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Applications of Pythagorean Uncertainty weighted Average Aggregation operator to Multiple attribute group Decision Making

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In this article, Pythagorean uncertainty weighted Averaging Aggregation operator has been introduced along their several properties, namely idempotency, boundedness and monotonicity. Secondly, we applied this proposed operator to deal with multiple attribute group decision making problem under Pythagorean uncertainty information. Finally, we constructed an algorithm for multiple attribute group decision making problems with suitable example.

Keywords: fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, Fermatean fuzzy set, boundaries, weighted average, decision making, idempotency algorithm.

1. Introduction

The concept of fuzzy set was first introduced by Zadeh in 1965 [24]. In 1986, Atanassov [2] presented the concept of intuitionistic fuzzy set (IFS), which is a general form of the fuzzy set [2]. Bustince and Burillo [9] developed that Vague sets are mathematically equal to intuitionistic fuzzy set. Pythagorean fuzzy set introduced by [12]. Chen and Tan [10] explained multi criteria paper fuzzy decision making based on Vague set. Xu and Yager [20] demonstrated several operators such as intuitionistic fuzzy weighted averaging (IFWA), intuitionistic fuzzy ordered weighted averaging (IFOWA) and intuitionistic fuzzy hybrid averaging (IFHA) operators. Xu and Yager [21] explained geometric aggregation operators, such as intuitionistic fuzzy weighted geometric (IFWG) operator and intuitionistic fuzzy hybrid geometric (IFHG) operator. They also applied them to multiple attribute group decision making (MAGDM) based on intuitionistic fuzzy set (IFS). Xu [20] was developed technique for order of preference by similarity to ideal solution (TOPSIS) method for multiple attribute group decision making. The advantage of the aggregation operators in this work. We familiarize the notion of (3, 2) uncertainty weighted averaging aggregation operator and also discuss some of their basic properties. The concept of (3,2)-uncertainty set proposed by [13]. Pythagorean fuzzy subsets was discussed by [22].

To illustrate the importance of Pythagorean uncertainty collection to extend the grade of membership and non-membership degrees, assume that $\alpha_D(x) = 0.9$ and

$\beta_D(x) = 0.8$ for $X = \{x\}$. We obtain $0.9 + 0.8 = 1.7 > 1$,

$(0.9)^2 + (0.8)^2 = 1.45 > 1$ and $(0.9)^3 + (0.8)^3 = 1.241 > 1$ which means

that $D = (0.9,0.8)$ neither following the condition of Fermatean uncertainty set

nor follows the condition of Pythagorean uncertainty set.

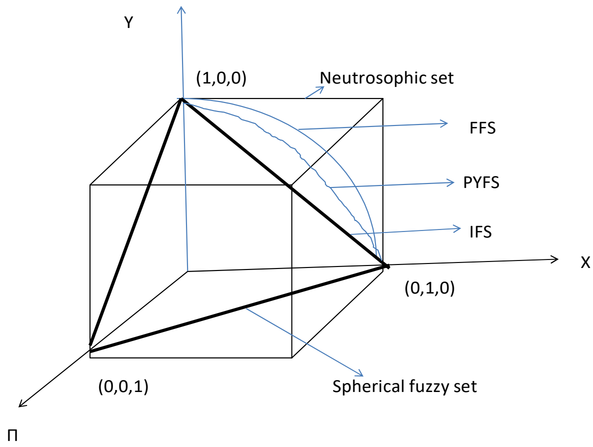
This paper contains of six sections. In section 2, we give some core explanations and effects which can be used in our discussion later. In section 3, we develop Pythagorean weighted averaging (PWA) operator and also some of their properties. In Section 4, consists an algorithm for multiple attribute group decision making (MAGDM). In section 5, we have conclusion.

2. Preliminaries:

Definition 2.1 (Fuzzy set): Let U be a non-empty set. Then by a fuzzy set on U is meant a function $A : U \to [0,1]$. A is called the membership function, A(x) is called the membership grade of x in A. We also write A= {(x, A(x)): x∈U}.

Example 2.2: Consider U = { a, b, c ,d } and $A : U \to [0,1]$ defined by A(a) = 0, A(b) = 0.7, A(c)=0.4, A(d)=1.

Definition 2.3 (Pythagorean Fuzzy set (PFS)): A Pythagorean uncertainty set D on a set X is defined by $D = \{(x, (\alpha_D(x), \beta_D(x))/x \in X\}$ where $\alpha_D: X \to [0,1]$ is the degree of membership and $\beta_D: X \to [0,1]$ is the degree of non – membership of $x \in X$, respectively which fulfill the condition $0 \leq \alpha_D^2(x) + \beta_D^2(x) \leq 1$ for all $x \in X$. The degree of indeterminacy $\pi_P(x) = \sqrt{1 - (\alpha_D(x))2 - (\beta_D(x))2}$.



Definition 2.4 (Fermatean fuzzy set) [Senapti and Yager, 2020]: Let 'X' be a universe of discourse A. Fermatean uncertainty set "F" in X is an object having the form $F = \{x, m_{F}(x), n_{F}(x) / x \in X\}$ where $m_{F}(x) : X \rightarrow 0,1$ and $n_{F}(x) : X \rightarrow 0,1$, including the condition $0 \leq m_F(x^3) + n_F(x^3) \leq 1$ for all $x \in X$. The numbers (x) signifies the level (degree) of membership and $n_F(x)$ indicate the non-membership of the element 'x' in the set F.

