Why the most risk-averse take the biggest risks: a quantitative reanalysis of Atkinson's and Litwin's "hoop-the-peg" experiments. Part 2: Establishing the raw data

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The psychologists Atkinson and Litwin set out to explore the tendency for anxious people to set their aspirations either very low or, apparently paradoxically, extremely high. They designed an experiment around the hoop-the-peg game. Their results, based on the observed behaviour of male university students, provide quantitative data on the tendency of more anxious people to select either very safe or very risky options. Unfortunately Atkinson's and Litwin's results were presented in filtered form, with the definition of the filter not fully specified. This paper will show that a recursive rolling average was almost certainly used. Once the form of the filter is established, the unfiltered, raw numbers of shots taken from each throwing position can be back-calculated. Arguments are presented for the players' subjectively assessed probability of failure being a linear function of throwing distance. The probability so derived may be used in conjunction with the raw shots data to produce probability distributions for selecting a task of given degree of difficulty. These results feed directly into the immediately preceding Part 1, where the concept of risk-aversion is used in a mathematical description of the hoop-the-peg exercise. Part 1 tests and validates the model against the empirical probability distributions found here.

Keywords: hoop-the-peg, psychological experiment, psychological measurement, risk-aversion

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Nanotechnology Perceptions **17** (2021) 229–250
Received 17 November 2019; *accepted* 30 November 2021

doi: 10.4024/N25TH19A.ntp.17.03

1. Introduction

Atkinson and Litwin[1] arranged for 45 male university students to take part in a specially designed game of hoop-the-peg, in which each participant could choose his own throwing position according to his own preference for difficulty. Lines were drawn at 1 foot intervals from the target, starting at 1 foot and ending at 15 feet from the peg.

Atkinson and Litwin divided up their cohort into four groups based on prior, independent measurements of the subjects' desire to achieve and their level of anxiety. French's Insight Test, where the subject is asked to provide an explanation for the documented behaviour of fictitious characters, was used to measure nAchievement, while Test Anxiety scores were obtained from a Mandler–Sarason Test Anxiety Questionnaire. (Further details and references are provided in the source paper.[1]) The four groups were:

1. Group 1, high desire to achieve and low level of anxiety: "Hi–Lo" (13 members)
2. Group 2, high desire to achieve and high level of anxiety: "Hi–Hi" (10 members)
3. Group 3, low desire to achieve and low level of anxiety: "Lo–Lo" (9 members)
4. Group 4, low desire to achieve and high level of anxiety: "Lo–Hi" (13 members).

Each participant was asked to take 10 throws, and the pitching positions were recorded.

Unfortunately Atkinson's and Litwin's paper did not tabulate the numbers of shots taken from each distance from the target. Instead their most comprehensive set of data was given in graphical form in their Figure 1. But the figure showed positional results only after they had passed through a filter for which no precise description was given. Aggregated measurements were also given in their Table 1, although the authors did not make it clear whether or not they had passed through the same filter.

The uncertainty associated with the way the experimental records were presented made it a first priority to establish the form of the filter most likely to have been used. A back-calculation from the graphical output could then determine the numbers of throws ("shots") taken from each distance. This would, *inter alia*, allow data held in Atkinson's and Litwin's Table 1 to be compared and reconciled with their graphically presented results.

A restricted set of measured probabilities of failure associated with each throwing distance was provided by Atkinson and Litwin. These measurements could be used as a check on the throwers' likely subjective judgments of the probability of missing the target, or "degree of difficulty", from each of the throwing positions. The probability of choosing such a degree of difficulty could then be judged from the relative frequency of raw shots taken from that position.

These topics form the focus for this paper. The results feed into the immediately preceding Part 1, where models for the probability distributions for degree of difficulty are derived and compared against the corresponding distributions based on observations.

The layout of Part 2 is as follows. Following this Introduction, §2 explains the exercise set up by Atkinson and Litwin and introduces the filtered graphical data that contain their results. §3 defines three candidate filters, while §4 selects the filter that performs best and provides optimal estimates of the raw numbers of shots based on the filtered results contained in Atkinson's and Litwin's Figure 1. §5 explores the data contained in Atkinson's and Litwin's

[1] Atkinson, J.W. and Litwin, G.H. Achievement motive and test anxiety conceived as motive to approach success and motive to avoid failure. *Journal of Abnormal and Social Psychology* **60** (1960) 52–63.

Table 1 and reconciles them with their Figure 1. §6 estimates the degree of difficulty of a shot taken from a given distance, as judged by the thrower. A probability distribution is then advanced for the players' choice of degree of difficulty. Conclusions for Part 2 are given in §7.

Mathematical details are covered in appendices. Appendix A demonstrates that the sum of the filtered number of shots from all pitching positions must be less than the sum of the raw shots for the recursive rolling average filter. Appendix B rationalizes the tabular data provided by Atkinson and Litwin in their Table 1. Appendix C provides a mathematical proof that the estimated average absolute error, ε_k, between model predictions and experimental observations will always be less than or equal to the actual average absolute error, E_k, if some of the records are based on sums of measurements.

2. The exercise devised by Atkinson and Litwin and the graphical data provided in their Figure 1

Atkinson and Litwin were interested in how the subjects would choose their throwing distances, and they presented the measurements graphically in the first figure in their paper. Their Figure 1 gives a "smoothed" or filtered version of the percentage of shots made by each of the four groups at discrete, one-foot intervals from 1 to 15 feet from the peg.

The data contained in Atkinson's and Litwin's Figure 1 are reproduced in Figure 1 of this article. The smoothed percentages depicted there were measured from a printed copy of Atkinson's and Litwin's paper using a standard office ruler. The measurements so obtained may be taken to be subject to an inaccuracy of approximately half the smallest division, viz. ±0.5 mm. This implies an estimated maximum error of ±0.18 of a percentage point on each of the percentages plotted in Figure 1. The filtered percentages were then converted into a filtered number of shots at each distance through multiplying by the number of shots taken by each group (130 for both Hi–Lo and Lo–Hi, 100 for Hi–Hi and 90 for Lo–Lo) and then dividing by 100. A tolerance of between ±0.16 shots and ±0.23 shots is thus associated with the filtered number of shots calculated for each distance in the present paper's Figure 1.

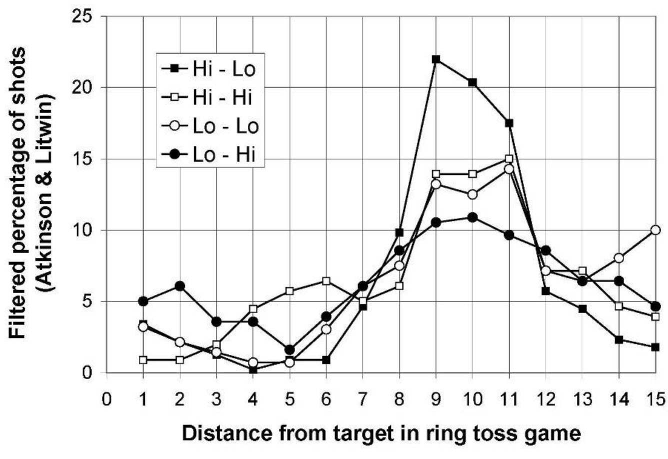

Figure 1. Atkinson & Litwin's Figure 1 showing filtered percentage of shots made in each of the four groups (Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi) against the distance/feet from which the shot was taken.

It is obviously desirable in the interests of accuracy to work with the raw measurements, but the curves contained in Figure 1 illustrate data that has been filtered using a method for which we have only Atkinson's and Litwin's description:

"The curves are smoothed by the method of running averages."

