J. and Morgenstern, O. *Theory of Games and Economic Behavior* (60th anniversary edn), Appendix: The axiomatic treatment of utility. Princeton: University Press (2007) (1st edn publ. 1944).
[11] Ramsey, F.P. A mathematical theory of saving. *Economic Journal* **38** (1928) 543–559.
[12] Kahneman, D. *Thinking Fast and Slow*, ch. 26, Prospect theory. London: Penguin (2011).

disutility of failure, v_m, is given by:

$$v_m = a_m\left(1-p_m\right)^{1+\varepsilon} \tag{5}$$

where the constant, a_m, reflects the fact that the utility function is correct to a positive multiplier.

The disutility function degenerates to Atkinson's incentive to avoid failure, viz. $v_m = I_m$ when $\varepsilon = 0$ and $a_m = 1$, as may be seen by substituting those values into equation (5) and comparing the result with equation (3). This parallels the situation where a disutility function applied to debt becomes simply the debt when $\varepsilon = 0$; the utility of failure is now simply the negative of the disutility of failure: $u_m = -v_m$.

Applying the utility model developed above to a task with degree of difficulty p_m allows the expected utility resulting from its execution, $z(\varepsilon, p_m)$, to be calculated for a person with risk-aversion ε as:

$$z\left(\varepsilon, p_m\right) = E\left[U\left(\varepsilon\right)\right] = p_s u_s + \left(1-p_s\right)u_m = a_s\left(1-p_m\right)p_m^{1-\varepsilon} - a_m p_m\left(1-p_m\right)^{1+\varepsilon}. \tag{6}$$

4.1 A fair game

The average utility expected over tasks of all difficulty levels, p_m, may be found after assuming there to be a continuum of tasks with failure probability, p_m, conforming to a uniform distribution, $f(p_m)$, between 0 to 1, which will obey: $f(p_m) = 1$ for $0 \le p_m \le 1$. Hence the expected utility, $z_T(\varepsilon)$, across tasks of all difficulty levels, p_m, for a person with risk-aversion ε will be:

$$\begin{aligned} z_T\left(\varepsilon\right) &= \int_{p_m=0}^{1} z\left(\varepsilon, p_m\right) f\left(p_m\right) dp_m = a_s \int_{p_m=0}^{1} \left(1-p_m\right) p_m^{1-\varepsilon} dp_m - a_m \int_{p_m=0}^{1} p_m\left(1-p_m\right)^{1+\varepsilon} dp_m = \\ &\frac{a_s}{\left(2-\varepsilon\right)\left(3-\varepsilon\right)} - \frac{a_m}{\left(2+\varepsilon\right)\left(3+\varepsilon\right)} \end{aligned} \tag{7}$$

where equation (6) has been used in the second line.

In betting, a fair game may be defined as one where the expected monetary gains are exactly offset by the expected monetary losses. Arbitrage is now impossible, so that, for example, backing both mounts equally in a two-horse race at evens cannot produce a profit (whether any commercial bookmaker would be prepared to offer such favourable odds is a separate question, likely to be answered in the negative!). By analogy, we may define a fair game where the rewards may be other than monetary as one for which the expected net reward is zero for a risk-neutral person. Since $\varepsilon = 0$ for such an individual, a fair game implies, from equation (7):

$$z_T\left(0\right) = \frac{1}{6}\left(a_s - a_m\right) = 0. \tag{8}$$

This will be satisfied for any pairing (a_s, a_m) such that $a_s = a_m$. The risk-neutral person will regard taking part in such a game with equanimity since, if he undertook each of the tasks within it, he would not expect overall to lose, even if he would not expect to win overall either.

The idea of a fair game just described may be expanded to encompass the notion of a "fair game *according to his lights*" if we replace the condition $z_T(0) = 0$ by the more general $z_T(\varepsilon) = 0$.

This will imply that a person with risk-aversion ε would regard the game as fair in that if he undertook each of the tasks in it, the person would get out of the game, in terms of utility (rather than money), what he put in. He could expect to break even in the sense that his psychological pain would be matched by his psychological gain. This condition may be satisfied by setting the weighting of utility of success (arbitrarily) at $a_s = 1$ and varying a_m. Solving equation (7) under these conditions gives the general solution:

$$a_m = \left(\frac{2+\varepsilon}{2-\varepsilon}\right)\left(\frac{3+\varepsilon}{3-\varepsilon}\right). \tag{9}$$

Assigning this value to a_m will mean that the game will be regarded by a player with risk-aversion ε as fair—it will be "fair according to his lights".

Substituting from equation (9) back into equation (6) gives the expected utility for a task of success probability, p_s, for someone with risk-aversion ε under the condition that he will regard the game as fair according to his lights. The requisite expression for the expected utility from performing such a task is then:

$$z(\varepsilon, p_m) = (1-p_m)\,p_m^{1-\varepsilon} - \left(\frac{2+\varepsilon}{2-\varepsilon}\right)\left(\frac{3+\varepsilon}{3-\varepsilon}\right) p_m (1-p_m)^{1+\varepsilon}. \tag{10}$$

It may be noted that this equation depends solely on the task's degree of difficulty and the individual's risk-aversion.

The expected utility given by equation (10) may be graphed against the degree of difficulty of the task, with risk-aversion as a parameter. Figure 3 plots curves where it takes a negative value, indicating a risk-confident individual, while Figure 4 plots a set of curves where the risk-aversion is positive, implying a cautious person. A single lobe, with a single mode, is observed in the positive half-plane for negative risk-aversions, but two positive lobes are seen to occur when the risk-aversion is positive.

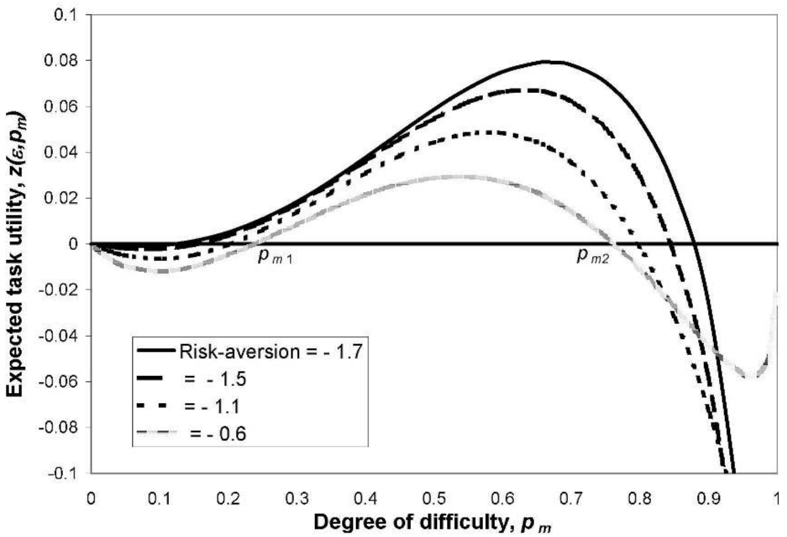

Figure 3. Expected task utility against degree of difficulty for negative risk-aversions.