Definition 2.5: Let X be a universal set. Then the Pythagorean uncertainty set (briefly, Pythagorean uncertainty is defined by the following; $D = \{\langle x, \alpha_D(x), \beta_D(x)\rangle / x \in X\}$------------(1) where $\alpha_D: X \rightarrow [0,1]$ is the degree of membership and $\beta_D: X \rightarrow [0,1]$ is the degree of non – membership of $x \in X$ to D, with the condition

$$0 \leq (\alpha_D(x))^2 + (\beta_D(x))^2 \leq 1 \text{-----------} (2)$$

The Degree of indeterminacy of $x \in X$ to D is defined by

$$\pi_D(x) = \sqrt[2]{1 - [(\alpha_D(x))^2 + (\beta_D(x))^2]} \text{.------------} (3)$$

It is clear that $(\alpha_D(x))^2 + (\beta_D(x))^2 + (\pi_D(x))^4 = 1$ and $\pi_D(x) = 0$ whenever

$(\alpha_D(x))^2 + (\beta_D(x))^2 = 1$. In the case of simplicity, we shall mention the symbol $D = (\alpha_D, \beta_D)$ for the Pythagorean uncertainty set $D = \{(x, (\alpha_D(x), \beta_D(x)) / x \in X\}$.

Here, $\alpha_D^2(x) = (\alpha_D(x))^2$ and $\beta_D^2(x) = (\beta_D(x))^2$ for all $x \in X$.

Example-2.6: Let D be Pythagorean fuzzy set and $x \in X$ such that $\beta_D(x) = 0.82$ and $\pi_D(x) = 0$.Then,

$$|\alpha_D(x)| = \sqrt[3]{|(\beta_D(x) - 1)(\beta_D(x) + 1)|}$$

$$= \sqrt[3]{|(-0.18)(1.82)|}$$

$$= \sqrt[3]{0.3276}$$

In 2013, Yager defined Pythagorean uncertainty subset (PUS) as a generalization of intuitionistic uncertainty set (IUS).

Definition 2.7: Let $\sigma = (\alpha_\sigma, \beta_\sigma)$, $\sigma_1 = (\alpha_{\sigma_1}, \beta_{\sigma_1})$ and $\sigma_2 = (\alpha_{\sigma_2}, \beta_{\sigma_2})$ are three Pythagorean uncertainty numbers and $\gamma > 0$. Then

(i) $\sigma^c = (\beta_\sigma, \alpha_\sigma)$

(ii) $\sigma_1 \oplus \sigma_2 = (\sqrt{\alpha\sigma_1^2 + \alpha\sigma_2^2 - \alpha\sigma_3^2}, \beta_{\sigma_1}\beta_{\sigma_2})$

(iii) $\sigma_1 \otimes \sigma_2 = (\alpha\sigma_1\alpha\sigma_2, \sqrt{(\beta\sigma_1^2 + \beta\sigma_2^2 - \beta\sigma_1^2\beta\sigma_2^2)})$

(iv) $\gamma_\sigma = (\sqrt{(1-(1-\alpha_\sigma^2)}, \beta_\sigma^\gamma)$

(v) $\sigma^\gamma = (\alpha_\sigma^\gamma, \sqrt{(1-(1-\beta_\sigma^2)^\gamma})$

Definition 2.8: Let $\sigma = (\alpha_\sigma, \beta_\sigma)$ be a Pythagorean uncertainty value. Then we can find the score of 'σ' as the following,

$S(\sigma) = \alpha_\sigma^2 - \beta_\sigma^2$ where $S(\sigma) \in [-1, 1]$ ----------------- (1)

Definition 2.9: Let $\sigma = (\alpha_\sigma, \beta_\sigma)$ be a Pythagorean uncertainty number. Then the accuracy degree 'σ' can be defined as follows:

$H(\sigma) = \alpha_\sigma^2 + \beta_\sigma^2$ where $H(\sigma) \in [0, 1]$ -----------------(2)

Definition 2.10: Let $\sigma_1 = (\alpha_{\sigma 1}, \beta_{\sigma 1})$ and $\sigma_2 = (\alpha_{\sigma 2}, \beta_{\sigma 2})$ be the two Pythagorean uncertainty numbers. Then

$S(\sigma_1) = \alpha_{\sigma_1}^2 - \beta_{\sigma_1}^2$ $S(\sigma_2) = \alpha_{\sigma_2}^2 - \beta_{\sigma_2}^2$

$H(\sigma_1) = \alpha_{\sigma_1}^2 + \beta_{\sigma_1}^2$ $H(\sigma_2) = \alpha_{\sigma_2}^2 + \beta_{\sigma_2}^2$ are the score and accuracy of σ_1 and σ_2 respectively. The following are the holds:

(i) If $S(\sigma_2) > S(\sigma_1)$, then σ_2 is greater than σ_1 represented by $\sigma_1 < \sigma_2$.