This suggests an assumption that the filter will be well known to the readers of the paper. We may also deduce that it is likely to be simple. However, since Atkinson and Litwin do not actually specify the algorithm they used, it necessary to assess possible filters.

3. Candidate filters

Using the deduction of probable simplicity, it is assumed that the filter is linear, which means it may be applied as well to the raw shots as to the percentages. The value of first data point (the number of throws taken from 1 foot) for each of the groups may then be used to provide a clue to the filter's mathematical form. There are four such data points, one per group, as given in Table 1. It is clear that they are or are very close to being integer multiples of 0.5, which would be consistent with the filter used by Atkinson and Litwin assigning a notional number of zero to the shots taken from 0 feet, adding this to the number recorded from 1 feet and then dividing the sum by two to find an arithmetic average. The first 3 values in the top row of Table 1 may be converted into exact numbers of halves by the addition of 0.09 or 0.11, small additions that lie well within the measurement tolerances estimated in Section 2. The fourth value, 6.5, is already a multiple of 0.5 and hence needs no addition, of course.

Table 1. Estimated filtered number of shots taken from a distance of 1 foot for the four groups.

Group	Hi–Lo	Hi–Hi	Lo–Lo	Lo–Hi
Filtered number of shots taken from 1 foot away	4.41	0.89	2.89	6.50
Adjusted filtered number	4.50	1.00	3.00	6.50
Amount of adjustment	0.09	0.11	0.11	0.00
Maximum permissible adjustment	0.23	0.18	0.16	0.23

Three simple linear filters would conform with this procedure

- the cumulative average
- the rolling average over two data points
- the "recursive rolling average" to be defined below.

Mathematical descriptions are given in the next three subsections for each filter. where it is assumed that a_{fk}, $f = 0,1,..., F$ is the number of shots taken by group k from a distance of f feet from the target, where the final distance is $F = 15$. The groups are assigned the numbers 1, 2, 3 and 4, with $k = 1$ corresponding to group Hi–Lo, $k = 2$ denoting Hi–Hi, $k = 3$ signifying Lo–Lo, and $k = 4$ identifying Lo–Hi. Meanwhile the filtered number of shots made by group k at distance f is denoted c_{fk}. For each group, the zeroth output of the filter is taken to be zero:

$$c_{0,k} = a_{0,k} = 0\,. \tag{1}$$

The a_{fk} must also satisfy the constraint of taking a non-negative integer value for all $f : f = 0,1,..., F$ and $k = 1,2,3,4$.

3.1 Candidate filter 1: the cumulative average

The equation for the cumulative average is:

$$c_{fk} = \frac{a_{fk} + fc_{f-1,k}}{f+1} \quad \text{for } 1 \le f \le F. \tag{2}$$

Thus the estimated raw number, $\hat{a}_{fk}$, of shots at distance *f* may be found as:

$$\hat{a}_{fk} = (f+1)c_{fk} - fc_{f-1,k} \quad \text{for } 1 \le f \le F. \tag{3}$$

Hence, combining equations (1) and (3), $\hat{a}_{1k} = 2c_{1k}$. The integer requirement on a_{1k} will be fulfilled as long as the filtered value c_{1k}, $k = 1,2,3,4$ is an integer multiple of a half. From Table 1, after allowing for measurement error, this can be taken to be the case.

3.2 Candidate filter 2: the rolling average

The rolling average is given by:

$$c_{fk} = \frac{a_{fk} + a_{f-1,k}}{2} \quad \text{for } 1 \le f \le F. \tag{4}$$

Thus the estimated raw number of shots at distance *f* may be found as:

$$\hat{a}_{fk} = 2c_{fk} - a_{f-1,k} \quad \text{for } 1 \le f \le F. \tag{5}$$

Combining equations (1) and (5) gives the number of shots made by group *k* from a distance 1 foot as $\hat{a}_{1k} = 2c_{1k}$, the same as for the cumulative average. The integer requirement on a_{1k} will be fulfilled, under the constraint discussed at the end of the previous subsection.

3.3 Candidate filter 3: the recursive rolling average

The recursive rolling average is a variant on the rolling average, which is consistent with the view that the best estimate of the number of shots at the previous throwing distance is the filtered estimate. It is described by:

$$c_{fk} = \frac{a_{fk} + c_{f-1,k}}{2} \quad \text{for } 1 \le f \le F. \tag{6}$$

Thus the estimated raw number of shots at distance *f* may be found as:

$$\hat{a}_{fk} = 2c_{fk} - c_{f-1,k} \quad \text{for } 1 \le f \le F. \tag{7}$$

From equations (1) and (7) the number of shots made by group *k* from 1 foot is given by $\hat{a}_{1k} = 2c_{1k}$, the same as found for the previous two filters considered. Thus this filter will fulfil the integer requirement on $\hat{a}_{fk}$ once again, provided that c_{1k} is an integer multiple of 0.5.

4. Comparison of the filters: raw numbers of shots derived from Atkinson's and Litwin's Figure 1

It will be shown in this section that the recursive rolling average outperforms the other two filters by a substantial margin.

The estimates of the numbers of shots, $\hat{a}_{fk}$, $f : f = 0,1,...,F$; k = 1,2,3,4, for distance, f, and group, k, are subject to two constraints. The first has been mentioned previously, namely that each a_{fk} should be a non-negative integer. But there is an additional requirement, namely that the sum of all the group's shots should match the group's known total of throws, N_k:

$$\sum_{f=1}^{F} a_{fk} = N_k \tag{8}$$

where $N_1 = 13 \times 10 = 130$, $N_2 = 10 \times 10 = 100$, $N_3 = 9 \times 10 = 90$, $N_4 = 13 \times 10 = 130$.

While the application of the back-estimation process for the three filters, viz. equations (3), (5) and (7), will not, in general, produce shot numbers $\hat{a}_{fk}$ that conform precisely to the two constraints, nevertheless the closer that they come to conformance the greater must be the confidence that the corresponding filter is the one actually used by Atkinson and Litwin.

Table 2 compares the back-estimated figures against the corresponding numbers associated with the Atkinson and Litwin filter. The first numerical column contains the estimated raw total of shots covering the range 1 foot to 15 feet, found after applying equations (3), (5) and (7) and then summing the results: $\sum_{f=1}^{15} \hat{a}_{fk}$, k = 1,...,4. It is clear that the recursive rolling average performs best, producing the total of estimated shots that comes closest to the actual number for all four groups. For group 1, Hi–Lo, the cumulative average performed particularly badly, producing a sum of 37.13 as opposed to the 130 desired. This anomalously low figure was occasioned by the presence of no less than eight negative estimated values, the most egregious of which was –176.36 shots. The rolling average chalked up five negative values, including one of –32.86 and another of –27.29, but the recursive rolling average produced only two, and these were of significantly smaller magnitude: –1.07 and –7.89.

The second column of numbers is found by replacing negative values of $\hat{a}_{fk}$ by zero and rounding all the other values to integers. The elimination of the negative numbers can cause the totals in the second column to exceed those in the first column, sometimes by a large margin. Most strikingly, replacing by zero the large number of high-magnitude negatives associated with the cumulative average for Group 1, Hi–Lo, boosts the sum of estimated shots to 300, more than twice the actual number of 130 throws. But once again the recursive rolling average gets substantially closer to the actual figures than the other filters, an outcome that applies across all four groups.

The third column of figures comprises the total number of shots taken by each group for the three candidate filters after applying equations (2), (4) and (6) to the non-negative, rounded estimates of the number of shots at each distance. These summations are contrasted against those resulting from the Atkinson and Litwin filter, as calculated from Figure 1. It is clear from Table 2 that the recursive rolling average is once again the best performer out of the three filters.

