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Nano Ideal Topological Spaces with the Set Operator ψ

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The idea of this paper we introduce the operator ψ is nano ideal topological space and study its basic properties and characterizations in terms of ideal. Also, we discuss *-equivalence property with respect to ideal.
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1. Introduction

An ideal $\mathcal{J}$ on a topological space (X, τ) is a collection of subsets of X which satisfies the following two properties: (1) $H \in \mathcal{J}$ and $T \subseteq H$ implies $T \in \mathcal{J}$ (heredity) and (2) $H \in \mathcal{J}$ and $T \in \mathcal{J}$ implies $H \cup T \in \mathcal{J}$ (finite additivity).Let $\wp(X)$ denotes the power set of X. Given a topological space (X, τ) and an ideal $\mathcal{J}$ on X, a set operator $(\cdot) * (\mathcal{J}, \tau): \mathcal{P}(X) \longrightarrow \mathcal{P}(X)$, called the local function of $\mathcal{J}$ with respect to τ in [11] is defined as follows:For $H \subseteq X$, $(H) * (\mathcal{J}, \tau) = \{x \in X: V \cap H \notin \mathcal{J} \text{ for every } v \in W(x)\}$ where $\mathcal{N}(x) = \{V \in \tau: x \in V\}$. A Kuratowski's closure operator cl^* for a topology $\tau^*(\mathcal{J})$ finer with τ is defined as follows: $cl^*(H) = H \in (H)^*(\mathcal{J}, \tau)$ [11].When there is no chance for confusion, we will simply write H^* for $H^*(\mathcal{J}, \tau)$. A basis $\beta(\mathcal{J}, \tau)$ for $\tau^*(\mathcal{J})$ can be described as follows: $\beta(\mathcal{J}, \tau) = \{V - I: v \in \tau, I \in \mathcal{J}\}$. When there is no chance for confusion, we will write β for $\beta(\mathcal{J}, \tau)$.The notion of a nano topological space was introduced by Lellis Thivagar and Richard [2] which was defined in terms of approximation and boundary region of a subsets of an universe using equivalence relation on it and also they defined nano closed sets, nano interior and nano closure. The notion of a nano ideal topological space was introduced by Parimala et al. [7]. They studied its properties and characterizations.

In what follows, $(X, \tau, \mathcal{J})$ will denote a topological space (X, τ) and an ideal $\mathcal{J}$ on X with no separation properties assumed. For $H \subseteq X$, $c\ell(H)$ and $Int(H)$ will respectively denote the closure and interior of H with respect to τ.cl*(H) and Int^{*}(H) will denote the closure and interior of H with respect to $\tau^{*}(\mathcal{J})$.

2. Preliminaries

Definition 2.1. [5] .Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U named as indiscernibility relation. Then U is divided into disjoint equivalence classes. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair (U, R) is said to be the approximation space.Let $X \subseteq U$, then.

1) The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and is denoted by $L_R(X)$. $L_R(X) = U\{R(x): R(x) \subseteq X, x \in U\}$ where R(X) denotes the equivalence class determined by $x \in U$.

2) The upper approximation of X with respect to R is set of all objects can be possibly classified as X with respect to R and is denoted by $U_R(X)$. $U_R(X) = U\{R(x): R(x) \cap X \neq \emptyset, x \in U\}$.

3) The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not X with respect to R and it is denoted by $B_R(X)$. $B_R(X) = U_R(X) - L_R(X)$

Property 2.2. [5].If (U, R) is an approximation space and $X, Y \subseteq U$, then

1) $L_R(X) \subseteq X \subseteq U_R(X)$

2) $L_R(\phi) = U_R(\phi) = \phi$

3) $L_R(U) = U_R(U) = U$

4) $U_R(X \cup Y) \supseteq U_R(X) \cup U_R(Y)$

5) $U_R(X \cap Y) \subseteq U_R(X) \cap U_R(Y)$

6) $L_R(X \cup Y) \supseteq L_R(X) \cup L_R(Y)$

7) $L_R(X \cap Y) = L_R(X) \cap U_R(Y)$

8) $L_R(X) \subseteq L_R(Y)$ and $U_R(X) \subseteq U_R(Y)$ whenever $X \subseteq Y$

9) $U_R(X^C) = [L_R(X)]^C$ and $L_R(X^C) = [U_R(X)]^C$

10) $U_R[U_R(X)] = L_R[U_R(X)] = U_R(X)$

11) $L_R[L_R(X)] = U_R[L_R(X)] = L_R(X)$

Definition 2.3. [5]. Let U be the universe R be an equivalence relation on U and $\tau_R(X) = \{U, \phi, L_R(X), U_R(X), B_R(X)\}$ where $X \subseteq U$. Then $\tau_R(X)$ satisfies the following axioms:

1) U and $\phi \in \tau_R(X)$ 2) The union of the elements of any sub-collection of $\tau_R(X)$ is in $\tau_R(X)$.