(ii) If $S(\sigma_1) = S(\sigma_2)$, then, (a) if $H(\sigma_1) = H(\sigma_2)$, then σ_1 and σ_2 have the same information (ie). $\alpha_{\sigma_1} = \alpha_{\sigma_2}$ and

$\beta_{\sigma_1} = \beta_{\sigma_2}$ represented by $\sigma_1 = \sigma_2$.

(iii) If $H(\sigma_1) < H(\sigma_2)$, then σ_2 is greater than σ_1.

3. Pythagorean uncertainty Weighted averaging aggregation operator

Pythagorean fuzzy set was introduced by [22] but in this paper, we familiarize uncertainty weighted averaging operator with their properties.

Definition 3.1: Let $\sigma_j = (\alpha_{\sigma j}, \beta_{\sigma j})$ $(j = 1,2, \dots, n)$ be Pythagorean uncertainty variables and let Pythagorean uncertainty weighed average is a mapping from $\Delta^n \rightarrow \Delta$. Then the Pythagorean uncertainty weighed averaging aggregation operator can be defined as,

Pythagorean $FWA_r(\sigma_1, \sigma_2, \dots \sigma_n) = r_1\sigma_1 \oplus r_2\sigma_2 \oplus \dots \dots \oplus r_n\sigma_n$ ....(3)

Where $r = (r_1, r_2, r_3, \dots, r_n)$ is the weighted vector of σ_j with condition, $r_j \in [0,1]$ and $\sum_{j=1}^{n} r_j = 1$.

If $\mathbf{r} = (\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \dots, \frac{1}{n})$, then the Pythagorean FWA is converted to Pythagorean uncertainty average which is defined as,

Pythagorean FA $(S_1, S_2, \dots, S_n) = \frac{1}{n}(\sigma_1 \oplus \sigma_2 \oplus \dots\dots \oplus \sigma_n)$.

Example 3.2: Let $\sigma_1 = (0.4, 0.3)$, $\sigma_2 = (0.6, 0.4)$, $\sigma_3 = (0.7, 0.5)$, $\sigma_4 = (0.8, 0.4)$ and $r = (0.1, 0.2, 0.3, 0.4)^T$ be the weighted vector of σ_j (j=1,2,3,4). Then Pythagorean $FWA_r(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$.

$$= \left(\sqrt{1 - \prod_{j=1}^{4} (1 - \alpha_{\sigma j}^2)^{r_j}}, \prod_{j=1}^{4} (\beta_{\sigma_j})^{r_j}\right)$$

$$= \left(\sqrt{1 - (1 - \alpha_{\sigma 1}^2)^{r_1}}, (\beta_{\sigma_1})^{r_1}\right) + \left(\sqrt{1 - (1 - \alpha_{\sigma 2}^2)^{r_2}}, (\beta_{\sigma_2})^{r_2}\right) + \left(\sqrt{1 - (1 - \alpha_{\sigma 3}^2)^{r_3}}, (\beta_{\sigma_3})^{r_3}\right) + \left(\sqrt{1 - (1 - \alpha_{\sigma 4}^2)^{r_4}}, (\beta_{\sigma_4})^{r_4}\right).$$

= (0.5247, 0.4275)

Theorem 3.3: Let $\sigma_j = (\alpha_{\sigma j}, \beta_{\sigma j})$ (j=1, 2,….,n) be Pythagorean uncertainty variables, Then their aggregated value by applying Pythagorean uncertainty weighted average operator is also a Pythagorean uncertainty value Pythagorean $FWA_r(\sigma_1, \sigma_2, \dots, \sigma_n) = \sqrt{1 - \prod_{j=1}^{n} (1 - \alpha_{\sigma j}^2)^{r_j}}, \prod_{j=1}^{n} (\beta_{\sigma_j})^{r_j}$ …… (4) and also the weighted vector of σ_j (j=1, 2,…,n) is $r_n = (r_1, r_2, \dots, r_n)^T$ with some conditions $\mathbf{r_j} \in [\mathbf{0}, \mathbf{1}]$ and $\sum_{j=1}^{n} \mathbf{r_j} = \mathbf{1}$.

Proof: By mathematical induction, we can prove that equation (4) holds for all n.

First we can show that equation (4) holds for n=2. Since,

$$\mathbf{r_1}\sigma_1 = \left(\left(\sqrt{1 - (1 - \alpha_{\sigma 1}^2)^{r_1}}, (\beta_{\sigma 2})^{r_1}\right)\right.$$

$$\mathbf{r_2}\sigma_2 = \left(\sqrt{1 - (1 - \alpha_{\sigma 2}^2)^{r_2}}, (\beta_{\sigma 2})^{r_2}\right)$$

So $\mathbf{r_1}\sigma_1 \oplus \mathbf{r_2}\sigma_2$

$$=\left(\left(\sqrt{1-(1-a^2_{\sigma_1})^{r_1}}, (\beta_{\sigma_2})^{r_1}\right) \oplus \left(\left(\sqrt{1-(1-a^2_{\sigma_2})^{r_2}}, (\beta_{\sigma_2})^{r_2}\right)\right)\right)$$

$$=\left(1-(1-a^2_{\sigma_1})^{r_1} + (1-(1-a^2_{\sigma_2})^{r_2} - (1-(1-a^2_{\sigma_1})^{r_1}\left(1-(1-a^2_{\sigma_2})^{r_2}, \quad (\beta_{\sigma_2})^{r_1} (\beta_{\sigma_2})^{r_3}\right).$$

$$=\left(\sqrt{1-\prod_{j=1}^{2}(1-a^2_{\sigma_j})^{r_j}}, \prod_{j=1}^{2}(\beta_{\sigma_j})^{r_j}\right)$$