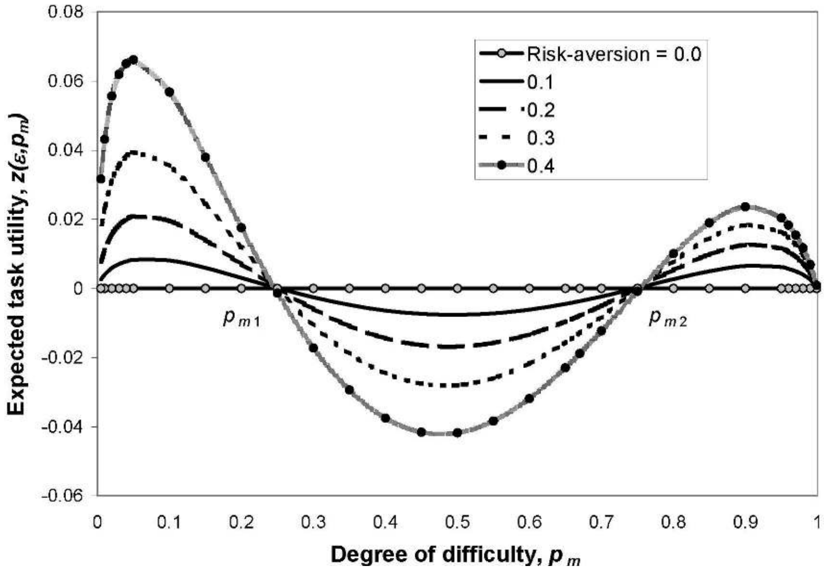

Figure 4. Expected task utility against degree of difficulty for non-negative risk-aversions.

It is reasonable to suppose that an individual will select a task of a specified difficulty only if he expects to gain utility from performing that task. For a risk-confident person (negative risk-aversion), it is clear from Figure 3 that this rules out all tasks with degrees of difficulty between 0 and p_{m1} and between p_{m2} and 1.0. Hence the only difficulty levels, p_m, for the task permitted to a risk-confident person ($\varepsilon < 0$) are governed by: $p_{m1} < p_m < p_{m2}$.

The situation is reversed for the cautious person, whose risk-aversion will be positive. From Figure 4, the risk-averse individual will avoid all tasks of difficulty level, p_m, such that $p_{m1} \leq p_m \leq p_{m2}$, and will choose instead a failure probability lying in either of the ranges $0 < p_m < p_{m1}$ or $p_{m2} < p_m < 1.0$.

These conclusions on the modes of the expected task utility, $z(\varepsilon, p_m)$, are derived using a different method in Appendix A, which also finds the two values, p_{m1}, p_{m2}, of the failure probabilities at intermediate zero crossings for non-risk-neutral cases.

5. Probability density for selecting a task of a given degree of difficulty

It is natural to assume that any individual will be more likely to select tasks with higher expected task utilities. Moreover, it is reasonable to suppose that tasks associated with expected utilities that are negative, implying an adverse effect on the person, will have no chance of being chosen, as discussed at the end of Section 4. Such behavioural choices may be modeled by assigning a probability density, $h\left(p_m \middle| \varepsilon\right)$, for task difficulty, that will be proportional to the expected task utility, provided this is positive, but will be zero for non-positive task utilities. The probability density will be conditioned on risk-aversion, ε, as indicated. Appendix B provides the derivation of $h\left(p_m \middle| \varepsilon\right)$, which emerges as:

$$h\left(p_m|\varepsilon\right)=\frac{1}{D}\begin{cases}\max\begin{pmatrix}0,(2-\varepsilon)(3-\varepsilon)(1-p_m)p_m^{1-\varepsilon}\\-(2+\varepsilon)(3+\varepsilon)p_m(1-p_m)^{1+\varepsilon}\end{pmatrix} & \text{for } \varepsilon<0\\ \\ \max\begin{pmatrix}0,(2+\varepsilon)(3+\varepsilon)p_m(1-p_m)^{1+\varepsilon}\\-(2-\varepsilon)(3-\varepsilon)(1-p_m)p_m^{1-\varepsilon}\end{pmatrix} & \text{for } \varepsilon>0\end{cases} \tag{B.12}$$

where

$$D=-\begin{pmatrix}(3-\varepsilon)\left(p_{m1}^{2-\varepsilon}-p_{m2}^{2-\varepsilon}\right)-(2-\varepsilon)\left(p_{m1}^{3-\varepsilon}-p_{m2}^{3-\varepsilon}\right)\\+p_{m2}^{2+\varepsilon}\left((2+\varepsilon)p_{m1}+1\right)-p_{m1}^{2+\varepsilon}\left((2+\varepsilon)p_{m2}+1\right)\end{pmatrix}. \tag{B.8}$$

Figure 5 illustrates the probability densities calculated for risk-aversions of – 0.5 and + 0.2. There is a single lobe for the negative risk-aversion, but there are two distinct lobes when the risk-aversion is positive.

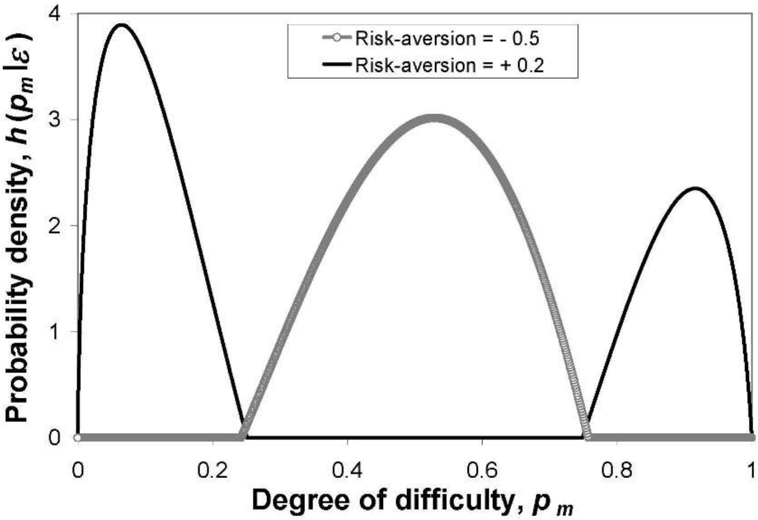

Figure 5. Probability densities given by equation (B.12) at risk-aversions, ε, of –0.5 and + 0.2.

6. Matching probability densities based on risk-aversion to the empirical probability distributions

As noted previously, Atkinson reported in his 1957 paper that confident 5-year old children displayed unimodal behaviour,[3] choosing a central throwing position of medium difficulty. But diffident children of the same age were found to display a bimodal choice of a throwing position: this could be either very close to the target or a great distance away. These basic forms of behaviour are captured in the probability distribution of equation (B.12) by the appropriate choice of risk-aversion, negative for confident people and positive for the less confident, strictly risk-averse, as illustrated in Figure 5.

However, as evident from Figures 1 and 2, the university students' empirical probability distributions are trimodal. They seem sometimes to take a risk-confident stance by choosing a central throwing position, but to act in a risk-averse way at other times by adopting a throwing position near one end or the other of the range. Such behaviour may be modeled by a new probability density that is the expected value of a new, random probability density, $G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)$:

$$E\left(G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)\right) = \alpha h\left(p_m \middle| \varepsilon_1\right) + (1-\alpha) h\left(p_m \middle| \varepsilon_2\right). \tag{11}$$

By this formulation, risk-aversion is regarded as a random variable, E, that may take one of two values, ε_1 and ε_2, while α is the probability of adopting a risk-aversion E = ε_1; its complement, $(1 - \alpha)$, is the probability of the person displaying a risk-aversion, E = ε_2.

To establish a good estimate of ε_1, it may first be noted from Figures 1 and 2 that the central mode for every group occurs when the degree of difficulty is 0.57. Equation (B.12) is found to reproduce this mode at the risk-confident value of $\varepsilon_1 = -1.02$.