The final column gives the root mean squared (rms) differences between the values $\hat{a}_{fk}$ of shots at each distance f as produced by each of the three candidate filters and the corresponding filtered value c_{fk} recorded for the Atkinson and Litwin filter. Once again the recursive rolling average performs much better than its two rivals. The low values in the final column associated with the recursive rolling average indicate that it should provide a close simulation of the performance of the Atkinson and Litwin filter.

Table 2. Comparing the performance of each of the possible filters in predicting the filtered number of shots taken within each group against the Atkinson & Litwin filter.

	Calculated total number of raw shots, equations (3), (5), (7), $\hat{N}_k^{(1)}$	Total number of shots after replacing negatives with zero and rounding to integers, $\hat{N}_k^{(2)}$	Total number of shots after filtering, equations (2),(4),(6)	Rms error between the outputs of the specified filter and those of the Atkinson and Litwin filter
Group 1: Hi–Lo ((Total number of shots in the group = 130)				
Cumulative average	37.13	300	304.56	36.98
Rolling average	145.26	220	201.50	7.93
Recursive rolling average	128.83	136	133.09	1.15
Atkinson & Litwin filter			126.51	0.00
Group 2. Hi–Hi (Total number of shots in the group = 100)				
Cumulative average	62.83	193	195.87	17.81
Rolling average	107.10	126	116.00	2.51
Recursive rolling average	101.03	102	98.19	0.13
Atkinson & Litwin filter			97.10	0.00
Group 3. Lo–Lo (Total number of shots in the group = 90)				
Cumulative average	143.94	221	189.95	14.61
Rolling average	99.60	122	109.00	2.43
Recursive rolling average	95.75	97	87.83	0.14
Atkinson & Litwin filter			86.75	0.00
Group 4. Lo–Hi (Total number of shots in the group = 130)				
Cumulative average	96.53	180	184.96	11.13
Rolling average	123.45	123	123.00	0.21
Recursive rolling average	130.18	131	124.84	0.16
Atkinson & Litwin filter			124.15	0.00

In an additional step, the estimated total of raw shots after rounding, back-calculated from the recursive rolling average filter, was made to equal the sum of shots taken for each of groups 1 to 4 by:

(i) distributing the rounded value of any negative $\hat{a}_{fk}$ by reducing the preceding and succeeding values, $\hat{a}_{f-1,k}$ and $\hat{a}_{f+1,k}$, accordingly;

(ii) adjusting the remaining $\hat{a}_{fk}$ by a maximum of 1 shot at each distance f to allow the satisfaction of $\sum_{f=1}^{15} \hat{a}_{fk} = N_f$.

Table 3 gives the resulting final estimate of the raw numbers of shots taken from each distance. Figure 2 shows them expressed as a percentage of the throws of each group to allow comparisons with Figure 1. It is noteworthy that lopped and broadened peaks are evident in Figure 1 as opposed to the more jagged curves seen in Figure 2. The reduction in sharpness between Figure 2 and Figure 1 is typical of the interposition of a linear filter.

Figure 2 illustrates that the students taking part in the exercise were displaying the elements of behaviour identified by Atkinson in his 1957 paper, namely throws clustering around a central value but also at the extremes of the range. All four groups exhibited a variant of this

Table 3. Best estimate of the actual number of shots taken at each distance based on the conclusion that Atkinson & Litwin used recursive rolling averaging as defined in §3.3.

	Distance/feet															
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
Group																Total
1. Hi–Lo	9	1	0	0	1	1	11	20	45	25	13	0	2	0	2	130
2. Hi–Hi	2	1	3	7	7	7	4	7	22	14	15	0	6	2	3	100
3. Lo–Lo	5	1	1	0	0	5	8	8	16	10	13	0	5	8	10	90
4. Lo–Hi	13	9	1	5	0	7	11	14	16	15	11	10	5	9	4	130
Total	29	12	5	12	8	20	34	49	99	64	52	10	18	19	19	450

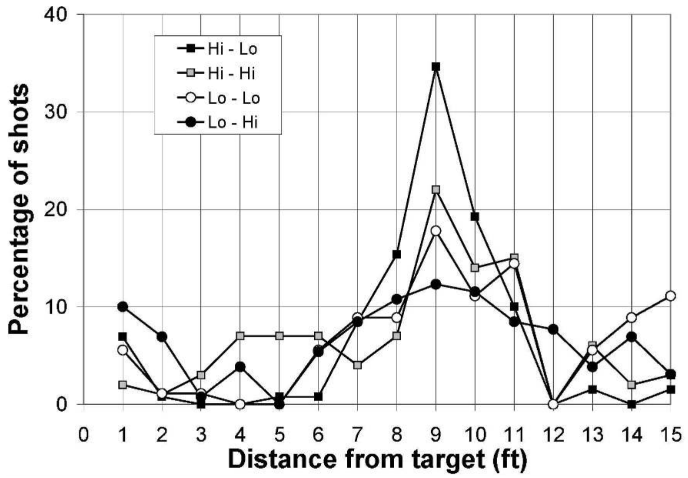

Figure 2. Estimate of the percentage of raw shots made in each of the four groups (Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi) against the distance from which the shot was taken.

approximately trimodal behaviour, although those in Group 1, Hi–Lo, tended to take their throws from central throwing positions more often than those in the other groups.

The integer numbers of shots given in Table 3 were tested further by passing them through the recursive rolling average filter and comparing the results graphically with the output of the Atkinson and Litwin filter. Figure 3 demonstrates a near match between the respective curves for all four groups. It confirms the hypothesis that the filter employed by Atkinson and Litwin was a recursive rolling average. The close correspondence provides validation for the estimates of the raw number of shots from the various throwing positions contained in Table 3.

The action of recursive rolling averaging is analysed mathematically in Appendix A, where it is shown that the sum of the filtered values will always be less than the total of actual shots:

$$\sum_{f=1}^{F} c_{fk} < \sum_{f=1}^{F} a_{fk} \tag{9}$$

for any finite end distance F. Very similar behaviour is displayed by the Atkinson and Litwin filter, providing further corroboration that the two filters, that of Atkinson and Litwin and the

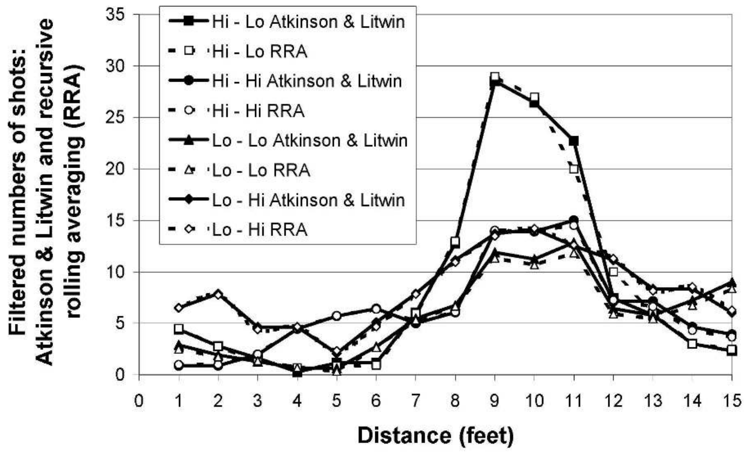

Figure 3. Filtered number of shots made in each of the four groups (Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi) against the distance from which the shot was taken: Atkinson & Litwin filter compared with recursive rolling averaging.

recursive rolling average, are equivalent (see Table 4). It may be concluded that, in all probability, Atkinson and Litwin filtered their raw results using a recursive rolling average, as defined mathematically in §3.3.