Thus equation (4) is true for n=2. Let us suppose that equation (4) is true for n=k. Then we have Pythagorean FW$\mathbf{A_r}(\sigma_1, \sigma_2, \sigma_3 \ldots . \sigma_n)$

$$= \left(\sqrt{1-\prod_{j=1}^{k}(1-a^2_{\sigma_j})^{r_j}}, \ \prod_{j=1}^{k}\beta_{\sigma_j}^{r_j}\right)$$

Now we show that equation (4) is true for n=k+1.

Pythagorean FW$\mathbf{A_r}(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_{k+1})$

$$= \left(\sqrt{1-\prod_{j=1}^{k}(1-a^2_{\sigma_j})^{r_j}}, \ \prod_{j=1}^{k}\beta_{\sigma_j}^{r_j}\right) \oplus \left(\sqrt{1-(1-a^3_{\sigma_{k+1}})^{r_{k+1}}}, (\beta\sigma_{k+1})^{r_{k+1}}\right)$$

$$\left(\sqrt{1-\prod_{j=1}^{k+1}(1-a^2_{\sigma_j})^{r_j}}, \prod_{j=1}^{k+1}(\beta_{\sigma_j})^{r_j}\right)$$

Hence equation (4) holds for n=k+1. Thus equation (4) holds for all n.

Theorem 3.4: Let $\sigma_j = (a\sigma_j, \beta\sigma_j)(\mathbf{j = 1, 2, .., n})$ be the Pythagorean uncertainty variables and the weight vector of $\sigma_j(\mathbf{j = 1, 2, 3, \ldots, n})$ is $\mathbf{r = (r_1, r_2, r_3, \ldots, r_n)}$ with some conditions $\mathbf{r_j} \in [\mathbf{0, 1}]$ and $\sum_{j=1}^{n}\mathbf{r_j = 1}$. If $\sigma_j(\mathbf{j = 1, 2, 3, \ldots, n})$ are mathematically equal, then Pythagorean uncertainty $\mathbf{FWA_r}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \sigma - - - (\mathbf{5})$

Proof: As we know that, Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \mathbf{r_1}\sigma_1 \oplus \mathbf{r_2}\sigma_2 \oplus \ldots \oplus \mathbf{r_n}\sigma_n$.

Let $\sigma_j(\mathbf{j = 1, 2, \ldots, n}) = \sigma$ then Pythagorean uncertainty $\mathbf{FWA_r}(\sigma_1, \sigma_2, \ldots, \sigma_n) = \mathbf{r_1}\sigma_1 \oplus \mathbf{r_2}\sigma_2 \oplus \ldots \oplus \mathbf{r_n}\sigma_n = \sigma\sum_{j=1}^{n}\mathbf{r_j} = \sigma$

Theorem 3.5: Let σ_j be Pythagorean uncertainty variables and let the weighted vector of σ_j **be** $\mathbf{r = (r_1, r_2, \ldots, r_n)^T}$ such that $\mathbf{r_j} \in [\mathbf{0, 1}]$ **and** $\sum_{j=1}^{n}\mathbf{r_j = 1}$.

$$\sigma^- = (\min_j(a\sigma_j), \max_j(\beta\sigma_1))$$

$$\sigma^+ = (\max_j(a\sigma_j), \min_j(\beta\sigma_j)),$$

Then $\sigma^- \leq \textbf{Pythagorean FWA}_\mathbf{r}(\sigma_1, \sigma_2, \ldots, \sigma_n) \leq \sigma^+ - - - - (\mathbf{6})$

Proof: we know that

$\min_j(a\sigma_j) \leq a\sigma_j \leq \max_j(a\sigma_j)$ --------------- (7)

$\min_j(\beta\sigma_j) \le \beta\sigma_j \le \max_j(\beta\sigma_j)$ ---------------(8)

From equation (7), we have

$$-\sqrt{\min_j(a\sigma_j)^2} \le \sqrt{(a\sigma_j)^2} \le \sqrt{\max_j(a\sigma_j)^2}$$

$$-\sqrt{(1-\max_j(a\sigma_j)^2)^{r_j}} \le \sqrt{(1-a^2\sigma_j)^{r_j}} \le \sqrt{(1-\min_j(a\sigma^2_j))^{r_j}}$$

$$-\sqrt{(-1+\min_j(a\sigma^2))} \le \sqrt{-\prod_{i=1}^{n}(1-a^2\sigma_j)^{r_j}} \le \sqrt{(-1+\min_j(a\sigma^2))}$$

$$-\min_j(a\sigma_j) \le \sqrt{1-\prod_{i=1}^{n}(1-a^2\sigma_j)^{r_j}} \le \max_j(a\sigma_j)$$