Meanwhile the upper and lower modes for the selection of degree of difficulty may be observed from Figures 1 and 2 to be approximately symmetrical about the midpoint of the range of degree of difficulty. This symmetry, which applies to all four groups, is predicted by equation (B.12) as $\varepsilon \rightarrow 0+$, where 0+ indicates that the risk-aversion approaches zero from above. Accordingly the finite but very small risk-averse value, $\varepsilon_2 = +1.0 \times 10^{-5}$ was chosen as representative of every group and of the combination of all four groups (see Figure 6). A set of optimal values of probability α were chosen by minimizing the average absolute difference between the theoretical, probability distributions for the four groups and the empirical probability distributions found in Part 2, as shown in Figures 1 and 2. A match was also made for the combination of all four groups.

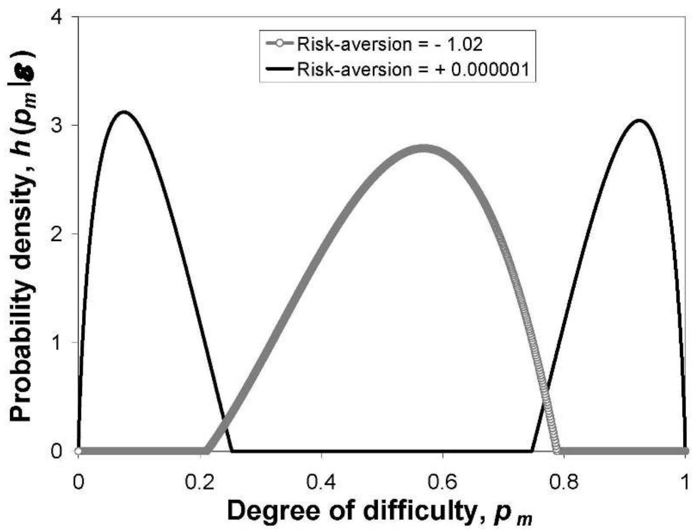

Figure 6. Probability density given by equation (B.12) at risk-aversions, ε, of –1.02 and 1.0×10^{-5}.

Figure 7 shows how the resultant probability, α, of risk-aversion taking the risk-confident value, $\varepsilon_1 = -1.02$, is very high for Group 1, Hi–Lo, but falls by 14% for Group 2, Hi–Hi, before dropping a further approximately 20% for the two low-achievement motivation groups, 3, Lo–Lo, and 4, Lo–Hi.

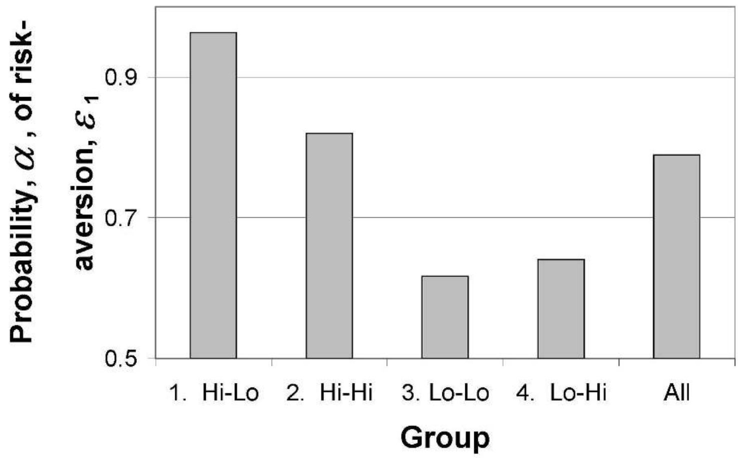

Figure 7. The optimal value of probability, α, of risk-aversion, ε_1, based on minimising the sum of absolute errors between the model probabilities and the estimates contained in Figures 1 and 2.

For comparison purposes, Figure 8 shows the probability, α, of risk-aversion taking the risk-confident value, $\varepsilon_1 = -1.02$, for all the groups and for the total cohort when the sum of the squared errors is minimized rather than the sum of absolute errors. The probability, α, can be seen to be very close to the value shown in Figure 7 for groups, 1, 2 and 4 and for the full cohort. However it rises from 0.62 to 0.72 for Group 3, Lo–Lo, thus reversing the ordering of groups 3 and 4.

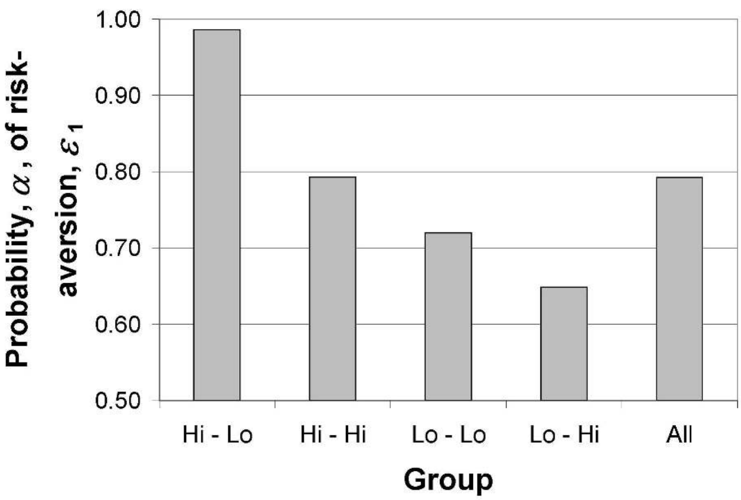

Figure 8. The optimal value of probability, α, of risk-aversion, ε_1, based on minimising the sum of the squared errors between the model probabilities and the estimates contained in Figures 1 and 2.

Figure 9 shows, for the complete cohort of students, the model predications for the probability of choosing a task with a given degree of difficulty against the corresponding figures derived from the estimated raw shots data illustrated in Figures 1 and 2. Figure 10 gives the graph for Group 1, Hi–Lo, Figure 11 for Group 2, Hi–Hi, Figure 12 for Group 3, Lo–Lo, and Figure 13 presents a comparison for Group 4, Lo–Hi.

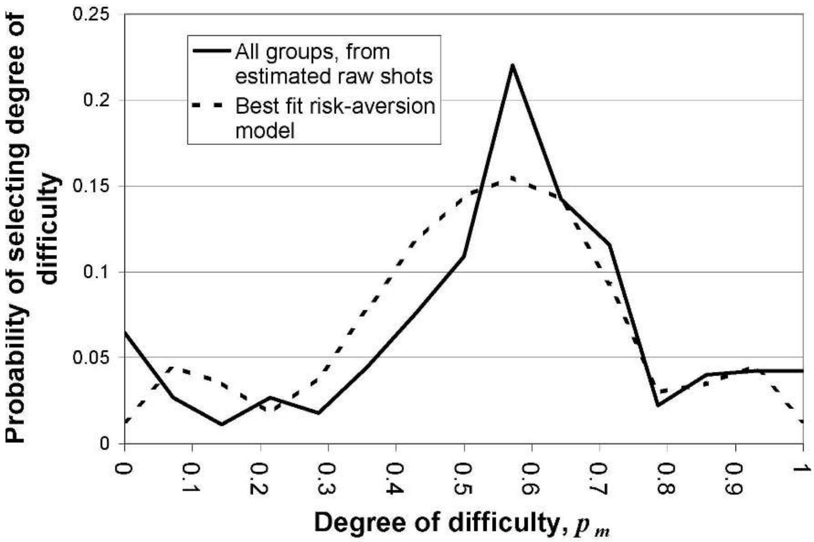

Figure 9. Comparison between the probabilities based on the risk-aversion model and those found from the estimated numbers of raw shots for the whole cohort; $\alpha = 0.79$.