Table 4. Comparison of the sum C_k of the filtered estimates of the shots at each distance with the actual total number of shots. Notice that both the recursive rolling average filter and the Atkinson & Litwin filter both produce sum totals below the actual number of shots taken by each group.

	Group			
	1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi
Actual total number of shots	130	100	90	130
Sum of filtered estimates C_k				
Recursive rolling average	127.50	96.34	81.64	123.71
Atkinson & Litwin filter	126.51	97.10	86.75	124.15

5. The data contained Atkinson's and Litwin's Table 1

In addition to providing their Figure 1, Atkinson and Litwin presented in their Table 1 a set of cumulative percentages of shots taken over ranges of distances from the target. A full disaggregation of the data is not possible, but the initial 10 data values may be reduced to 7 nonoverlapping records, which consist of the following 3 percentages and 4 contiguous cumulative percentages:

$$\sum_{f=1}^{5} p_{fk}, \sum_{f=6}^{7} p_{fk}, p_{8,k}, \sum_{f=9}^{10} p_{fk}, p_{11,k}, p_{12,k}, \sum_{f=13}^{15} p_{fk} \,. \tag{10}$$

Here p_{fk} is the percentage of shots taken from distance f by the members of group k, as given by Atkinson's and Litwin's Table 1 and, in extended form, in Table 5 of the present paper. The process of disentanglement, which allows Table 5 to be deduced from Atkinson's and Litwin's Table 1, is described in Appendix B.

Table 5. Probabilities implied by Atkinson's and Litwin's Table 1 (probabilities expressed as percentages).

	Group				
	1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi	All groups combined
$\sum_{f=1}^{5} p_{fk}$	8%	15%	9%	21%	14%
$\sum_{f=6}^{7} p_{fk}$	3%	11%	9%	11%	8%
$p_{8,k}$	11%	5%	10%	7%	8%
$\sum_{f=9}^{10} p_{fk}$	56%	39%	30%	26%	38%
$p_{11,k}$	6%	4%	8%	8%	7%
$p_{12,k}$	9%	12%	10%	8%	10%
$\sum_{f=13}^{15} p_{fk}$	7%	14%	24%	20%	16%
$\sum_{f=1}^{15} p_{fk}$	100%	100%	100%	101%	100%
No of shots	130	100	90	130	450
Mean distance/feet	9.36	9.12	9.95	9.03	9.33
Median distance/feet	9.50	9.49	9.73	9.42	9.52

The match between Atkinson's and Litwin's tabulated and graphical data was assessed by plotting the numbers contained in Table 5 against the corresponding probabilities found from Figure 1 on a group-by-group basis. In addition, comparisons were made (i) with the percentages found after applying the recursive rolling average filter to the estimated numbers of raw shots and (ii) with the percentages found directly from the relative frequencies of the estimated raw shots themselves. The criterion for goodness of fit was the estimated average absolute error, ε_k, for each group k defined as:

$$\varepsilon_k = \frac{1}{15}\left\{\begin{array}{l}\left|\sum_{f=1}^{5} p_{fk} - \sum_{f=1}^{5} \hat{p}_{fk}\right| + \left|\sum_{f=6}^{7} p_{fk} - \sum_{f=6}^{7} \hat{p}_{fk}\right| \\ + \left|p_{8,k} - \hat{p}_{8,k}\right| + \left|\sum_{f=9}^{10} p_{fk} - \sum_{f=9}^{10} \hat{p}_{fk}\right| + \left|p_{11,k} - \hat{p}_{11,k}\right| \\ + \left|p_{12,k} - \hat{p}_{12,k}\right| + \left|\sum_{f=13}^{15} p_{fk} - \sum_{f=13}^{15} \hat{p}_{fk}\right|\end{array}\right\} \text{ for } k = 1,2,3,4. \qquad (11)$$

Combining values within the same summation rationalizes equation (11) to:

$$\varepsilon_k = \frac{1}{15}\left\{\begin{array}{l}\left|\sum_{f=1}^{5}\left(p_{fk} - \hat{p}_{fk}\right)\right| + \left|\sum_{f=6}^{7}\left(p_{fk} - \hat{p}_{fk}\right)\right| \\ + \left|p_{8,k} - \hat{p}_{8,k}\right| + \left|\sum_{f=9}^{10}\left(p_{fk} - \hat{p}_{fk}\right)\right| + \left|p_{11,k} - \hat{p}_{11,k}\right| \\ + \left|p_{12,k} - \hat{p}_{12,k}\right| + \left|\sum_{f=13}^{15}\left(p_{fk} - \hat{p}_{fk}\right)\right|\end{array}\right\} \text{ for } k = 1,2,3,4. \qquad (12)$$

Here $\hat{p}_{fk}$ is the probability estimate, which may come from:

- Figure 1;
- the application of the recursive rolling average filter to the estimated raw shots; and
- the estimated raw shots.

In the absence of fully disaggregated data, the estimated average absolute error, ε_k, of Equation (12) is used as an approximation to the actual average absolute error, E_k, where

$$E_k = \frac{1}{15}\sum_{f=1}^{15}\left|p_{fk} - \hat{p}_{fk}\right| \qquad \text{for } k = 1,2,3,4\,, \tag{13}$$

which cannot be calculated due to lack of data. It is shown in Appendix C that ε_k will be less than or equal to E_k:

$$\varepsilon_k \le E_k\,. \tag{14}$$

Figure 4 plots the data for Group 4, Lo–Hi, which gave the lowest level of mismatch between the data given in Table 5 and the corresponding percentages and cumulative percentages provided from the other sources described above. Meanwhile Figure 5 shows the situation for Group 1, Hi–Lo, for which the match was poorest. Table 6 shows the full comparison for each group and for all groups combined using ε_k as the index. The size of ε_k for each group lies between 0.7% and 3.3%, with an average of between 1.1% and 1.6% across all groups combined.

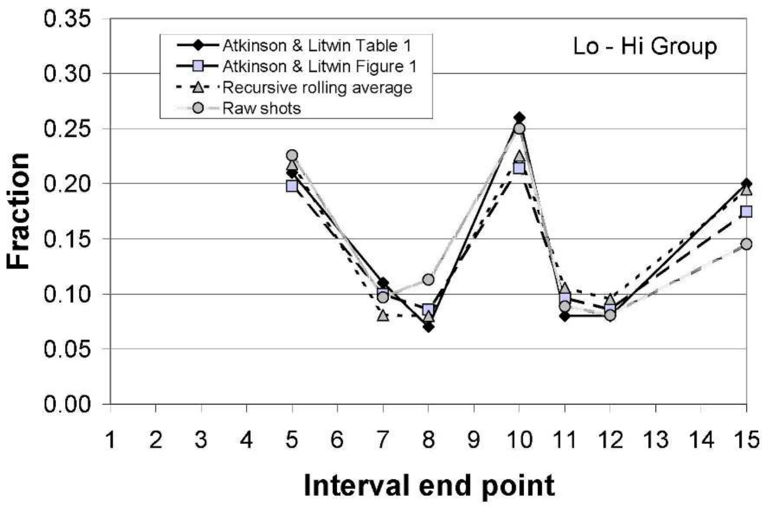

Figure 4. Comparing the cumulative probabilities, expressed as fractions, from Table 5 against the equivalent numbers deduced from (i) the Atkinson & Litwin filtered output of Figure 1, (ii) the recursive rolling filter, and (iii) the estimated numbers of raw shots: Lo–Hi group.