Now from equation (8), we have

$$-\min_j(\beta\sigma_j^{r_j})^{r_j} \le \mathop{\mathbf{G}}_{j=1}^{n}(\beta\sigma_j)^{r_j} \le \max_j(\beta\sigma_j)^{r_j}$$

$$-\min_j(\beta\sigma_j)^{\sum_{j=1}^{n} r_j} \le \mathop{\mathbf{G}}_{j=1}^{n}(\beta\sigma_j)^{r_j} \le \max_j(\beta\sigma_j)^{\sum_{j=1}^{n} r_j}$$

$$-\min_j(\beta\sigma_j) \le \mathop{\mathbf{G}}_{j=1}^{n}(\beta_j)^{r_j} \le \max_j(\beta\sigma_j) - - - - - (9)$$

Let Pythagorean $\mathbf{FWA_r}(\sigma_1,\sigma_2,\sigma_3,..,\sigma_n) = \sigma$. Then $\mathbf{S}(\sigma) = a_\sigma^2 - \beta_\sigma^2$

$$\le \max_j(a_\sigma)^2 - \min_j(\beta_\sigma)^2 = \mathbf{S}(\sigma^+)$$

Thus $\mathbf{S}(\sigma) \le \mathbf{S}(\sigma^+)$. Again

$$\mathbf{S}(\sigma) = a_\sigma^2 - \beta_\sigma^2$$

$$\ge \min_j(a\sigma)^2 - \max_j(\beta\sigma)^2 = \mathbf{S}(\sigma^-).$$

Thus $\mathbf{S}(\sigma) \ge \mathbf{S}(\sigma^-)$. If $\mathbf{S}(\sigma) < \mathbf{S}(\sigma^+)$ **and** $\mathbf{S}(\sigma) > \mathbf{S}(\sigma^-)$, then

$\sigma^- < \mathbf{PFWA_r}(\sigma_1,\sigma_2,\sigma_3,..,\sigma_n) < \sigma^+$--------(10)

If $(\sigma) = \mathbf{S}(\sigma^+)$, then

$$-\ a_\sigma^2 - \beta_\sigma^2 = \max_j(a_{\sigma_j})^2 - \min_j(\beta\sigma_j)^2$$

$$-\ a_\sigma^2 = \max_j(a\sigma_j)^2,\ \beta_\sigma^2 = \min_j(\beta\sigma_j)^2$$

$$-\ a_\sigma = \max_j(a\sigma_j),\ \beta_\sigma = \min_j(\beta\sigma_j)$$

Since

$$\mathbf{H}(\sigma) = \alpha_\sigma^2 + \beta_\sigma^2 = \max_j(\alpha\sigma_j)^2 + \min_j(\beta\sigma_j)^2 = \mathbf{H}(\sigma^+)$$

Thus, Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) = \sigma^+ --- (\mathbf{11})$

If S (σ) = S(σ^-), then

$$- \ \alpha_\sigma^2 - \beta_\sigma^2 = \min_j(\beta_{\sigma_j})^2 - \min_j(\alpha\sigma_j)^2$$

$$\alpha_\sigma^2 = \min_j(\beta\sigma_j)^2, \beta_\sigma^2 = \min_j(\alpha\sigma_j)^2$$

$$- \ \alpha_\sigma = \min_j(\beta\sigma_j), \beta_\sigma = \max_j(\alpha\sigma_j)$$

Since

$$\mathbf{H}(\sigma) = \alpha_\sigma^2 + \beta_\sigma^2 = \min_j(\beta\sigma_j)^2 + \min_j(\alpha\sigma_j)^2 = \mathbf{H}(\sigma^-)$$

Thus , Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n) = \sigma^- --- (\mathbf{12})$

Thus from the equations (11) and (12), we have

$\sigma^- \leq$ **Pythagorean** $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n) \leq \sigma^+$. Hence the proof.

Theorem 3.6: Let $\sigma_j(\mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n})$ and $\sigma_j^*(\mathbf{j} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{n})$ be the two collection of Pythagorean uncertainty variables. If $\alpha\sigma_j \leq \alpha\sigma_j^*$ and $\beta\sigma_j \geq \beta\sigma_j^*$. Then Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) \leq$ **Pythagorean** $\mathbf{FWA_r}(\sigma_1^*, \sigma_2^*, \sigma_3^*, .., \sigma_n^*)$

Proof: Since $\alpha\sigma_j \leq \alpha\sigma_j^*$ and $\beta\sigma_j \geq \beta\sigma_j^*$.