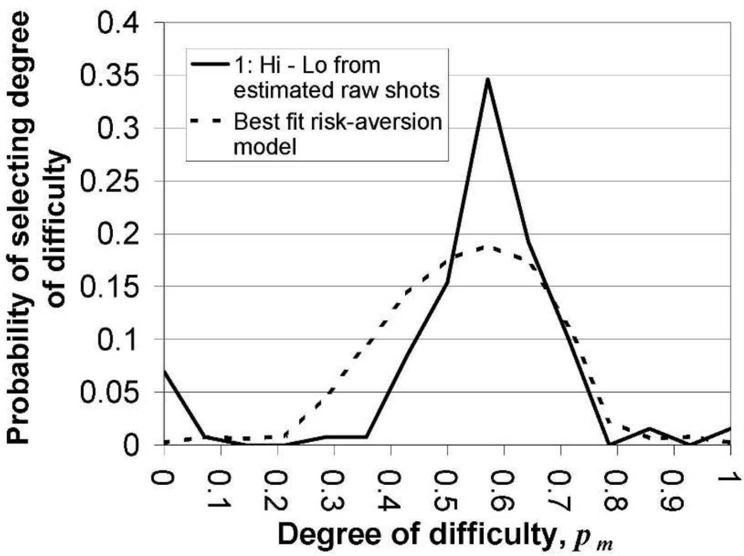

Figure 10. Comparison between the probabilities based on the risk-aversion model and those found from the estimated numbers of raw shots for Group 1, Hi–Lo; $\alpha = 0.96$.

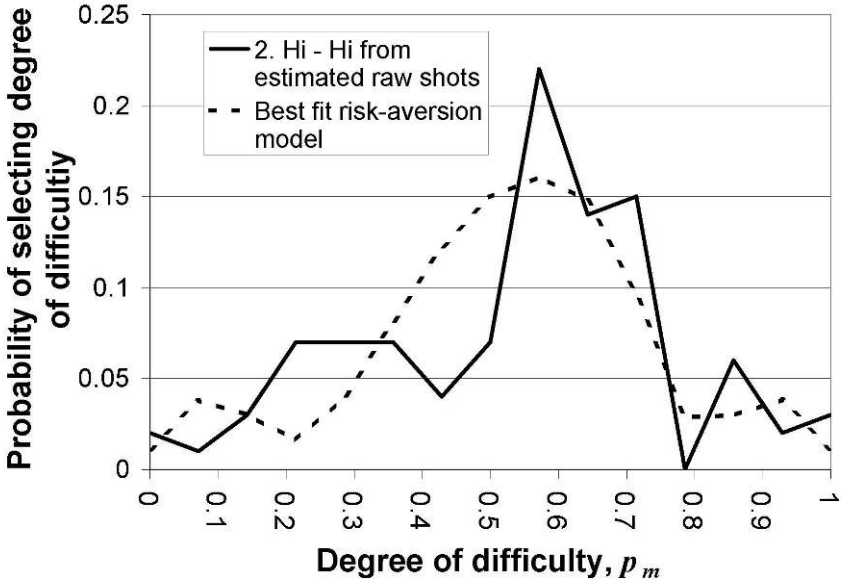

Figure 11. Comparison between the probabilities based on the risk-aversion model and those found from the estimated numbers of raw shots for Group 2, Hi–Hi; $\alpha = 0.82$.

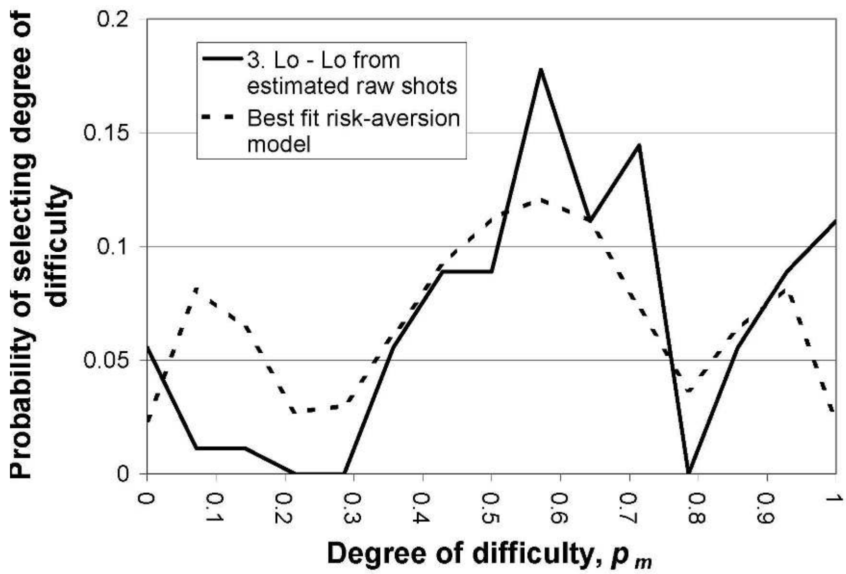

Figure 12. Comparison between the probabilities based on the risk-aversion model and those found from the estimated numbers of raw shots for Group 3, Lo–Lo; $\alpha = 0.62$.

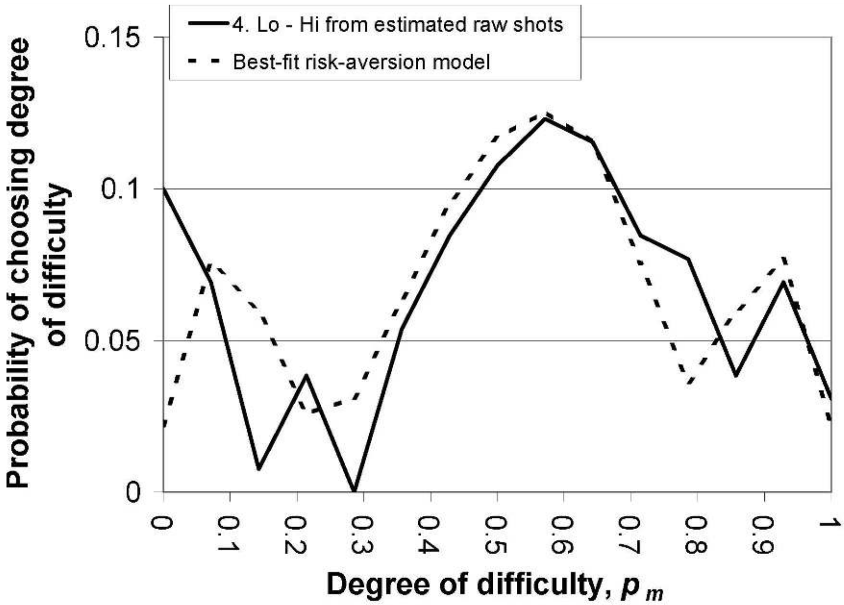

Figure 13. Comparison between the probabilities based on the risk-aversion model and those found from the estimated numbers of raw shots for Group 4, Lo–Hi ; $\alpha = 0.64$.

The closeness of the matches may be judged by eye from Figures 9–13, and numerical measures are provided in Table 1. The average absolute error between the model prediction and the probability of choosing a given throwing position with its associated level of difficulty varies from 0.02 for Group 4, Lo–Lo, to 0.035 for Group 1, Hi–Lo. The overall average absolute error per throwing position is 0.024.

Table 1. Summary of the residual differences after optimization between the probabilities found by applying $E\left(G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)\right)$ from equation 11 and the estimated probabilities based on the estimated raw shots. Comparison with the estimated differences between Atkinson's and Litwin's Figure 1 and their Table 1.[13] (See Part 2 of this paper[9].)