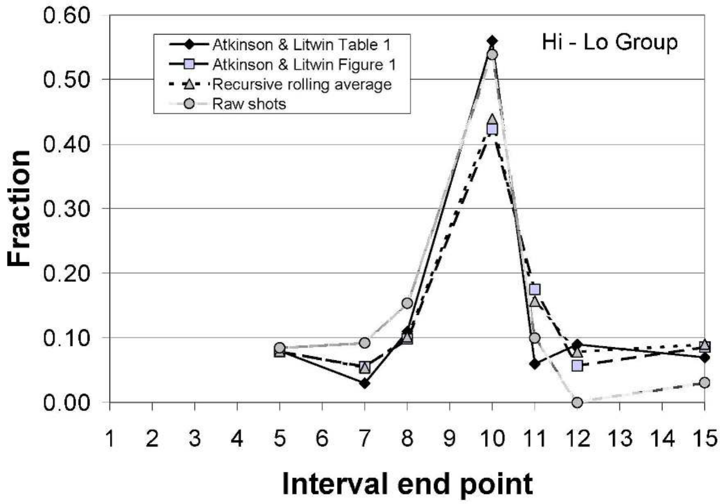

Figure 5. Comparing the cumulative fractions from Table 5 against the equivalent numbers deduced from (i) the Atkinson & Litwin filtered output of Figure 1, (ii) the recursive rolling filter, and (iii) the estimated numbers of raw shots: Hi–Lo group.

Table 6. Absolute error, measured from the baseline percentage probabilities given in Table 5, in the percentage probability that someone in a specified group will choose a particular position to throw from, averaged across all throwing positions.

	Groups				
	1. Hi–Lo	2. Hi–Hi	3. Lo–Lo	4. Lo–Hi	All groups combined
Recursive rolling average	1.88	1.94	1.13	0.85	1.15
Atkinson & Litwin Fig. 1 transcribed	3.21	2.08	1.15	0.74	1.32
Estimated raw shots	2.01	2.40	1.79	0.97	1.60

The percentages generated from the two filters performed better than those derived from the estimated raw shots. The fact that the closest correspondence to Atkinson's and Litwin's Table 1 occurred when the recursive rolling filter was applied to the estimated numbers of raw shots provides further corroboration for the numbers of raw shots from each pitching (shooting) position contained in Table 3.

The results confirm the view that Atkinson and Litwin, having once filtered their data using the recursive rolling filter described in §3.3, regarded the filtered readings as the best indicators of their results and based the rest of their analysis on them. This proposition receives further backing in Atkinson's and Litwin's section entitled "The measure of the subjective probability of success", where they suggest:

> "In the present experiment, the modal point of shots by the high achievement–
> low test anxiety group [1: Hi–Lo] was 9–10 feet",

a remark that assumes implicitly that the throwing distances were continuously variable, as opposed to the discrete distances implied by the lines at 1 foot intervals. In any case, the comment fits the filtered percentage of shots (Figure 1, where, assuming a continuum, the mode would probably lie between 9 and 10 feet) much better than it does the raw numbers (Figure 2, where the mode is sharply defined as 9 feet).

The hypothesis that Atkinson and Litwin used the filtered values in their subsequent analysis receives additional corroboration in the penultimate paragraph of the section of their paper entitled "Descriptive analysis of goal-setting data", where they suggest that the "obtained median" is "9.8 feet". The filtered data from Atkinson's and Litwin's Figure 1 support an overall median of 9.2 feet, while their Table 1, most likely based on filtered data also, suggests a median of 9.5 feet. While discrepancies remain between these numbers and the 9.8 feet quoted, nevertheless the value cited by Atkinson and Litwin lies much closer to these filtered figures than to the median distance derived from the estimated raw shots, namely 8.6 feet.

All the evidence points in the direction of Atkinson's and Litwin's Table 1 containing filtered data.

6. The degree of difficulty of a shot from a given distance as judged by the thrower

Atkinson and Litwin provided a limited number of observed probabilities, p_s, of successfully hooping the peg from a number of distances in their section entitled "The measure of subjective probability of success". They stated that success was certain from 1 foot but the probability of success from 7 feet was down to 0.52 while there was no chance from 15 feet. Moreover, the average probability of success between 8 feet and 12 feet was stated as 0.23.

The simplest model that can accommodate these data is a kinked linear model. This allows the probability of failure, p_m, to rise linearly with slope $m_1 = 0.08$ over distances 1 to 7 feet, reaching 0.48 at the last-mentioned distance, and thereafter to increase with a steeper gradient m_2. The second slope may be found by assuming that it begins from the point (7.0,0.48), with the coordinates at greater distances given by $(8, 0.48 + m_2), (9, 0.48 + 2m_2), \dots (12, 0.48 + 5m_2)$. If the average value of the probability of success for these five points is 0.23, then the average value of the corresponding failure probabilities will be 1 – 0.23 = 0.77. Hence:

$$\frac{1}{5}\left(5 \times 0.48 + m_2\left(1+2+3+4+5\right)\right) = 0.77 . \tag{15}$$

Thus the second slope will be: $m_2 = (0.77 - 0.48)/3 = 0.0967$. But the full data needed to calculate these empirical probabilities could not have become available until after the hoop-the-peg exercise had been completed. Those engaged in playing the game would have had at best partial access to this information, based on their assessment of how people coming before them had played the game (but not on the performance of those coming after). Accordingly the players would have needed to rely heavily on whatever cues they could pick up from the design of the game and thereafter on their own subjective judgement as they weighed up the chances of failure at each throwing distance.

Some degree of "framing"[2] might have been at work also, with the subjects taking their cue from the experimenters and probably regarding the throwing limits they set, namely 1 foot, and 15 feet, as representing certainty of success or failure respectively, with a linear rise in failure

[2] See e.g. Kahneman, D. *Thinking, Fast and Slow*, ch. 7, last p. London: Penguin (2011).

probability between these extremal throwing lines. Such a "strictly linear" model would have a slope of $m_1 = 1/14 = 0.071$.

Figure 6 compares the models for probability of failure versus distance based on the strictly linear model on the one hand and the kinked linear model on the other, where the latter incorporates the (limited) knowledge available for the observed probabilities. There is relatively good agreement between the two characterizations of failure probability, especially up to the halfway mark of 8 feet. The strictly linear model gives a 50% chance of success occurring at a distance of $1 + \frac{1}{2}(15 - 1) = 8$ feet from the peg. Such a figure lies close to the overall average distance of throws, 8.7 feet, calculated from the estimates of the raw numbers of shots (Table 3). Meanwhile, as shown in Figure 2, the throwing position from which most shots were taken is 9 feet in each group. Hence two central values of throwing distance based on empirical estimates, the mean and the mode, have values close to the distance that the strictly linear model would predict as the median.

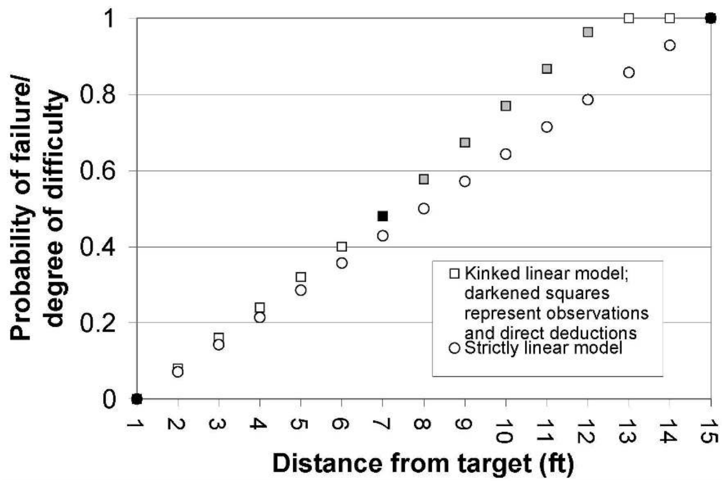

Figure 6. Probability of failure versus distance from the target: strictly linear model and kinked linear model. Observations are superimposed in dark squares. The dark squares between 8 feet and 12 feet are inferred from the stated average for those five distances under the assumption of linearity, which explains why they fit exactly on the line; individual values, had they been available, would have been subject to scatter.