Then $\Longleftrightarrow \alpha^2\sigma_j^* \leq \alpha^2\sigma_j^*$

$$- \sqrt{\mathbf{1} - \alpha^2\sigma_j^*} \leq \sqrt{\mathbf{1} - \alpha^2\sigma_j}$$

$$- \sqrt{(\mathbf{1} - \alpha^2\sigma_j^*)^{r_j}} \leq \sqrt{(\mathbf{1} - \alpha^2\sigma_j)^{r_j}}$$

$$- \sqrt{(\mathbf{1} - \textstyle\prod_{i=1}^{n} (\mathbf{1} - \alpha^2\sigma_j)^{r_j}} \leq \sqrt{(\mathbf{1} - \textstyle\prod_{j=1}^{n} (\mathbf{1} - \alpha^2\sigma_j^*)^{r_j}} \text{---------(13)}$$

Now $\beta\sigma_j \geq \beta\sigma_j^*$,

$\beta^{r_j}\sigma_j \geq \beta^{r_j}\sigma_j^*$

$- \prod_{i=1}^{n} \beta^{r_j}\sigma_j \geq \prod_{i=1}^{n} \beta^{r_j}\sigma_j^*$ ---------(14)

Let

Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) = \sigma$ -----------(15)
Pythagorean $\mathbf{FWA_r}(\sigma^*_1, \sigma^*_2, \sigma^*_3, .., \sigma^*_n) = \sigma^*$ ---------(16)

Then form equation (15) and (16), we have $\mathbf{S}(\sigma) \leq \mathbf{S}(\sigma^*)$.

If $\mathbf{S}(\sigma) < \mathbf{S}(\sigma^*)$, then
Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) <$ **Pythagorean** $\mathbf{FWA_r}(\sigma^*_1, \sigma^*_2, \sigma^*_3, ..., \sigma^*_n)$ -------(17)

If $\mathbf{S}(\sigma) = \mathbf{S}(\sigma^*)$,

Then $\Longleftrightarrow a\sigma_j^2 - \beta\sigma_j^2 = a\sigma_{j^*}^2 - \beta\sigma_{j^*}^2$

- $a\sigma_j^2 = a^2\sigma^*_j, \beta\sigma_j^2 = \beta^2\sigma^*_j$
- $a\sigma_j = a\sigma^*_j, \beta\sigma_j = \beta\sigma^*_j$

Since

$\mathbf{H}(\sigma) = a\sigma_j^2 + \beta\sigma_j^2$
$= a^2\sigma^*_j + \beta^2\sigma^*_j$
$= \mathbf{H}(\sigma^*)$

Thus, Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) =$ (**Pythagorean**)$\mathbf{FWA_r}(\sigma^*_1, \sigma^*_2, \sigma^*_3, .., \sigma^*_n)$-------(18)

Thus from equations (17) and (18), we have $\mathbf{A_r}(\sigma_1, \sigma_2, \sigma_3, .., \sigma_n) \leq$ **Pythagorean** $\mathbf{FWA_r}(\sigma^*_1, \sigma^*_2, \sigma^*_3, .., \sigma^*_n)$.

Example 3.7: $\sigma_1 = (\mathbf{0.4}, \mathbf{0.6})$, $\sigma_2 = (\mathbf{0.4}, \mathbf{0.7})$
$\sigma_3 = (\mathbf{0.5}, \mathbf{0.7})$, $\sigma_4 = (\mathbf{0.6}, \mathbf{0.6})$

and

$\sigma^*_1 = (\mathbf{0.7}, \mathbf{0.6})$, $\sigma^*_2 = (\mathbf{0.8}, \mathbf{0.6})$, $\sigma^*_3 = (\mathbf{0.9}, \mathbf{0.6})$, $\sigma^*_4 = (\mathbf{0.8}, \mathbf{0.3})$

Where $\mathbf{r} = (\mathbf{0.1}, \mathbf{0.2}, \mathbf{0.3}, \mathbf{0.4})$.

Now using the Pythagorean **FWA** operator, we get the following result

Pythagorean $\mathbf{FWA_r}(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \left(\sqrt{1 - \prod_{j=1}^{4} (1 - a^2_{\sigma_j})^{r_j}}, \prod_{j=1}^{4} (\beta_{\sigma_j})^{r_j}\right)$.

= (0.527,0.5210)

Again **Pythagorean** $\mathbf{FWA_r}(\sigma^*_1, \sigma^*_2, \sigma^*_3, \sigma^*_4) = \left(\sqrt{1 - \prod_{j=1}^{4} (1 - a^2\sigma^*_j)^{r_j}}, \prod_{j=1}^{4} (\beta_{\sigma^*_j})^{r_j}\right)$

= (0.7267, 0.1297)

Theorem 3.8: (Commutative law). If $\sigma_{\mathbf{ij}} = (a_{\sigma_{\mathbf{ij}}}, \beta_{\sigma_{\mathbf{ij}}})$ $(\mathbf{j} = \mathbf{1}, \mathbf{2})$ be two Pythagorean uncertainty numbers, then

(i) $\sigma_{\mathbf{11}} \oplus \sigma_{\mathbf{12}} = \sigma_{\mathbf{12}} \oplus \sigma_{\mathbf{11}}$

(ii) $\sigma_{\mathbf{11}} \otimes \sigma_{\mathbf{12}} = \sigma_{\mathbf{12}} \otimes \sigma_{\mathbf{11}}$

Proof: It is obvious

Theorem 3.9: (Associative law). If $\sigma_{\mathbf{ij}} = (a_{\sigma_{\mathbf{ij}}}, \beta_{\sigma_{\mathbf{ij}}})$ $(\mathbf{j} = \mathbf{1}, \mathbf{2}, \mathbf{3})$ be three Pythagorean uncertainty numbers, then