3) The intersection of the elements of any finite sub collection of $\tau_R(X)$ is in $\tau_R(X)$.

Then $\tau_R(X)$ is a topology on U called the nano topology on U with respect to $X \cdot$ $(U, \tau_R(X))$ is called the nano topological space. Elements of the nano topology are known as nano open sets in U. Elements of $[\tau_R(X)]^C$ being called dual nano topology of $\tau_R(X)$.

Definition 2.4.Let $(U, \tau, \mathcal{J})$ be a nano ideal topological space, then $H \subseteq U$ is called nano-regular open if $H = \mathrm{Int}\big(c\ell(H)\big)$.

Definition 2.5.Let $(U, \tau, \mathcal{J})$ be a nano ideal topological space then $H \subseteq U$ is called nano semi-open if $H \subseteq \mathrm{Int}(c\ell(H))$.

3. ψ operator

Definition 3.1.An operator $\psi: \mathcal{P}(X) \longrightarrow \tau$ as follows:

For every $H \in U$; ψ(H)={x: there exist $V \in W$, such that $V - H \in \mathcal{J}$}. i.e., $\psi(H) = X - (X - H)^*$.

Example 3.2.

Let $U = [a, b, c, d\}$, $X = \{a, b\}$,

$U|R = \{\{a\}, \{c\}, \{b, d\}\}$, $\tau^* = \{U, \phi, \{a\}, \{a, b, d\}\}$

$\tau_R(X) = \{U, \phi, \{a\}, \{b, d\}, \{a, b, d\}\}$, and $\mathcal{J} = \{\phi, \{c\}\}$.

Let $H = \{a, c\}$, then $\psi(H) = \{a\}$. Also

$(X - H)^* = \{b, c, d\}$. This implies $X - (X - H)^* = \{a\}$.Several basic facts concerning the behaviour of the operator ψ are includes in the following:

Theorem 3.3.Let $(U, \tau, \mathcal{J})$ be a nano topological space.

(i) If $H \subseteq U$, then $\psi(H)$ is open.

(ii) If $H \subseteq T$, then $\psi(H) \subseteq \psi(T)$.

(iii) If $H, T \in \mathcal{P}(X)$, then $\psi(H \cap T) = \psi(H) \cap \psi(T)$.

(iv) If $H \in \tau^*$, then $H \subseteq \psi(H)$.

(v) If $H \subseteq U$, then $\psi(H) \subseteq \psi(\psi(H))$.

(vi) If $H \subseteq U$, then $\psi(H) \subseteq \psi(\psi(H))$ iff $(U - H)^* = (U - H)^{**}$.

(vii) If $\boldsymbol{H} \in \boldsymbol{\mathcal{J}}$, then $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{U} - \boldsymbol{U}^*$.

(viii) If $\boldsymbol{H} \subseteq \boldsymbol{U}$, then $\boldsymbol{H} \cap \boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{Int}^*(\boldsymbol{H})$.

(ix) If $\boldsymbol{H} \subseteq \boldsymbol{U}, \boldsymbol{I} \in \boldsymbol{\mathcal{J}}$, then $\boldsymbol{\psi}(\boldsymbol{H} - \boldsymbol{I}) = \boldsymbol{\psi}(\boldsymbol{H})$.

(x) If $\boldsymbol{H} \subseteq \boldsymbol{U}, \boldsymbol{I} \in \boldsymbol{\mathcal{J}}$, then $\boldsymbol{\psi}(\boldsymbol{H} \cup \boldsymbol{I}) = \boldsymbol{\psi}(\boldsymbol{H})$

(xi) If $(\boldsymbol{H} - \boldsymbol{T}) \cup (\boldsymbol{T} - \boldsymbol{H}) \in \boldsymbol{\mathcal{J}}$, then $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{\psi}(\boldsymbol{T})$.

Proof.Proof of (i) and (ii) are trivial.

(iii) $\boldsymbol{H \cap T \subseteq H}$. This implies $\boldsymbol{\psi(H \cap T) \subseteq \psi(H)}$.

$\boldsymbol{H \cap T \subseteq H}$. This implies $\boldsymbol{\psi(H \cap T) \subseteq \psi(T)}$.