Given the relative closeness just described of the strictly linear model to the observed data, and noting that the throwers would not have had access to the kinked linear model in any case, it is concluded that the strictly linear model provides the best feasible model for the subjects' assessment of the difficulty of hitting the target from the fifteen throwing distances.

Using their experience of running another, related, game, Atkinson and Litwin conjectured that the players might think they had a 50% probability of success two thirds of the way from the lower to the upper limit in the hoop-the-peg exercise. The effect would again rely on "framing"; that is, with the subjects relying on the judgement of the experimenters to inform their own. Atkinson and Litwin seemed to think the $\frac{2}{3}$ point would occur at a distance of "9–10 feet" from the target and they adduced evidence based on the filtered number of shots (see penultimate paragraph of §4). However, the true ⅔ distance is $1 + \frac{2}{3}(15 - 1) = 10.33$ feet. This is some distance away from the average distance at which shots were taken, 8.7 feet based on the estimated raw data, and also from the mode of the distribution of shots which was 9 feet for all four groups. This conjecture of Atkinson's and Litwin's seems, therefore, to lack any real foundation.

Applying the strictly linear model allows a subjective failure probability or "degree of difficulty", p_m (m for miss), to be associated with each of the throwing positions. The graph of percentage of raw shots versus throwing distance of Figure 2 may then be easily transformed into an equivalent probability distribution for the chosen degree of difficulty (see Figure 7).

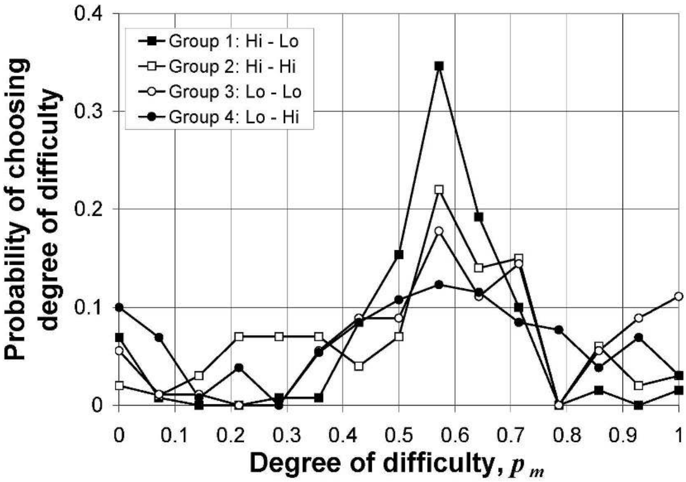

Figure 7. Probability distributions for the degree of difficulty.

It seems likely that the subjects, when assessing the probability for success from each throwing position, took their cue from the way the experiment was framed and adopted the strictly linear model. Hence they would have shared broadly the same perception of the difficulty of hitting the target from the various throwing positions. This would account for members of all groups, viz. Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi, being most attracted to throwing from 9 feet, irrespective of their motivation to achieve and their level of anxiety. The throwers would have assessed that shots taken from this distance would have a 43% chance of success (57% chance of failure). But the more cautious and anxious chose to take a greater fraction of their shots from either extremity of the range, where they thought success was either certain or else impossible. Choosing to throw from very close to the target would spare them the embarrassment of failing to hit their mark, while missing from a great distance would not be condemned, because no one would expect them to succeed anyway.

7. Conclusions

It is most likely that Atkinson and Litwin smoothed their data before presenting it in their Figure 1 (equivalent to Figure 1 of this paper) using the recursive rolling average filter defined in §3.3.

The raw numbers of shots taken from all throwing positions have been deduced by back calculation under the constraint that the estimated total for each group should match the total number of throws recorded by its members. Table 3 of this paper contains the best estimates of the number of shots taken from each throwing distance.

It is concluded that the data Atkinson and Litwin gave in their Table 1 (equivalent information is contained in rationalized form in Table 5 of the present paper) were probably smoothed using the same, recursive rolling average filter, because such a filter gives the best match between their Figure 1 and their Table 1, even though some discrepancies remain.

The strictly linear model illustrated in Figure 6 is argued to give the best characterization of the players' subjective probability of failure as a function of throwing position. Subjects in all groups, Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi, are likely to have had a similar perception of the chances of success and failure from each throwing position, but they chose markedly different spreads of throwing positions, and this indicates that the groups had different appetites for risk.

Figure 7 gives best estimates of the probability distributions for selecting a task of given degree of difficulty as seen by the throwers in all four groups. This result feeds directly into Part 1 of this study.

Appendix A. Demonstration that the sum of the filtered number of shots from all pitching positions must be less than the sum of the raw shots for the recursive rolling average filter

Equation (6), repeated below, gives the filtered number, c_{fk}, of shots taken from distance f calculated using the recursive rolling average filter:

$$c_{fk} = \frac{c_{f-1,k} + a_{fk}}{2} \quad \text{for } 1 \le f \le F \tag{6}$$

where $c_{0,k} = 0$, while $F = 15$. Thus

$$\begin{aligned}
c_{1k} &= \frac{a_{1k}}{2} \\
c_{2k} &= \frac{a_{1k}}{2^2} + \frac{a_{2k}}{2} \\
c_{3k} &= \frac{a_{1k}}{2^3} + \frac{a_{2k}}{2^2} + \frac{a_{3k}}{2} \\
&\vdots \\
c_{fk} &= \frac{a_{1k}}{2^f} + \frac{a_{2k}}{2^{f-1}} + \frac{a_{3k}}{2^{f-1}} + \cdots + \frac{a_{f-1,k}}{2^2} + \frac{a_{fk}}{2} \\
&\vdots
\end{aligned} \tag{A.1}$$

Summing these terms yields the total, C_k, of filtered values over all throwing distances f for group k:

$$C_k = \sum_{f=1}^{F} c_{fk} = \frac{a_{1k}}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{F-1}\right)+\frac{a_{2k}}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{F-2}\right)+$$
$$\ldots+\frac{a_{fk}}{2}\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^{F-f}\right)+\ldots+\frac{a_{F-1,k}}{2}\left(1+\frac{1}{2}\right)+\frac{a_{Fk}}{2} \quad . \qquad \text{(A.2)}$$