(i) $(\sigma_{\mathbf{11}} \oplus \sigma_{\mathbf{12}}) \oplus \sigma_{\mathbf{13}} = \sigma_{\mathbf{11}} \oplus (\sigma_{\mathbf{12}} \oplus \sigma_{\mathbf{13}})$

(ii) $(\sigma_{\mathbf{11}} \otimes \sigma_{\mathbf{12}}) \otimes \sigma_{\mathbf{13}} = \sigma_{\mathbf{11}} \otimes (\sigma_{\mathbf{12}} \otimes \sigma_{\mathbf{13}})$

Proof: It is obvious

Theorem 3.10: Let $\sigma = (a, \beta)$ and $\sigma_{\mathbf{ij}} = (a_{\sigma_{\mathbf{ij}}}, \beta_{\sigma_{\mathbf{ij}}})$ $(\mathbf{j} = \mathbf{1}, \mathbf{2})$ be Pythagorean uncertainty numbers and a real number $\lambda > \mathbf{o}$, we have

(**i**)$\lambda(\sigma_{\mathbf{11}} \oplus \sigma_{\mathbf{12}}) = \lambda\sigma_{\mathbf{11}} \oplus \lambda\sigma_{\mathbf{12}}$

(ii)$(\sigma_{\mathbf{11}} \otimes \sigma_{\mathbf{12}})^{\lambda} = \sigma^{\lambda}_{\mathbf{11}} \otimes \sigma^{\lambda}_{\mathbf{12}}$

(iii) $\lambda_{\mathbf{1}}\sigma \oplus \lambda_{\mathbf{2}}\sigma = (\lambda_{\mathbf{1}} + \lambda_{\mathbf{2}})\sigma$

(iv)$\sigma^{\lambda_1} \otimes \sigma^{\lambda_2} = \sigma^{\lambda_1 + \lambda_2}$.

Proof: Here, we prove the parts (i) and (iii) only and the proof of others are similar.

$$\lambda\sigma_{11} = (\mathbf{1} - (\mathbf{1} - a_{11})^{\lambda}, (\mathbf{1} - a_{11})^{\lambda} - (\mathbf{1} - a_{11} - \beta_{11})^{\lambda}))$$

and

$$\lambda\sigma_{12} = (\mathbf{1} - (\mathbf{1} - a_{12})^{\lambda}, (\mathbf{1} - a_{12})^{\lambda} - (\mathbf{1} - a_{12} - \beta_{12})^{\lambda}))$$

Thus, we have

$$\lambda\sigma_{11} \oplus \lambda\sigma_{12} = (\mathbf{1} - (\mathbf{1} - a_{11})^{\lambda}(\mathbf{1} - a_{12})^{\lambda}, (\mathbf{1} - a_{11})^{\lambda} (\mathbf{1} - a_{12})^{\lambda} - \{(\mathbf{1} - \mathbf{1} + (\mathbf{1} - a_{11})^{\lambda} - (\mathbf{1} - a_{11})^{\lambda} + (\mathbf{1} - a_{11} - \beta_{11})^{\lambda}) \times (\mathbf{1} - \mathbf{1} + (\mathbf{1} - a_{12})^{\lambda} - (\mathbf{1} - a_{12})^{\lambda} + (\mathbf{1} - a_{12} - \beta_{12})^{\lambda})\})$$

$$=(\mathbf{1} - (\mathbf{1} - a_{11})^{\lambda}(\mathbf{1} - a_{12})^{\lambda}, (\mathbf{1} - a_{11})^{\lambda}(\mathbf{1} - a_{12})^{\lambda} - (\mathbf{1} - a_{11} - \beta_{11})^{\lambda}(\mathbf{1} - a_{12} - \beta_{12})^{\lambda})$$

$$= \lambda(\sigma_{\mathbf{11}} \oplus \sigma_{\mathbf{12}}).$$

For $\lambda_{\mathbf{1}}, \lambda_{\mathbf{2}} > \mathbf{0}$ and the Pythagorean uncertainty numbers $\sigma = (a, \beta)$, we have

$$\lambda_{\mathbf{1}}\sigma = (\mathbf{1} - (\mathbf{1} - a)^{\lambda_1}, (\mathbf{1} - a)^{\lambda_1} - (\mathbf{1} - a - \beta)^{\lambda_1})$$

and

$$\lambda_2\sigma = (\mathbf{1} - (\mathbf{1} - a)^{\lambda_2}, (\mathbf{1} - a)^{\lambda_2} - (\mathbf{1} - a - \beta)^{\lambda_2})$$

$$\lambda_1\sigma \oplus \lambda_2\sigma = (\mathbf{1} - (\mathbf{1} - a)^{\lambda_1} (\mathbf{1} - a)^{\lambda_2} (\mathbf{1} - a)^{\lambda_1} (\mathbf{1} - a)^{\lambda_2} - (\mathbf{1} - a - \beta)^{\lambda_1} \times (\mathbf{1} - a - \beta)^{\lambda_2})$$

$$= (\mathbf{1} - (\mathbf{1} - a)^{\lambda_1+\lambda_2}, (\mathbf{1} - a)^{\lambda_1+\lambda_2} - (\mathbf{1} - a - \beta)^{\lambda_1+\lambda_2})$$