This implies,

$$\boldsymbol{\psi(H \cap T) \subseteq \psi(H) \cap \psi(B)}$$

Let $\boldsymbol{x \in \psi(H) \cap \psi(T)}$. Then there exist a $\boldsymbol{U_1 \in \mathcal{N}(x)}$, such that $\boldsymbol{U_1 - H \in \mathcal{I}}$ and there exist $\boldsymbol{U_2 \in \mathcal{N}(x)}$, such that $\boldsymbol{U_2 - H \in \mathcal{I}}$. Let $\boldsymbol{U_3 = U_1 \cap U_2}$, we have $\boldsymbol{U_3 - H \in \mathcal{I}}$ and $\boldsymbol{U_3 - T \in \mathcal{I}}$. Thus $\boldsymbol{U_3 - (H \cap T) = (U_3 - H) \cup (U_3 - T) \in \mathcal{I}}$ and hence $\boldsymbol{x \in \psi(H \cap T)}$.

$$\boldsymbol{i.e., \psi(H) \cap \psi(T) \subseteq \psi(H \cap T)}$$

From (1) and (2)

$$\boldsymbol{\psi(H \cap T) = \psi(H) \cap \psi(T)}.$$

(iv) If $\boldsymbol{H \in \tau^*}$, then $\boldsymbol{U - H}$ is τ^*-closed. This implies $\boldsymbol{(U - H)^* \subseteq U - H}$.

This implies $\boldsymbol{H \subseteq U - (U - H)^* = \psi(H)}$.

(v) $\boldsymbol{\psi(H)}$ is open, $\boldsymbol{\psi(H) \in \tau^*}$. By using (iv) $\boldsymbol{\psi(H) \subseteq \psi(\psi(H))}$.

(vi) $\boldsymbol{\psi(H) = U - (U - H)^*}$

$\boldsymbol{\psi(\psi(H)) = U - (U - H)^{**}}$. This implies $\boldsymbol{\psi(H) = \psi(\psi(H))}$

iff $\boldsymbol{(U - H)^* = (U - H)^{**}}$

(vii) Let $\boldsymbol{H \in \mathcal{I}}$. This implies $\boldsymbol{(U - H)^* = U^*}$. By using (ii), This implies $\boldsymbol{\psi(H) = U - (U - H)^* = U - U^*}$.

(viii) Let $\boldsymbol{x \in H \cap \psi(H)}$. There exist a $\boldsymbol{V_x \in \mathcal{N}(x)}$, such that $\boldsymbol{V_x - H \in \mathcal{I}}$. This implies $\boldsymbol{V_x - (V_x - H)}$ is τ^*-open neighbourhood of x and $\boldsymbol{x \in Int^*(H)}$. Conversely, if $\boldsymbol{x \in Int^*(H)}$, there exist τ^*-open neighbourhood $\boldsymbol{V_x - I}$ of x, $\boldsymbol{I \in \mathcal{I}}$ such that $\boldsymbol{x \in V_x - I \subseteq H}$. This implies $\boldsymbol{V_x - H \subseteq I}$ and hence $\boldsymbol{V_x - H \in \mathcal{I}}$. i.e., $\boldsymbol{x \in \psi(H)}$. Hence $\boldsymbol{H \cap \psi(H) = Int^*(H)}$.

(ix) $\boldsymbol{\psi(H - I) = U - (U - (H - I))^* = U - ((U - H) \cup I)^* = U - (U - H)^*}$. Since $\boldsymbol{(T \cup I)^* = T^* = \psi(H)}$.

(x) $\boldsymbol{\psi(H \cup I) = U - (U - (H - I))^* = U - ((U - H) - I)^* = U - (U - A)^* = \psi(H)}$.

(xi) Assume that $\boldsymbol{(H - T) \cup (T - H) \in \mathcal{I}}$ and let $\boldsymbol{H - T = I}$ and $\boldsymbol{T - H = J}$. Also $\boldsymbol{T = (H - I) \cup J}$. Therefore $\boldsymbol{\psi(T) = \psi(H - I) \cup J) = \psi(H - I)\psi(H)}$.

The above theorem explained the following example.

Example 3.4.If $\boldsymbol{H \subseteq X}$, $\boldsymbol{\psi(H)}$ is open, see example (1).