	Groups				
	All	1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi
Average absolute error per throwing position between estimated probability based on raw shots and that derived from $E\left(G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)\right)$	0.0244	0.0354	0.0340	0.0341	0.0200
Estimated average absolute error per throwing position between Atkinson's and Litwin's Figure 1 and their Table 1	0.0132	0.0221	0.0208	0.0115	0.0074

[13] Table 5 of Part 2 of this paper.[9]

The estimated absolute difference found in Part 2 between the probabilities presented in Atkinson's and Litwin's Figure 1 and those given in their Table 1, averaged over all throwing positions, are between a half and two thirds of the residual differences shown in the top line of Table 1. Moreover, the estimated average differences, as given by equation (11) of Part 2 of this paper,[9] are shown there to be underestimates of the true average differences, even though those cannot be calculated due to lack of data. Inaccuracies in the underlying data may therefore explain a significant fraction of the discrepancies revealed between the risk-aversion-based and the experimental probability distributions for selecting degree of difficulty.

7. Sensitivity studies

Sensitivity studies were carried out over a wide range for both the negative risk-aversion, ε_1, and the positive risk-aversion, ε_2.

Risk-aversion, ε_1, was increased from its base value, $\varepsilon_1 = -1.02$, to $\varepsilon_1 = -0.5$ and then decreased to $\varepsilon_1 = -1.5$. The effect was modest on the values of the propensity, α, that yielded the optimal match between the predicted probabilities for the degree of difficulty chosen and the empirical estimates of Figures 1 and 2 (Table 2). That table also gives the average absolute error per throwing position between the empirically estimated probability and that derived from equation (11). The effect of varying ε_1 over the large range selected is clearly quite small.

Table 2. Sensitivity study: the effect on α and on $E\left(G\left(p_m|\varepsilon_1,\varepsilon_2\right)\right)$ of varying ε_1 from its baseline value of – 1.02 although keeping it negative.

		1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi	All
Probability, α	$\varepsilon_1 = -0.5$	0.964	0.821	0.664	0.599	0.802
	$\varepsilon_1 = -1.02$	0.964	0.821	0.617	0.640	0.789
	$\varepsilon_1 = -1.5$	1.000	0.824	0.746	0.676	0.792
Average absolute error per throwing position between estimated probability based on raw shots and that derived from $E\left(G\left(p_m\|\varepsilon_1,\varepsilon_2\right)\right)$	$\varepsilon_1 = -0.5$	0.041	0.040	0.038	0.028	0.032
	$\varepsilon_1 = -1.02$	0.035	0.034	0.034	0.020	0.024
	$\varepsilon_1 = -1.5$	0.039	0.033	0.035	0.017	0.023

Table 3 shows similar comparisons when risk-aversion, ε_2, was increased from its base value, $\varepsilon_2 = 10^{-5}$, to $\varepsilon_2 = 0.5$ and then to $\varepsilon_2 = 0.9$, thus covering 90% of the possible range of positive risk-aversions allowable in this model. The effects on both the probability, α, and on the average absolute error per throwing position can once again be seen to be small.

It may be concluded that the mathematical model for the choice of throwing positions in hoop the peg, as summarized in equations (11), (B.8) and (B.12), is robust against variations in

Table 3. Sensitivity study: the effect on α and on $E\left(G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)\right)$ of varying ε_2 from its baseline value of $+1.0 \times 10^{-5}$ while maintaining it positive.

		1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi	All
Probability, α	$\varepsilon_2 = 10^{-5}$	0.964	0.821	0.617	0.640	0.789
	$\varepsilon_2 = 0.5$	0.978	0.861	0.632	0.628	0.789
	$\varepsilon_2 = 0.9$	0.981	0.929	0.804	0.647	0.789
Average absolute error per throwing position between estimated probability based on raw shots and that derived from $E\left(G\left(p_m \middle| \varepsilon_1, \varepsilon_2\right)\right)$	$\varepsilon_2 = 10^{-5}$	0.035	0.034	0.034	0.020	0.024
	$\varepsilon_2 = 0.5$	0.035	0.036	0.041	0.025	0.028
	$\varepsilon_2 = 0.9$	0.035	0.038	0.041	0.027	0.029

the values adopted for the baseline values of risk-aversions, ε_1 and ε_2. Provided that ε_1 is negative and ε_2 is positive, the precise values used will make little difference to the goodness of fit between model and observations.

8. Discussion

It is clear from Figures 9 to 13 that the risk-aversion-based model derived here provides a good explanation for how many shots are taken from each throwing position by each group in the hoop-the-peg exercise. The thrower is guided in his choice of pitching position by a risk-aversion that may be either positive, indicating caution, or negative, indicating confidence. Negative risk-aversions will be chosen with probability α, while positive values will be selected with probability $1 - \alpha$. The model's demonstrated insensitivity to the values of the two baseline risk-aversions $\varepsilon_1 < 0$ and $\varepsilon_2 > 0$ means that the precise size of either is of small importance, as long as the sign is not changed.

The merit of using a model that incorporates a probability α of choosing a negative risk-aversion is that it is a quantitative measure distinguishing between the risk-confident groups in the cohort and those made up of inherently more cautious people. In the hoop-the-peg exercise described by Atkinson and Litwin the most confident students chose a negative risk-aversion more than 95% of the time, whereas the least confident students selected a negative risk-aversion less than 65% of the time.[14]

The present model has been validated against empirical data in the sense that it is able to explain observed behaviour in the hoop-the-peg game. The model assumes that individuals are likely to apply more than one value of risk-aversion when judging how to react in a given situation, oscillating between being risk-confident and risk-averse. Cautious people will not always take a strictly risk-averse stance but they will show a higher propensity to adopt such an

[14] See Part 2[9] for a summary of the independent assessment of the degree of confidence.

attitude than risk-confident people. Similarly, more confident people will have an increased likelihood of striking a risk-confident attitude, but may still become strictly risk-averse at times. This may be a new, general and important psychological finding.

The caveat is offered, that, although the propensity α of adopting a risk-confident stance is likely to decline with increasing risk-aversion in the way displayed in Figure 7, it cannot be assumed that the propensity values, α, will cover the same range of values across all cohorts of throwers.

The fact that a negative risk-aversion appears to have been chosen most of the time within all four groups in the hoop-the-peg game deserves comment. People tend to be strictly risk-averse, viz. $\varepsilon > 0$, for decisions involving significant amounts of money,[15] and for decisions on extending human life.[16] However, no money was at risk one way or the other in the hoop-the-peg exercise, nor was there much by way of conventional sporting status involved. For example, no prowess in track and field athletics was being judged, nor would any skills on display be directly applicable to baseball or American football, for example. Hence, any kudos accruing to the best-performing players is likely to have been relatively low and short-lived. The high probability of acting according to a negative risk-aversion, indicating boldness, in all of the groups can thus be explained by the low stakes involved.

However, strictly risk-averse decisions, where risk-aversion is positive, should not be taken always to imply low risk in the conventional sense. On the contrary, strict risk-aversion on the part of the decision-maker can lead to very high risks of failure being accepted, as illustrated by the right-hand lobes on display in Figures 5 and 6. Indeed, the strictly risk-averse person will be as likely to adopt an apparently high-risk strategy as take a course of action that is, on the face of it, safer. By contrast, a person who was entirely risk-confident would never take such a large risk of failure.

The results reported in the Atkinson and Litwin study[2] confirm earlier findings for young children playing the same game, as reported in Atkinson's earlier paper.[3] Moreover, a model based on the mathematically defined concept of risk-aversion has been shown to provide a good explanation for the behaviour of Atkinson's and Litwin's cohort as a whole and its four constituent groups. But it is important to bear in mind that only 45 male university students were included in the research.