$$=(\lambda_1 + \lambda_2)\sigma$$

4. Pythagorean uncertainty weighted averaging Aggregation operator to multiple attribute group decision making

Let $\mathbf{P} = \{\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \ldots\ldots, \mathbf{p_n}\}$ **be a set of n alternatives and** $\mathbf{Q} = \{\mathbf{q_1}, \mathbf{q_2}, \ldots, \mathbf{q_m}\}$ be a set of m alternatives and $\mathbf{r} = (\mathbf{r_1}, \mathbf{r_2}, \ldots, \mathbf{r_m})^\mathbf{T}$ be the weighted vector of the attributes $\mathbf{Q_i}(\mathbf{i} = \mathbf{1}, \mathbf{2}, \ldots, \mathbf{m})$ such that $\mathbf{w_i} e [\mathbf{0}, \mathbf{1}]$ **and** $\sum_{i=1}^{m} w_i = 1$.

4.1 : Algorithm

Step-1: The decision makers provide the information in the form of a matrix.

Step-2: Compute $\sigma_j (j = 1,2 \ldots, n)$ using Pythagorean fuzzy weighted averaging (Pythagorean FWA) aggregation operator.

Step-3: Compute the scores of $\sigma_j (j = 1,2 \ldots, n)$. If there is no difference between two or more than two scores, then we must have to calculate the degrees of accuracy.

Step-4: Arrange the score function of all alternatives in the form of descending order and select the alternatives, which has the highest score function value.

4.2 Numerical example: we consider an example for selecting a watch from different cell phones.

Suppose a customer wants to buy a cell phone from different cell phones. Let p_1, p_2, p_3, p_4, p_5 represent the five cellphones of different companies. Let Q_1, Q_2, Q_3 be the criteria of these cellphones. In the process of choosing one of the cellphones; three factors are considered.

Q_1: Price of each cellphone

Q_2: Model of each cellphone

Q_3: Design of each cell phone

Suppose the weight vector of $Q_j(j = 1,2,3)$ is $r = (0.3,0.4,0.5)^T$ and the Pythagorean uncertainty values of the alternatives $P_j(j = 1,2,3,4,5)$ are represented by the following decision matrix.

Step-1: The decision maker gives the decision in Table

Table: Pythagorean uncertainty decision matrix

	P_1	P_2	P_3	P_4	P_5
Q_1	(0,0)	(0.2,0.1)	(0.4,0.2)	(0.5,0.4)	(0.6,0.3)
Q_2	(0.3,0.1)	(0.4,0.2)	(0.4,0.3)	(0.3,0.2)	(0.7,0.5)
Q_3	(0.6,0.4)	(0.4,0.3)	(0.5,0.2)	(0.6,0.3)	(0.4,0.2)

Step-2: Compute $\sigma_j(j = 1,2,3,4,5)$ by applying Pythagorean uncertainty weighted average operator

$\sigma_1 = (0.5220,0.3020)$ $\sigma_2 = (0.5000,0.3267)$

$\sigma_3 = (0.5348,0.2147)$ $\sigma_4 = (0.7592,0.2016)$

$\sigma_3 = (0.5201,0.3020)$

Step-3: We can find the scores of $\sigma_j(j = 1,2,3,4,5)$

$S(\sigma_1) = (0.5220)^2 - (0.3020)^2 = 0.27248 - 0.0912 = 0.181284$

$S(\sigma_2) = (0.5000)^2 - (0.3267)^2 = 0.25 - 0.1067 = 0.1433$

$S(\sigma_3) = (0.5348)^2 - (0.2147)^2 = 0.28601 - 0.046 = 0.24001$

$S(\sigma_4) = (0.7592)^2 - (0.2016)^2 = 0.57638 - 0.0406 = 0.53578$

$S(\sigma_5) = (0.5201)^2 - (0.3020)^2 = 0.27050 - 0.0912 = 0.17930$

and the accuracy function,

$H(\sigma_1) = (0.5220)^2 + (0.3020)^2 = 0.3637$, $H(\sigma_2) = 0.3567$, $H(\sigma_3) = 0.3320$, $H(\sigma_4) = 0.6169$, $H(\sigma_5) = 0.3617$.

Step 4: Arrange the scores of the alternatives in the form of descending order and select the alternatives, which has the highest score function. Since $\sigma_4 > \sigma_3 > \sigma_1 > \sigma_5 > \sigma_2$.

Hence $P_4 > P_1 > P_5 > P_2 > P_3$. Thus the type of cellphone P_4 is the best option for the customer.

5. Conclusion:

An aggregation operator based on Pythagorean fuzzy number and applied them to the multivariable decision making problem, where the values are Pythagorean uncertainty numbers is to be presented. Firstly, we have developed Pythagorean uncertainty weighted averaging aggregation operator along with their properties namely idempotency, boundedness and monotonically. Finally, we have developed a method for multi criteria decision making based on the proposed operator and the operational process have illustrated in detail. The main advantage of using the proposed method and operator is that this method provides more general, accurate and precise results. Therefore, the suggested methodology can be used for any type of selection problem involving any number of selection attributes. This method plays a vital role in real world situations. We ended the paper with an application of Pythagorean uncertainty decision making problem.

Future Work:

In future, some author may develop this given operators in various fuzzy Environment.

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