(i) $\boldsymbol{H = \{a, c\}, T = \{a, b, c\}, \psi(H) = \{a\}}$,

$\boldsymbol{\psi}(\boldsymbol{T}) = \{\boldsymbol{a}\}$. i.e., if $\boldsymbol{H} \subset \boldsymbol{T}$ then $\boldsymbol{\psi}(\boldsymbol{H}) \leq \boldsymbol{\psi}(\boldsymbol{T})$

(ii) $\boldsymbol{H} \cap \boldsymbol{T} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{\psi}(\boldsymbol{H} \cap \boldsymbol{T}) = \{\boldsymbol{a}\}$. This implies $\boldsymbol{\psi}(\boldsymbol{H}) \cap \boldsymbol{\psi}(\boldsymbol{T}) = \boldsymbol{\psi}(\boldsymbol{H} \cap \boldsymbol{T})$.

(iii) Let $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{b}\} \in \boldsymbol{\tau}^*$, $\boldsymbol{\psi}(\boldsymbol{H}) = \{\boldsymbol{b}, \boldsymbol{d}\}$, $\boldsymbol{H} \subseteq \boldsymbol{\psi}(\boldsymbol{H})$.

(iv) $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{\psi}(\boldsymbol{H}) = \{\boldsymbol{a}\}$. This implies $\boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big) = \{\boldsymbol{a}\}$.

This implies $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big)$.

(v) $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big)$. $\boldsymbol{X} - \boldsymbol{H} = \{\boldsymbol{b}, \boldsymbol{d}\}$. This

implies $(\boldsymbol{X} - \boldsymbol{H})^* = (\boldsymbol{b}, \boldsymbol{d})^* = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$. Also

$(\boldsymbol{X} - \boldsymbol{H})^{**} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$. Therefore $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big)$

iff $(\boldsymbol{X} - \boldsymbol{H})^* = (\boldsymbol{X} - \boldsymbol{A})^{**}$.

(vi) Let $\boldsymbol{H} = \{\boldsymbol{c}\} \in \boldsymbol{\mathcal{I}}$, $\boldsymbol{\psi}(\boldsymbol{H}) = \emptyset$, $\boldsymbol{U} - \boldsymbol{U}^* = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\} - \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\} = \emptyset$.

This implies $\boldsymbol{H} \in \boldsymbol{\mathcal{I}}$. This implies $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{U} - \boldsymbol{U}^*$.

(vii) $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{\psi}(\boldsymbol{H}) = \{\boldsymbol{a}\}$, $\boldsymbol{H} \cap \boldsymbol{\psi}(\boldsymbol{H}) = \{\boldsymbol{a}\}$.

$\boldsymbol{int}^*(\boldsymbol{H}) = \{\boldsymbol{a}\}$. This implies $\boldsymbol{H} \cap \boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{int}^*(\boldsymbol{H})$.

(viii) Let $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{I} = \{\boldsymbol{c}\}$, $\boldsymbol{H} - \boldsymbol{I} = \{\boldsymbol{a}\}$.

$\boldsymbol{\psi}(\boldsymbol{H} - \boldsymbol{I}) = [\boldsymbol{a}\} = \boldsymbol{\psi}(\boldsymbol{H})$.

(ix) $\boldsymbol{H} \cup \boldsymbol{I} = \boldsymbol{H}$. This implies $\boldsymbol{\psi}(\boldsymbol{H} \cup \boldsymbol{I}) = \boldsymbol{\psi}(\boldsymbol{H}) = \{\boldsymbol{a}\}$.

(x) Let $\boldsymbol{H} = \{\boldsymbol{a}, \boldsymbol{c}\}$, $\boldsymbol{T} = \{\boldsymbol{c}\}$, $(\boldsymbol{H} - \boldsymbol{T}) \cup (\boldsymbol{T} - \boldsymbol{H}) = \{\boldsymbol{c}\}$

This implies $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{\psi}(\boldsymbol{T}) = \{\boldsymbol{a}\}$.

Definition 3.5.Define the ideal $\boldsymbol{\mathcal{I}}$ is to be compatible with τ denoted by $\boldsymbol{\mathcal{I}} \sim \boldsymbol{\tau}$ if the following hold: For every subset $\boldsymbol{H} \subseteq \boldsymbol{U}$. If for every $\boldsymbol{x} \in \boldsymbol{H}$, there exist $\boldsymbol{v} \in \boldsymbol{\mathcal{N}}(\boldsymbol{x})$, such that $\boldsymbol{V} \cap \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$, then $\boldsymbol{H} \in \boldsymbol{\mathcal{I}}$.

Theorem 3.6.Let $(\boldsymbol{U}, \boldsymbol{\tau}, \boldsymbol{\mathcal{I}})$ be a nano ideal topological space. Then $\boldsymbol{\mathcal{I}} \sim \boldsymbol{\tau}$ iff $\boldsymbol{\psi}(\boldsymbol{H}) - \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$, for every subset $\boldsymbol{H} \subseteq \boldsymbol{X}$.