Noting that the sum of a geometrical progression is given by:

$$1+r+r^2+\ldots+\ldots+r^{n-2}+r^{n-1}=\frac{1-r^n}{1-r}\,, \qquad \text{(A.3)}$$

it follows that

$$C_k = \frac{a_{1k}}{2}\frac{1-\left(\frac{1}{2}\right)^F}{1-\frac{1}{2}}+\frac{a_{2k}}{2}\frac{1-\left(\frac{1}{2}\right)^{F-1}}{1-\frac{1}{2}}\ldots+\frac{a_{fk}}{2}\frac{1-\left(\frac{1}{2}\right)^{F-(f-1)}}{1-\frac{1}{2}}+\ldots+\frac{a_{F-1,k}}{2}\frac{1-\left(\frac{1}{2}\right)^{F-(F-2)}}{1-\frac{1}{2}}+\frac{a_{Fk}}{2}=$$
$$a_{1k}\left(1-\left(\frac{1}{2}\right)^F\right)+a_{2k}\left(1-\left(\frac{1}{2}\right)^{F-1}\right)\ldots+a_{fk}\left(1-\left(\frac{1}{2}\right)^{F-f+1}\right)+\ldots+a_{F-1,k}\left(1-\left(\frac{1}{2}\right)^2\right)+\frac{a_{Fk}}{2}$$
$$\text{(A.4)}$$

or

$$C_k = \sum_{f=1}^{F} c_{fk} = \sum_{f=1}^{F} w_f a_{fk} \qquad \text{(A.5)}$$

where w_f is the weighting accorded to the number of shots taken at distance f:

$$w_f = 1-\left(\frac{1}{2}\right)^{F-f+1} \quad . \qquad \text{(A.6)}$$

Clearly $w_f \to 1$ as $F-f \to \infty$, but $w_f < 1$ for finite F. The values of the weighting factor for $F = 15$ are given in Table 7. Therefore, from equation (A.5), the sum of the filtered numbers of shots over all pitching positions in the hoop-the-peg exercise must be less than the total number of shots taken by the group, as expressed by equation (9), repeated below:

$$\sum_{f=1}^{F} c_{fk} < \sum_{f=1}^{F} a_{fk} \; . \qquad (9)$$

Table 7. Value of the weighting factor w_f for $1 \le f \le F$; $F = 15$.

Distance f /feet	1	2	3	4	5	6	7	
Weighting w_f	0.99997	0.99994	0.99988	0.9998	0.9995	0.9990	0.9980	
Distance f /feet	8	9	10	11	12	13	14	15
Weighting w_f	0.9961	0.9922	0.9844	0.9688	0.9375	0.8750	0.7500	0.5000

Appendix B. Rationalizing the tabular data provided by Atkinson and Litwin

In addition to the graphical presentation of smoothed percentages in their Figure 1, Atkinson and Litwin also gave, in their Table 1, a set of cumulative percentages for numbers of shots between different distances for the four groups Hi–Lo, Hi–Hi, Lo–Lo and Lo–Hi. Specifically, they provided the following ten aggregates:

$\sum_{f=1}^{5} p_{fk}, \sum_{f=6}^{10} p_{fk}, \sum_{f=11}^{15} p_{fk}, \sum_{f=1}^{8} p_{fk}, \sum_{f=9}^{10} p_{fk}, \sum_{f=1}^{7} p_{fk}, \sum_{f=8}^{11} p_{fk}, \sum_{f=12}^{15} p_{fk}, \sum_{f=8}^{12} p_{fk}, \sum_{f=13}^{15} p_{fk}$. Here p_{fk} is the percentage of throws taken by group *k* from distance *f*. Tighter ranges may be produced through noting that, for all *k*:

$$\sum_{f=6}^{7} p_{fk} = \sum_{f=1}^{7} p_{fk} - \sum_{f=1}^{5} p_{fk} \tag{B.1}$$

$$p_{8,k} = \sum_{f=1}^{8} p_{fk} - \sum_{f=1}^{7} p_{fk} \tag{B.2}$$

$$p_{11,k} = \sum_{f=11}^{15} p_{fk} - \sum_{f=12}^{15} p_{fk} \tag{B.3}$$

$$\sum_{f=9}^{10} p_{fk} = \sum_{f=8}^{11} p_{fk} - p_{8,k} - p_{11,k} \tag{B.4}$$

and

$$p_{12,k} = \sum_{f=12}^{15} p_{fk} - \sum_{f=13}^{15} p_{fk} \;. \tag{B.5}$$

The process of removing the overlaps allows the number of groupings to be reduced from ten to seven. Table 5 contains the results.

Appendix C. Demonstration that the estimated average absolute error, ε_k, will be less than or equal to the actual average absolute error, E_k

This appendix will consider how the spread of positive, negative and zero values taken by the x_k dictates the relationship between $\left|\sum_k x_k\right|$ and $\sum_k \left|x_k\right|$ and relate the results to the relative sizes of ε_k and E_k.

It is convenient to restate the absolute value of a typical summation with *M* terms in equation (12) as:

$$\left|\sum_{m=0}^{M-1}\left(p_{f_s+m,k} - \hat{p}_{f_s+m,k}\right)\right| = \left|\sum_{i=1}^{M}\delta_{i,k}\right| \tag{C.1}$$

where f_s is the starting distance and the difference, $\delta_{i,k}$, is given by

$$\delta_{i,k} = p_{f_s+i-1,k} - \hat{p}_{f_s+i-1,k} \qquad \text{for } i = 1,2,\ldots,M. \tag{C.2}$$

C.1 Case where none of the differences $\delta_{i,k}$ is negative

Consider first the condition where all the differences are greater than or equal to zero, so that $\delta_{i,k} = \alpha_{i,k}$, where $\alpha_{i,k} \geq 0$ for all $i : i = 1,...,M$. In this case,

$$\left|\sum_{i=1}^{M}\delta_{i,k}\right| = \sum_{i=1}^{M}\alpha_{i,k} = \sum_{i=1}^{M}\left|\alpha_{i,k}\right| = \sum_{i=1}^{M}\left|\delta_{i,k}\right| . \tag{C.3}$$

Substituting from equations (C.2) and (C.3) into equation (C.1),

$$\left|\sum_{m=0}^{M-1}\left(p_{f_s+m,k} - \hat{p}_{f_s+m,k}\right)\right| = \sum_{m=0}^{M-1}\left|p_{f_s+m,k} - \hat{p}_{f_s+m,k}\right| . \tag{C.4}$$

C.2 Case where none of the differences $\delta_{i,k}$ is positive

Now consider the condition where all the differences are less than or equal to zero, so $\delta_{i,k} = \alpha_{i,k}$, and $\alpha_{i,k} \leq 0$ for all i: $i = 1,...,M$. In this case,

$$\left|\sum_{i=1}^{M}\delta_{i,k}\right| = -\sum_{i=1}^{M}\alpha_{i,k} = \sum_{i=1}^{M}-\alpha_{i,k} = \sum_{i=1}^{M}\left|\alpha_{i,k}\right| = \sum_{i=1}^{M}\left|\delta_{i,k}\right| . \tag{C.5}$$

On comparing the end-points of equations (C.3) and (C.5), it is clear that equation (C.4) will hold once again.

Hence, if the differences within each of the individual summations in equation (12) are of the same sign or zero, then the estimated average absolute error ε_k will be equal to the actual average absolute error E_k:

$$\varepsilon_k = \frac{1}{15}\sum_{f=1}^{15}\left|p_{fk} - \hat{p}_{fk}\right| = E_k \qquad \text{for } k = 1,2,3,4 . \tag{C.6}$$

C.3. Case where some of the differences $\delta_{i,k}$ may be positive or zero while others will be negative