The insensitivity to the values used for baseline risk-aversions, ε_1 and ε_2, as long as their sign is maintained, suggests that these values could be used in modeling a separate cohort of people. However, the propensities, respectively α and $1 - \alpha$, to select from the baseline risk-aversions might differ from those found here. In that sense, while the model can be seen to offer high explanatory power, further tests on different cohorts would be needed to assess its power to provide quantitative predictions.

Atkinson speculates in his 1957 paper[3] that the effect he identified may be a factor behind what he sees as the "sociological problems of mobility aspirations". The idea seems to be that an

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[15] Thomas, P.J., Jones, R.D. and Boyle, W.J.O. The limits to risk aversion: Part 1. The point of indiscriminate decision. *Process Safety and Environmental Protection* **88** (2010) 381–395; Thomas, P.J., Jones, R.D. and Boyle, W.J.O. The limits to risk aversion. Part 2: The permission point and examples. *Process Safety and Environmental Protection* **88** (2010) 396–406.

[16] Thomas, P. and Waddington, I. Validating the J-value safety assessment tool against pan-national data. *Process Safety and Environmental Protection* **112A** (2017) 179–197.

individual who is predominantly strictly risk-averse may reject the necessarily difficult route to acceptance into one of the high-status occupations, or may pursue a path that may be beyond his capabilities, or require more application that he is prepared to bring to bear.

An important complicating effect is that prestigious professions normally also bring high income, which might itself be a strong motivator. Nevertheless, economic reasoning would suggest that such high income will at least partly result from the scarcity of people prepared or able to carry out the difficult tasks involved. A simple mathematical model would take the size of the pool of capable people as inversely proportional to the degree of difficulty and the income as inversely proportional to the number of people available to do the job. This would make the attendant income proportional to the degree of difficulty. Taking the motivator associated with salary to be proportional to income would make it proportional also to the degree of difficulty.

Summing the two motivators of status and income would mean that equation (2) would be replaced by $I_s = \left(1 + a_I\right) p_m$ where α_I is an income-related constant. Equation (4) would then have the form: $u_s = a_s p_m^{1-\varepsilon}$. Applying the principle of a "fair game according to his lights" to the person considering each of his possible choices of occupation then allows equation (10) and the analysis thereafter to stand.

Another point needs to be considered, however, namely whether the behaviour of people (probably mainly young) deciding on whether or not to aim for a high-status occupation can be modeled using values of risk-aversion similar to those applicable to the students taking part in the hoop-the-peg exercise. Specifically, it is open to question whether a negative risk-aversion may be used to characterize a choice expected to have a major effect on future lifestyle. On the other hand, two factors likely to have an effect on the student's chances of entering a high-status occupation were found by Atkinson and Litwin[8] to be well correlated with the results of the hoop-the-peg exercise: (i) the number of minutes the student spent working on the final examination held later for the course (nominally 3 hours long) and (ii) the score achieved in that exam. The students with the highest propensity to show a negative risk-aversion ($\alpha = 0.96$ for Group 1, Hi–Lo; see Figure 10) were found to stick with the exam longest and to come away with the highest marks. Persistence fell away in the order of the decline in propensity, α, shown in Figure 7, while the reduction in exam marks followed the downward order shown in Figure 8 for the propensity, α, over the groups. It may be concluded that the hoop-the-peg exercise, quantified here by the risk-aversion model, provides information that might be relevant to students' later employment. Those characterized by a high propensity, α, to show a negative risk-aversion in the hoop-the-peg exercise may well have a higher chance of moving on to high-status occupations later.

A major contribution of Atkinson's work, as interpreted by the risk-aversion model, is the demonstration that strict risk-aversion can bring unexpected consequences. Strict risk-aversion ($\varepsilon > 0$) can lead to behaviour almost certain to bring failure. Such a finding accords well with the work of Thomas et al.,[14] who used mathematical utility theory to show that the decision-maker's risk-aversion may increase to the point of indiscriminate decision, where he or she will be unable to distinguish amongst the three options of (i) reducing the likelihood of damage in an accident, (ii) increasing the probability of harm, or (iii) doing nothing. At this point the person taking the decision would be equally likely to adopt any of the three courses of action just set out. Therefore, decisions should obviously not be taken when experiencing such a high level of risk-aversion. The clarification that strictly risk-averse people can adopt harmful strategies also has an echo in Kahneman's and Tversky's work.[7]

The point made in the previous paragraph is nonobvious and ought to be of interest to those seeking to manage and regulate both financial risks and the safety of highly hazardous industrial plant.

9. Conclusions

The model based on risk-aversion that has been derived in this paper has been shown to explain well the behaviour of the 45 university students who took part in the hoop-the-peg exercise staged by Atkinson and Litwin. It explains why the most risk-averse people may sometimes take the biggest risks.

More cautious individuals will not necessarily display a risk-aversion that is consistently positive, but instead exhibit a mixture of positive and negative risk-aversions. What distinguishes the more risk-averse from the risk-confident is simply that the former have a higher propensity to demonstrate a positive (cautious) level of risk-aversion.

Interpreting the empirical work of Atkinson and Litwin using the risk-aversion-based model has demonstrated that becoming ever more risk-averse should not be regarded as an unalloyed good. In particular, high levels of risk-aversion can lead to high likelihoods of failure.

Appendix A. The zero-crossing points of the locus of expected utility, $z(\varepsilon, p_m)$, versus degree of difficulty, p_m

By inspection of equation (10), the expected utility for a task, $z(\varepsilon, p_m)$, will be zero at $p_m = 0$ and at $p_m = 1$ for any value of risk-aversion. For the degenerate, risk-neutral case where $\varepsilon = 0$, $z(\varepsilon, p_m) = 0$ for all p_m: $0 \le p_m \le 1$. But for all risk-aversions in the range $-2 < \varepsilon < 1$, $\varepsilon \ne 0$, $z(\varepsilon, p_m)$ will also take the value zero at two further intermediate values of degree of difficulty, which may be found by setting the right-hand side of equation (10) to zero. Applying this constraint, we may apply equation (9) and, providing $0 < p_m < 1$, divide by $(1 - p_m)\, p_m^{1-\varepsilon}$ to produce:

$$1 - a_m \left(p_m - p_m^2\right)^{\varepsilon} = 0\,, \quad 0 < p_m < 1\,. \tag{A.1}$$

Rearranging gives the standard quadratic form:

$$p_m^2 - p_m + a_m^{-\frac{1}{\varepsilon}} = 0\,, \quad 0 < p_m < 1, \tag{A.2}$$

which dictates that there can be just two solutions:

$$p_m = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 - 4a_m^{-\frac{1}{\varepsilon}}}\,. \tag{A.3}$$

The lower zero-crossing value, p_{m1}, is thus:

$$p_{m1} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4a_m^{-\frac{1}{\varepsilon}}}\,. \tag{A.4}$$

Meanwhile it is clear from equation (A.3) that the upper crossing value, p_{m2}, will be the complement of the first:

$$p_{m2} = 1 - p_{m1}\,. \tag{A.5}$$

The existence of these crossover points is evident in Figures 3 and 4.

The crossover probabilities may be plotted as a function of risk-aversion, ε, as shown in Figure 14. The curves are fairly flat in the range $-1 \le \varepsilon < 1$, but p_{m1} and p_{m2} tend to zero and unity, respectively, as the risk-aversion tends towards the most risk-confident value of –2.0.