Proof.Assume that $\boldsymbol{\mathcal{I}} \sim \boldsymbol{\tau}$ and let $\boldsymbol{H} \subseteq \boldsymbol{X}$. Let $\boldsymbol{x} \in \boldsymbol{\psi}(\boldsymbol{H}) - \boldsymbol{H}$. This implies $\boldsymbol{x} \notin \boldsymbol{H}$ and $\boldsymbol{x} \notin (\boldsymbol{U} - \boldsymbol{H})^*$. This implies there exist $\boldsymbol{V_x} \in \boldsymbol{\mathcal{N}}(\boldsymbol{x})$ such that $\boldsymbol{V_x} - \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$. This implies $\boldsymbol{x} \in \boldsymbol{V_x} - \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$. Now, for each $\boldsymbol{x} \in \boldsymbol{\psi}(\boldsymbol{H}) - \boldsymbol{H}$ and $\boldsymbol{V_x} \in \boldsymbol{\mathcal{N}}(\boldsymbol{x})$, $\boldsymbol{V_x} \cap (\boldsymbol{\psi}(\boldsymbol{H}) - \boldsymbol{H}) \in \boldsymbol{\mathcal{I}}$, by heredity. So that $\boldsymbol{\psi}(\boldsymbol{H}) - \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$, by assumption.

Conversely, let $\boldsymbol{H} \subseteq \boldsymbol{X}$ and each $\boldsymbol{x} \in \boldsymbol{H}$, there exist $\boldsymbol{V_x} \in \boldsymbol{\mathcal{N}}(\boldsymbol{x})$, such that $\boldsymbol{V_x} \cap \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$. Observe that $\boldsymbol{\psi}(\boldsymbol{U} - \boldsymbol{H}) - (\boldsymbol{U} - \boldsymbol{H}) = \{\boldsymbol{x}$: there exist $\boldsymbol{V_x} \in \boldsymbol{N}(\boldsymbol{x})$, such that $\boldsymbol{x} \in \boldsymbol{V_x} \cap \boldsymbol{H} \in \boldsymbol{\mathcal{I}}\}$. We have $\boldsymbol{H} \subseteq \boldsymbol{\psi}(\boldsymbol{U} - \boldsymbol{H}) - (\boldsymbol{U} - \boldsymbol{H}) \in \boldsymbol{\mathcal{I}}$ and hence $\boldsymbol{H} \in \boldsymbol{\mathcal{I}}$.

Corollary 3.7.Let $(\boldsymbol{U}, \boldsymbol{\tau}, \boldsymbol{\mathcal{I}})$ be a nano ideal topological space with $\boldsymbol{\tau} \sim \boldsymbol{\mathcal{I}}$. Then $\boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big) = \boldsymbol{\psi}(\boldsymbol{H})$ for every $\boldsymbol{H} \subseteq \boldsymbol{U}$.

Proof.Clearly $\boldsymbol{\psi}(\boldsymbol{H}) \subseteq \boldsymbol{\psi}\big(\boldsymbol{\psi}(\boldsymbol{H})\big)$ by theorem 3.6. Since $\boldsymbol{\mathcal{I}} \sim \boldsymbol{\tau}$, $\boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{H} \cup \boldsymbol{I}$, for some $\boldsymbol{I} \in$

$\mathcal{J}$ and hence $\psi(\psi(H)) = \psi(H \cup I) = \psi(H)$.

Definition 3.8.Define an ideal $\mathcal{J}$ on a nano ideal topological space $(U, \tau, \mathcal{J})$ to be τ-boundary if $\mathcal{J} \cap \tau_N = \{\emptyset\}$.

Example 3.9.Let $U = \{a, b, c, d\}$, with $U|R = \{\{a, c\}, \{b\}, \{d\}\}$ and $X = \{a, b\}$, then $\tau = \{U, \emptyset, \{b\}, \{a, b, c\}, \{a, c\}\}$, $I = \{\emptyset, \{d\}\}$.

Theorem 3.10.Let $(U, \tau, \mathcal{J})$ be a nano ideal topological space. Then

(i) $\psi(H) = \cup \{M \in \tau: M - H \in \mathcal{J}\}$

(ii) $\psi(V) = \cup \{M \in \tau: (M - V) \cup (V - M) \in \tau\}, V \in \tau$.

Proof. (I) is trivial from the definition.