The ranges of the summations contained in equation (12) are relatively small, as each consists of the addition of 2 or 3 or, at most, 5 components. The short span of the summation may increase the chance of the successive differences sharing the same sign. But we need also to consider situations where this is not the case. Suppose that some of the differences are greater than or equal to zero but that some differences, $\delta_{g,k}$, $\delta_{h,k}$, $\delta_{j,k}$, ... are negative. We may prepare for this case by first postulating a set of values $\alpha_{i,k} \geq 0$ for all i: $i = 1,...,M$ except for $i = g,h,j$, ..., where the alpha values are strictly positive: $\alpha_{g,k} > 0$, $\alpha_{h,k} > 0$, $\alpha_{j,k} > 0$... . Now set $\delta_{i,k} = \alpha_{i,k}$ for all i: $i = 1,...,M$ except for $i = g, h, j$, ..., where $\delta_{g,k} = -\alpha_{g,k}$, $\delta_{h,k} = -\alpha_{h,k}$, $\delta_{j,k} = -\alpha_{j,k}$ ... will now be negative. Clearly $\left|\delta_{n,k}\right| = \left|-\alpha_{n,k}\right| = \alpha_{n,k}$ for all the negative differences: $n = g,h,j,...$, while $\left|\delta_{p,k}\right| = \left|\alpha_{p,k}\right| = \alpha_{p,k}$ for all the positive or zero differences, where $p \neq g,h,j,...$. Hence:

$$\left|\delta_{i,k}\right| = \alpha_{i,k} \qquad \text{for all } i. \tag{C.7}$$

Considering the negative differences, we may write:

$$\begin{aligned} \delta_{g,k} &= -\alpha_{g,k} = \beta_{g,k} \\ \delta_{h,k} &= -\alpha_{h,k} = \beta_{h,k} \\ \delta_{j,k} &= -\alpha_{j,k} = \beta_{j,k} \\ &\vdots \end{aligned} \tag{C.8}$$

where $\beta_{g,k}$, $\beta_{h,k}$, ... are all negative numbers. Summing the terms in equation set (C.8) gives:

$$\delta_{g,k} + \delta_{h,k} + \delta_{j,k} + \ldots = -A_{(g,h,j\ldots),k} = B_{(g,h,j\ldots),k} \tag{C.9}$$

where $A_{(g,h,j\ldots),k} = \alpha_{g,k} + \alpha_{h,k} + \alpha_{j,k} + \ldots$ and $B_{(g,h,j\ldots),k} = \beta_{g,k} + \beta_{h,k} + \beta_{j,k} + \ldots$.

Now

$$\left|\sum_{i=1}^{M} \delta_{i,k}\right| = \left|\sum_{\substack{i=1 \\ i \neq g,h,j\ldots}}^{M} \alpha_{i,k} + B_{(g,h,j\ldots),k}\right| = \left|\sum_{\substack{i=1 \\ i \neq g,h,j\ldots}}^{M} \alpha_{i,k} - A_{(g,h,j\ldots),k}\right| = \left|\sum_{i=1}^{M} \alpha_{i,k} - 2A_{(g,h,j\ldots),k}\right| \tag{C.10}$$

where in the last step the positive quantity $A_{(g,h,j\ldots),k}$ has been added to the first term and subtracted from the second.

To say more about $\left|\sum_{i=1}^{M} \delta_{i,k}\right|$, we need now to consider 3 exclusive and exhaustive possibilities:

(i) $\sum_{i=1}^{M} \alpha_{i,k} > 2A_{(g,h,j\ldots),k}$. (Note that, since $A_{(g,h,j\ldots),k} = \alpha_{g,k} + \alpha_{h,k} + \alpha_{j,k} + \ldots > 0$, condition (i) implies that $\alpha_{i,k}\big|_{i \neq g,h,j\ldots} > 0$ for at least one i: $i \neq g,h,j\ldots$). Expanding condition (i) gives $0 < \sum_{i=1}^{M} \alpha_{i,k} - 2A_{(g,h,j\ldots),k} < \sum_{i=1}^{M} \alpha_{i,k}$, where the last step follows from the fact that $A_{(g,h,j\ldots),k} > 0$. Therefore

$$\left|\sum_{i=1}^{M} \alpha_{i,k} - 2A_{(g,h,j\ldots),k}\right| < \left|\sum_{i=1}^{M} \alpha_{i,k}\right| = \sum_{i=1}^{M} \alpha_{i,k} \tag{C.11}$$

where the equality in the last step follows from the fact that $\alpha_{i,k} \geq 0$ for all i. But, from equation (C.10), $\left|\sum_{i=1}^{M} \delta_{i,k}\right| = \left|\sum_{i=1}^{M} \alpha_{i,k} - 2A_{(g,h,j\ldots),k}\right|$, and applying this along with equation (C.7) in condition (C.11) implies that the absolute value of the sum of differences will be less than the sum of the absolute differences:

$$\left|\sum_{i=1}^{M} \delta_{i,k}\right| < \sum_{i=1}^{M} \left|\delta_{i,k}\right| . \tag{C.12}$$

(ii) $\sum_{i=1}^{M}\alpha_{i,k} = 2A_{(g,h,j...),k}$. (Note again that since $A_{(g,h,j...),k} > 0$, condition (ii) implies that $\alpha_{i,k}\big|_{i\neq g,h,j...} > 0$ for at least one i:$i \neq g,h,j$...). In this second case $0 = \sum_{i=1}^{M}\alpha_{i,k} - 2A_{(g,h,j...),k} < \sum_{i=1}^{M}\alpha_{i,k}$. Thus $\left|\sum_{i=1}^{M}\alpha_{i,k} - 2A_{(g,h,j...),k}\right| < \sum_{i=1}^{M}\alpha_{i,k}$ as before. Therefore, using the same reasoning as for case (i), inequality (C.12) will apply again.

(iii) $\sum_{i=1}^{M}\alpha_{i,k} < 2A_{(g,h,j...),k}$, or, on subtracting $A_{(g,h,j...),k} > 0$ from both sides of condition (iii), and introducing the appropriate limitation on the subscript, i, in the summation of $\alpha_{i,k}$, $\sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} < A_{(g,h,j...),k}$. Further expanding condition (iii) gives:

$$0 < A_{(g,h,j...),k} - \sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} \leq A_{(g,h,j...),k} + \sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} \tag{C.13}$$

where the second step follows from adding $2\sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k}$ to the preceding expression. An equality now enters the condition as, since $\alpha_{i,k} \geq 0$, for all i: $i = 1,...,M$ except for $i = g,h,j...$, it is possible that $\sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} = 0$. The final summation on the right-hand side of inequality (C.13) will, by definition, obey:

$$A_{(g,h,j...),k} + \sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} = \sum_{i=1}^{M}\alpha_{i,k}\ . \tag{C.14}$$

Therefore, condition (C.13) implies that:

$$0 < A_{(g,h,j...),k} - \sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k} \leq \sum_{i=1}^{M}\alpha_{i,k}\ . \tag{C.15}$$

Since the two expressions in condition (C.15) are positive, we may take the modulus of each and write:

$$\left|A_{(g,h,j...),k} - \sum_{\substack{i=1 \\ i\neq g,h,j...}}^{M}\alpha_{i,k}\right| \leq \left|\sum_{i=1}^{M}\alpha_{i,k}\right|. \tag{C.16}$$

Since the $\alpha_{i,k}$ are all positive, $\left|\sum_{i=1}^{M}\alpha_{i,k}\right| = \sum_{i=1}^{M}\alpha_{i,k} = \sum_{i=1}^{M}\left|\delta_{i,k}\right|$, the second step reflecting equation (C.6).

Applying equation (C.10) now gives the final result:

$$\left|\sum_{i=1}^{M} \delta_{i,k}\right| \le \sum_{i=1}^{M} \left|\delta_{i,k}\right| . \tag{C.17}$$

Since all possibilities have now been considered, it is clear from a comparison of the results of cases (i), (ii) and (iii) that condition (C.17) applies as a general statement where there is no information on the signs and magnitudes of the differences $\delta_{i,k}$.

C.4 Conclusion

Since the estimated average absolute error ε_k includes the addition of a number of absolute values of sums, it is clear from condition (C.17) that the estimate so formed will be less than or equal to the actual average absolute error, E_k. Hence condition (14) must apply, repeated below:

$$\varepsilon_k \le E_k. \tag{14}$$