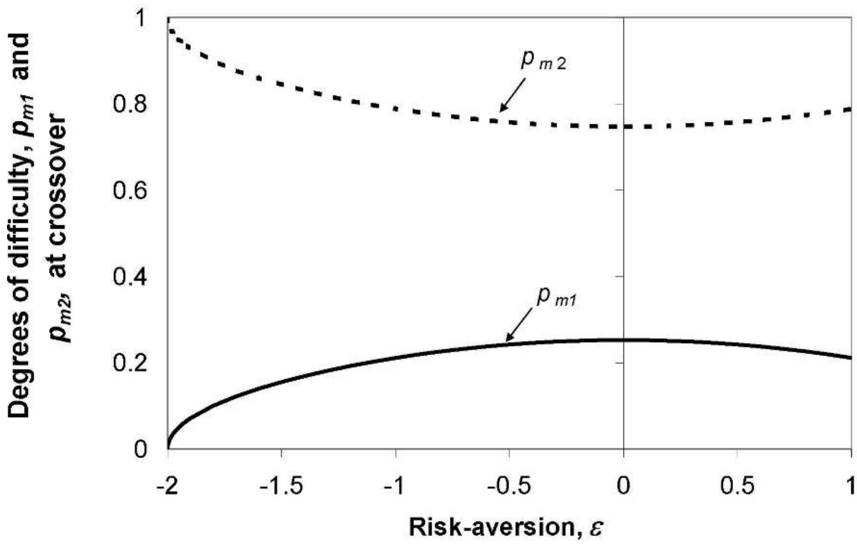

Figure 14. Crossover probabilities, p_{m1} and p_{m2}, versus risk-aversion, ε.

Further information on the locus of expected task utility, $z(\varepsilon, p_m)$, with degree of difficulty, p_m, may be found by taking the partial derivative:

$$\frac{\partial z}{\partial p_m}(\varepsilon, p_m) = p_m^{-\varepsilon}\left((1-\varepsilon)(1-p_m) - p_m\right) - a_m(1-p_m)^{\varepsilon}\left((1-p_m) - (1+\varepsilon)p_m\right). \tag{A.6}$$

Bearing in mind equation (A.5), we may write the partial differential at $p_m = p_{m1}$ as

$$\frac{\partial z}{\partial p_m}(\varepsilon, p_{m1}) = p_{m1}^{-\varepsilon}\left((1-\varepsilon)p_{m2} - p_{m1}\right) - a_m p_{m2}^{\varepsilon}\left(p_{m2} - (1+\varepsilon)p_{m1}\right) \tag{A.7}$$

and the partial differential at $p_m = p_{m2}$ as

$$\frac{\partial z}{\partial p_m}(\varepsilon, p_{m2}) = p_{m2}^{-\varepsilon}\left((1-\varepsilon)p_{m1} - p_{m2}\right) - a_m p_{m1}^{\varepsilon}\left(p_{m1} - (1+\varepsilon)p_{m2}\right). \tag{A.8}$$

The complementary nature of p_{m1} and p_{m2} (equation A.5) explains why equation (A.8) can be derived from equation (A.7) simply by interchanging the subscripts $m1$ and $m2$. Figure 15 plots the two curves corresponding to equations (A.7) and (A.8) as functions of risk-aversion, ε.

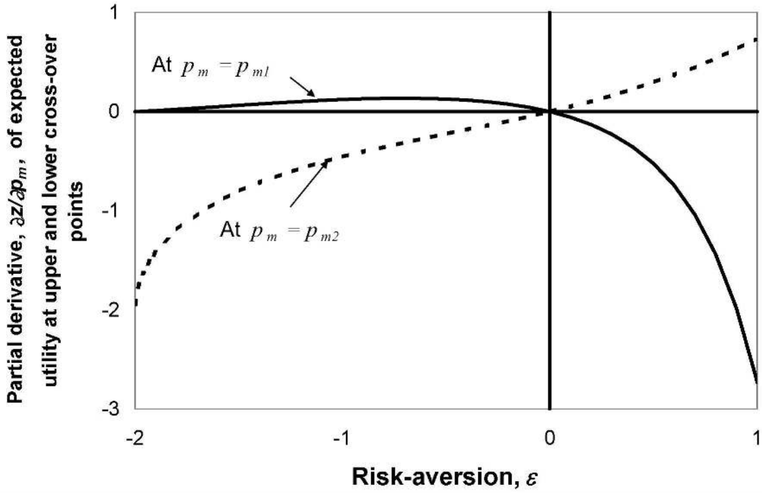

Figure 15. Partial derivatives, $\partial z\left(\varepsilon, p_m\right)/\partial p_m$, of expected task utility at the two zero-crossing points, $p_m = p_{m1}, p_{m2}$, plotted against the subject's risk-aversion.

It is clear from the figure that

$$\left.\begin{aligned} \frac{\partial z}{\partial p_m}\left(\varepsilon, p_{m1}\right) < 0 \\ \frac{\partial z}{\partial p_m}\left(\varepsilon, p_{m2}\right) > 0 \end{aligned}\right\} \text{ when } \varepsilon > 0. \tag{A.9}$$

Conversely,

$$\left.\begin{aligned} \frac{\partial z}{\partial p_m}\left(\varepsilon, p_{m1}\right) > 0 \\ \frac{\partial z}{\partial p_m}\left(\varepsilon, p_{m2}\right) < 0 \end{aligned}\right\} \text{ when } \varepsilon < 0, \tag{A.10}$$

but it has been established that the expected task utility, $z(\varepsilon, p_m)$, is zero both when $p_m = 0.0$ and when $p_m = 1.0$, irrespective of the value of risk-aversion. Moreover, for $\varepsilon \neq 0$, the locus of $z(\varepsilon, p_m)$ possesses only two intermediate, zero, crossings, which occur at $p_m = p_{m1}$ and $p_m = p_{m2}$. This means that the sign of the slope, $\partial z/\partial p_m\big|_{pm1,pm2}$, at the crossover points will determine whether the locus of the curve of $z(\varepsilon, p_m)$ v. p_m up to that point has lain either wholly in the positive or wholly in the negative quadrant. If the slope is negative then the preceding locus will have been positive; if the slope is positive, the locus of z in the interval up to that point will have been negative.

Taking the case of positive risk-aversions, $\varepsilon > 0$, the slope at p_{m1} is negative, which implies that z must have been positive over the range $0 < p_m < p_{m1}$. However, the slope at p_{m2} is positive, which means that z must have been negative for $p_{m1} < p_m < p_{m2}$. Moreover, z must necessarily become positive for the range $p_{m2} < p_m < 1.0$.

The situation is reversed for negative risk-aversions $\varepsilon < 0$. Now the slope at p_{m1} is positive, hence z must have been negative over the range: $0 < p_m < p_{m1}$. But the slope at p_{m2} is negative, which requires z to have been positive over the range $p_{m1} < p_m < p_{m2}$. Furthermore, z must necessarily become and remain negative over the range: $p_{m2} < p_m < 1.0$.