(ii) Take $\psi'(H) = \cup \{M \in \tau: (M - H) \cup (H - M) \in \mathcal{J}\}$

By heredity, $\psi'(H) \subseteq \psi(H)$, for every $H \subseteq U$.

Assume that $V \in \tau$, $x \in \psi(V)$. Then there exist $M \in \tau$

Such that $x \in M - V \in \mathcal{J}$. Let $N = M \cup V$, then $N \in \tau$

and $x \in (N - V) \cup (V - N) = (M - V) \cup \emptyset = (M - V) \in \mathcal{J}$.

Then $x \in \psi'(V)$ and hence $\psi(V) = \psi'(V)$ if $V \in \tau$.

Theorem 3.11. Let $(U, \tau, \mathcal{J})$ be a ideal topological space. The following are equivalent.

(i) $\mathcal{J}$ is τ-boundary

(ii) $\psi(\emptyset) = \emptyset$

(iii) If $H \subseteq X$ is closed, then $\psi(H) - H = \emptyset$

(iv) If $H \subseteq X$, then $int(cl(H)) = \psi(int(cl(H))$

(v) If H is regular open, then $H = \psi(H)$.

(vi) If $V \in \tau$, then $\psi(V) \subseteq int(cl(V)) \subseteq U^*$

(vii) If $I \in \mathcal{J}$, then $\psi(I) = \emptyset$

Proof. (i) ⇒ (ii) $\psi(\emptyset) = \cup \{M \in \tau: M - \emptyset \in \mathcal{J}\} = \cup \{M \in \tau: M \in \mathcal{J}\} = \emptyset$.

(ii) ⇒ (iii) If $x \in \psi(H) - H$, then there exist $V \in \mathcal{N}(x)$ such that $x \in V - H \in \mathcal{J}$. But $V - H \in \{M \in \tau: M \in \mathcal{J}\}$ which implies $\psi(\emptyset) \neq \emptyset$. Hence $\psi(H) - H = \emptyset$.

(iii) ⇒ (iv) We know that $int(cl(H)) \subseteq \psi(intcl(H))$. By (iii) $\psi(cl(H)) \subseteq cl(H)$. This implies $\psi(cl(H)) = int(\psi(cl(H)) \subseteq int(cl(H))$. Since $\psi(cl(H))$ is open. Now $\psi\left(int(cl(H))\right) \subseteq \psi(cl(H)) \subseteq int(cl(H))$. This implies $int(cl(H)) = \psi\left(int(cl(H))\right)$

(iv) ⇒ (v) If H is regular open, then $H = int(cl(H))$ we get, $H = \psi(H)$, by (iv).

(v) ⇒ (i) $\emptyset = \psi(\emptyset) = \psi(\tau \in \mathcal{J})$. This implies $\tau \cap \mathcal{J} = \emptyset$. This implies $\mathcal{J}$ is τ-boundary.

(v) ⇒ (vi) Let $\boldsymbol{V} \in \boldsymbol{\tau}$. This implies $\boldsymbol{V} \subseteq \boldsymbol{V}^*$ and $\boldsymbol{V}^*$ is closed. This implies $\boldsymbol{cl}(\boldsymbol{V}) \subseteq \boldsymbol{V}^*$.

Now $\boldsymbol{V} \subseteq \boldsymbol{int}(\boldsymbol{cl}(\boldsymbol{V})) \subseteq \boldsymbol{V}^*$. This implies $\boldsymbol{\psi}(\boldsymbol{V}) \subseteq \boldsymbol{\psi}\left(\boldsymbol{int}(\boldsymbol{cl}(\boldsymbol{V}))\right) = \boldsymbol{int}(\boldsymbol{cl}(\boldsymbol{V})) \subseteq \boldsymbol{V}^*$.

(vi) ⇒ (i) If $\boldsymbol{V} \in \boldsymbol{\mathcal{I}} \cap \boldsymbol{\tau}$, then we have $\boldsymbol{V} \subseteq \boldsymbol{\psi}(\boldsymbol{V}) \subseteq \boldsymbol{V}^* = \emptyset$ and hence $\boldsymbol{\mathcal{I}} \cap \boldsymbol{\tau} = \emptyset$.

Definition 3.12.A subset $\boldsymbol{H} \subseteq \boldsymbol{U}$ is closed nowhere dense in $\boldsymbol{U}$ if $\boldsymbol{int}(\boldsymbol{cl}(\boldsymbol{H})) = \emptyset$.

Example 3.13.In Example 3.8 if $\boldsymbol{H} = \{\boldsymbol{d}\}$ then $\boldsymbol{intcl}(\boldsymbol{H}) = \emptyset$. Hence $\boldsymbol{H}$ is nowhere dense set in $\boldsymbol{U}$.

Theorem 3.14.Let $(\boldsymbol{U}, \boldsymbol{\tau}, \boldsymbol{\mathcal{I}})$ be a nano ideal topological space with $\boldsymbol{\mathcal{I}_n}$, the ideal of nowhere dense subsets

(i) If $\boldsymbol{\mathcal{I}_n} \subseteq \boldsymbol{\mathcal{I}}$ and $\boldsymbol{H} = \boldsymbol{\psi}(\boldsymbol{H})$, then $\boldsymbol{H}$ is regular open.

(ii) If $\boldsymbol{\mathcal{I}_n} \subseteq \boldsymbol{\mathcal{I}}$ and $\boldsymbol{\mathcal{I}} \cap \boldsymbol{\tau} = \{\emptyset\}$ then $\boldsymbol{H} = \boldsymbol{\psi}(\boldsymbol{H})$ iff $\boldsymbol{H}$ is regular open.

(iii) If $\boldsymbol{\mathcal{I}_n} \subseteq \boldsymbol{\mathcal{I}}$, $\boldsymbol{\mathcal{I}} \sim \boldsymbol{\tau}$ and $\boldsymbol{H} \subseteq \boldsymbol{U}$, then $\boldsymbol{\psi}(\boldsymbol{H})$ is regular open.

Proof.(i) Let $\boldsymbol{H} \subseteq \boldsymbol{U}$ and assume $\boldsymbol{H} = \boldsymbol{\psi}(\boldsymbol{H})$. Since $\boldsymbol{H} = \boldsymbol{\psi}(\boldsymbol{H})$, $\boldsymbol{H}$ is open. This implies $\boldsymbol{H} \subseteq \boldsymbol{int}(\boldsymbol{cl}(\boldsymbol{H}))$. Let $\boldsymbol{x} \in \boldsymbol{cl}(\boldsymbol{H})$. This implies $\boldsymbol{V} - \boldsymbol{H} \subseteq \boldsymbol{cl}(\boldsymbol{H}) - \boldsymbol{H} \in \boldsymbol{\mathcal{I}_n}$ and $\boldsymbol{V} - \boldsymbol{H} \in \boldsymbol{\mathcal{I}}$. This implies $\boldsymbol{x} \in \boldsymbol{\psi}(\boldsymbol{H})$. This implies $\boldsymbol{Int}(\boldsymbol{cl}(\boldsymbol{H})) \subseteq \boldsymbol{\psi}(\boldsymbol{H}) = \boldsymbol{H}$. This implies **H** is regular open. (ii) is immediately from (i) as theorem. (iii) follows from (i).

Example 3.15.The converse of Theorem 3.14 (i) need not be true. In Example 3.13 $\mathrm{I_n} = \{\mathrm{d}\}$, $\mathrm{I} = \{\emptyset, \{\mathrm{d}\}\}$, $\mathrm{H} = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$ and so H is regular open but $\psi(\mathrm{H}) \neq \mathrm{H}$.

Definition 3.16.If β is a basis for a topology on U, then $\psi(\beta)$ is a basis for the topology coarser than τ and denoted as$< \psi(\beta) = \tau^{\#}$.

Recall that the regular open sets of a space $(\mathrm{U}, \tau, \mathcal{I})$ form a basis for a topology coarser than τ. This topology is called the semi regularization of τ and is denoted τ_S.

Theorem 3.17.Let $(\mathrm{U}, \tau, \mathcal{I})$ be a ideal topological space. If $\mathcal{I} \cap \tau = \{\emptyset\}$, then $\tau_S \subseteq \tau^{\#} \subseteq \tau$.

Proof.If H is regular open, then $\mathrm{H} = \psi(\mathrm{H})$. This implies $\tau_S \subseteq \tau^{\#} \subset \tau$.

Definition 3.18.Define $\mathrm{H} = \mathrm{T}(\mathrm{mod}\, \mathcal{I})$ if $(\mathrm{H} - \mathrm{T}) \cup (\mathrm{T} - \mathrm{H}) \in \mathcal{I}$.

Note:$\mathrm{H} = \mathrm{T}(\mathrm{mod}\, \mathcal{I})$. This implies $\psi(\mathrm{H}) = \psi(\mathrm{T})$. But the converse need not be true.

Here $\psi(\emptyset) = \psi(\mathrm{U}) = \emptyset$ but $\emptyset \neq \mathrm{U}$.