Appendix B. Derivation of the probability density, $h\left(p_m \middle| \varepsilon\right)$, for someone with risk-aversion ε, selecting a task of degree of difficulty p_m

A fundamental requirement for a probability density is that its integral over all possible values of its argument must equal unity. Since the probability density, $h\left(p_m \middle| \varepsilon\right)$, for choosing a task of difficulty, p_m, is taken to be proportional to positive values of expected task utility, $z(\varepsilon, p_m)$, and to be zero elsewhere, this may be achieved by setting

$$h\left(p_m \middle| \varepsilon\right) = \frac{\max\left(0, z\left(\varepsilon, p_m\right)\right)}{\int\limits_{p_m=0}^{1} \max\left(0, z\left(\varepsilon, p_m\right)\right) dp_m}. \tag{B.1}$$

It has been demonstrated in §4 and in Appendix A that $z(\varepsilon, p_m) < 0$ for some regions of p_m, as determined by the value of risk-aversion. Hence the max function in the integral in the denominator of equation (B.1) may be replaced as indicated in equation (B.2) below:

$$h\left(p_m \middle| \varepsilon\right) = \begin{cases} \dfrac{\max\left(0, z\left(\varepsilon, p_m\right)\right)}{\int\limits_{p_m=p_{m1}}^{p_{m2}} z\left(\varepsilon, p_m\right) dp_m} & \text{for } \varepsilon < 0 \\ \dfrac{\max\left(0, z\left(\varepsilon, p_m\right)\right)}{\int\limits_{p_m=0}^{p_{m1}} z\left(\varepsilon, p_m\right) dp_m + \int\limits_{p_m=p_{m2}}^{1} z\left(\varepsilon, p_m\right) dp_m} & \text{for } \varepsilon > 0 \end{cases}. \tag{B.2}$$

Using equation (10), the indefinite integral may be written

$$\int z\left(\varepsilon, p_m\right) dp_m = \frac{p_m^{2-\varepsilon}}{2-\varepsilon} - \frac{p_m^{3-\varepsilon}}{3-\varepsilon} + \frac{(2+\varepsilon)(3+\varepsilon)}{(2-\varepsilon)(3-\varepsilon)} \frac{\left(1-p_m\right)^{2+\varepsilon}\left(\left(2+\varepsilon\right)p_m + 1\right)}{(2+\varepsilon)(3+\varepsilon)} + const =$$
$$\frac{1}{(2-\varepsilon)(3-\varepsilon)} \begin{pmatrix} (3-\varepsilon)p_m^{2-\varepsilon} - (2-\varepsilon)p_m^{3-\varepsilon} \\ +\left(1-p_m\right)^{2+\varepsilon}\left(\left(2+\varepsilon\right)p_m + 1\right) \end{pmatrix} + const. \tag{B.3}$$

Hence the definite integral between p_{m1} and p_{m2} may be evaluated as:

$$\int\limits_{p_{m1}}^{p_{m2}} z\left(\varepsilon, p_m\right) dp_m = \frac{1}{(2-\varepsilon)(3-\varepsilon)} \begin{pmatrix} (3-\varepsilon)\left(p_{m2}^{2-\varepsilon} - p_{m1}^{2-\varepsilon}\right) - (2-\varepsilon)\left(p_{m2}^{3-\varepsilon} - p_{m1}^{3-\varepsilon}\right) \\ +\left(1-p_{m2}\right)^{2+\varepsilon}\left(\left(2+\varepsilon\right)p_{m2} + 1\right) \\ -\left(1-p_{m1}\right)^{2+\varepsilon}\left(\left(2+\varepsilon\right)p_{m1} + 1\right) \end{pmatrix}. \tag{B.4}$$

Meanwhile the definite integral between 0 and p_{m1} may be seen to be

$$\int_{p_m=0}^{p_{m1}} z(\varepsilon, p_m)\,dp_m = \frac{1}{(2-\varepsilon)(3-\varepsilon)}\begin{pmatrix}(3-\varepsilon)p_{m1}^{2-\varepsilon}-(2-\varepsilon)p_{m1}^{3-\varepsilon}\\ +(1-p_{m1})^{2+\varepsilon}\left((2+\varepsilon)p_{m1}+1\right)-1\end{pmatrix} \tag{B.5}$$

and that between p_{m2} and 1.0 is:

$$\int_{p_m=p_{m2}}^{1} z(\varepsilon, p_m)\,dp_m = \frac{1}{(2-\varepsilon)(3-\varepsilon)}\left\{1-\begin{pmatrix}(3-\varepsilon)p_{m2}^{2-\varepsilon}-(2-\varepsilon)p_{m2}^{3-\varepsilon}\\ +(1-p_{m2})^{2+\varepsilon}\left((2+\varepsilon)p_{m2}+1\right)\end{pmatrix}\right\}. \tag{B.6}$$

Adding equations (B.5) and (B.6) gives, after noting the complementarity equation (A.5):

$$\begin{pmatrix}\displaystyle\int_{p_m=0}^{p_{m1}} z(\varepsilon, p_m)\,dp_m\\ \displaystyle+\int_{p_m=p_{m2}}^{1} z(\varepsilon, p_m)\,dp_m\end{pmatrix} = \frac{1}{(2-\varepsilon)(3-\varepsilon)}\begin{pmatrix}(3-\varepsilon)\left(p_{m1}^{2-\varepsilon}-p_{m2}^{2-\varepsilon}\right)\\ -(2-\varepsilon)\left(p_{m1}^{3-\varepsilon}-p_{m2}^{3-\varepsilon}\right)\\ +p_{m2}^{2+\varepsilon}\left((2+\varepsilon)p_{m1}+1\right)\\ -p_{m1}^{2+\varepsilon}\left((2+\varepsilon)p_{m2}+1\right)\end{pmatrix}. \tag{B.7}$$

Putting

$$D = -\begin{pmatrix}(3-\varepsilon)\left(p_{m1}^{2-\varepsilon}-p_{m2}^{2-\varepsilon}\right)-(2-\varepsilon)\left(p_{m1}^{3-\varepsilon}-p_{m2}^{3-\varepsilon}\right)\\ +p_{m2}^{2+\varepsilon}\left((2+\varepsilon)p_{m1}+1\right)-p_{m1}^{2+\varepsilon}\left((2+\varepsilon)p_{m2}+1\right)\end{pmatrix} \tag{B.8}$$

allows equation (B.7) to be recast as:

$$\int_{p_m=0}^{p_{m1}} z(\varepsilon, p_m)\,dp_m + \int_{p_m=p_{m2}}^{1} z(\varepsilon, p_m)\,dp_m = \frac{-D}{(2-\varepsilon)(3-\varepsilon)}. \tag{B.9}$$

Similarly, after making use of the complementarity equation (A.5) once more, equation (B.4) may be rewritten:

$$\int_{p_{m1}}^{p_{m2}} z(\varepsilon, p_m)\,dp_m = \frac{D}{(2-\varepsilon)(3-\varepsilon)}. \tag{B.10}$$

A comparison of equations (B.9) and (B.10) shows that the areas above and below the p_m-axis are equal for any value of risk-aversion:

$$\int_{p_m=p_{m1}}^{p_{m2}} z(\varepsilon, p_m)\,dp_m = -\left(\int_{p_m=0}^{p_{m1}} z(\varepsilon, p_m)\,dp_m + \int_{p_m=p_{m2}}^{1} z(\varepsilon, p_m)\,dp_m\right). \tag{B.11}$$

Substituting from equations (10), (B.9) and (B.10) into equation set (B.2) gives the required probability density for selecting a task of given difficulty given the person has risk-aversion ε:

$$h(p_m|\varepsilon) = \frac{1}{D}\begin{cases}\max\begin{pmatrix}0,(2-\varepsilon)(3-\varepsilon)(1-p_m)p_m^{1-\varepsilon}\\ -(2+\varepsilon)(3+\varepsilon)p_m(1-p_m)^{1+\varepsilon}\end{pmatrix} & \text{for } \varepsilon<0\\ \max\begin{pmatrix}0,(2+\varepsilon)(3+\varepsilon)p_m(1-p_m)^{1+\varepsilon}\\ -(2-\varepsilon)(3-\varepsilon)(1-p_m)p_m^{1-\varepsilon}\end{pmatrix} & \text{for } \varepsilon>0\,.\end{cases} \tag{B.12}$$