Theorem 3.19.Let $(\mathrm{U}, \tau, \mathcal{I})$ be a nano ideal topological space with $\mathcal{I} \cap \tau$. If $\mathrm{M}, \mathrm{N} \in \tau$ and $\psi(\mathrm{M}) = \psi(\mathrm{N})$
then $\mathrm{M} = \mathrm{N}(\mathrm{mod}\, \mathcal{I})$

Proof.We know that $\mathrm{M} \subseteq \psi(\mathrm{M})$. This implies $\mathrm{M} - \mathrm{N} \subseteq \psi(\mathrm{M}) - \mathrm{N} = \psi(\mathrm{N}) - \mathrm{N} \in \mathcal{I}$.

This implies $\mathrm{M} - \mathrm{N} \in \mathcal{I}$. This implies $\mathrm{M} = \mathrm{N}\ (\mathrm{mod}\, \mathcal{I})$.

Definition 3.20.Let $(\mathrm{U}, \tau, \mathcal{I})$ be a nano ideal topological space and let $\mathrm{H} \subseteq \mathrm{X}$. H is said to be nano basis set with respect to $\mathcal{I}$ and τ denoted as $\mathrm{H} \in \mathcal{B}_{\mathrm{r}}(\mathrm{U}, \tau, \mathcal{I})$ if there exist open set $\mathrm{U} \in \tau$

such that $H = U \ (\mathrm{mod}\ \mathcal{J})$.

Theorem 3.21.Let $(U, \tau, \mathcal{J})$ be a nano ideal topological space with $\mathcal{J} \cap \tau$. If $H, T \in \mathcal{B}_r(U, \tau, \mathcal{J})$ and $\psi(H) = \psi(T)$, then $H = T \ (\mathrm{mod}\ \mathcal{J})$.

Proof.

Let $V, W \in \tau$ such that $H = V \ (\mathrm{mod}\ \mathcal{J})$ and $T = W \ (\mathrm{mod}\ \mathcal{J})$. Now $\psi(H) = \psi(V)$ and $\psi(T) = \psi(W)$. This implies $\psi(V) = \psi(W)$. This implies $V = W \ (\mathrm{mod}\ \mathcal{J})$ above theorem 3.19. This implies $H = T \ (\mathrm{mod}\ \mathcal{J})$.

Definition 3.22.Let σ and τ be topologies and let $\mathcal{J}$ be a fixed ideal on a set U. Define σ and τ are said to be ∗-equivalent with respect to $\mathcal{J}$, denoted as $\sigma = \tau \ (\mathrm{mod}\ \mathcal{J})$ if $\sigma^* = \tau^*$. Define [τ] = {σ : σ is topology on U with σ* = τ*} is called equivalence class for τ.

Theorem 3.23.If $(U, \tau, \mathcal{J})$ be a nano ideal topological space with $\mathcal{J} \cap \tau = \{\emptyset\}$ and $\sigma \in [\tau]$ then $\tau_S \subseteq \sigma$.

Proof.It is clear that $cl(W) = cl^*(W)$. For every $W \in \tau^*$ and hence $int(F) = int^*(F)$, for every τ^*-closed set F. Thus for every $W \in \tau^*$, we have $int^*(cl^*(W)) = int\,cl(W)$ and for every $V \in \tau$, we have $int^*(cl^*(V)) = int\,cl(V)$. This implies τ and τ* have precisely the same regular open sets and hence $\tau_S = \tau_S^*$.

If $\sigma \in [\tau]$, we have $\sigma^* = \tau^*$ and $\sigma_S^* = \sigma_S$. Therefore $\sigma^* = \tau^*$. This implies $\sigma_S^* = \tau_S^*$. This implies $\sigma_S = \tau_S$. This implies $\tau_S \subseteq \sigma$.

Theorem 3.24.Let $(U, \tau, \mathcal{J})$ be a nano ideal topological space, then $(\tau^*)^* = \tau^*$.

Proof.Observe that $\tau^*(\mathcal{J}) = \sigma$ iff every $I \in \mathcal{J}$ is closed in τ. Since every member of $\mathcal{J}$ is closed in τ^*, it follows that $(\tau^*)^* = \tau^*$.

Theorem 3.25.If $(U, \tau, \mathcal{J})$ be a nano ideal topological space then τ^* is the largest number of [τ].

Proof.By the above theorem, $\tau^* \in [\tau]$. If $\sigma \in [\tau]$, then $\sigma \subseteq \sigma^* = \tau^*$. This implies $\sigma \subseteq \tau^*$. This implies τ^* is largest member of τ.

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