Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

Coefficient Characterization For Some Spiral-Like Subclasses Of Generalized Rational Univalent Functions

S. Lalitha[1], G. Padma[2] , K.V.Vidyasagar[3]

[1.] *Department of Mathematics, Geethanjali College of Engineering and Technology, Hyderabad, Telangana, India.*
[2.] *Department of Mathematics, Vardhaman College of Engineering, Hyderabad, Telangana, India.*
[3.] *Department of Mathematics, SVLNS Government Degree College., Bheemunipatnam, Visakhapatnam, Andhra Pradesh, India.*
(Communicating Author: K.V.Vidyasagar, Lecturer in Mathematics, Mail Id: vidyavijaya08@gmail.com,)

The aim of this paper is to introduce Spiral-like subclasses $SP_{+}^{\lambda}(b_1, \alpha)$ and $SP_{+}(\lambda, \rho)$ of generalized rational functions and study the geometric properties like coefficient characterization, growth and distortion bounds.

Keywords: univalent, spiral-like, rational univalent

Introduction

A function f(z) which is normalized and analytic in the open unit disk around the origin, non-vanishing outside the origin and is of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ can take the form $\frac{z}{g(z)}$ where g(z) has Taylor coefficientsb_n's in unit disk U.

Sufficient condition for functions of the form

$$\frac{z}{1+b_1 z+\cdots+b_n z^n}, \quad b_n \neq 0$$

(1)

to be univalent in U was obtained by Mitrinovic [2]

Theorem[2] The function $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ is in S if

$\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$ and $\sum_{n=1}^{\infty}|b_n| \leq 1.$

Different subclasses of univalent rational functions were introduced by Reade et al. [5]. Also obtained sufficient conditions for $f(z) \in S$ to be in those subclasses.

Spacek [8] defined another interesting subclass of univalent functions, the class of spiral-like functions and proved that a function f(z) is λ-spiral-like if it satisfies

$$\operatorname{Re}\left[\frac{e^{i\lambda} z f'(z)}{f(z)}\right] > 0 \text{ for some real } \lambda \left(|\lambda| \le \frac{\pi}{2}\right) \text{ and for all } z \text{ in U.}$$

The class of λ-spiral-like functions is denoted by SP^{λ} or S^{λ} .
A function f(z) analytic on open unit disk U with the usual normalization is called λ-spiral-like of order α, if the inequality

$$\operatorname{Re}\left[\frac{e^{i\lambda} z f'(z)}{f(z)}\right] > \alpha \cos \lambda$$

holds for some real α, λ $\left(0 \le \alpha < 1,\ |\lambda| \le \frac{\pi}{2}\right)$ and for all z in U.
This class is known as the class of λ-spiral-like functions of order α and is denoted by $SP^{\lambda}(\alpha)$ or $S^{\lambda}(\alpha)$
And note that $SP^{\lambda}(\alpha) \subset SP^{\lambda}$, $SP^{0}(\alpha) = S^{*}(\alpha)$ and $SP^{0}(0) = S^{*}$.

Ahuja and Jain [1] proved the following sufficient condition on $b_n{}'s$ for the function f(z) of the form (1) to be $\lambda -$ Robertson function of order α:

Theorem : Let $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and α, λ be constants $0 \le \alpha < 1$, $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. f(z) is said to be $\lambda -$ Robertson function of order α if the coefficients $b_n{}'s$ satisfy

$$\frac{3+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}|b_1| + \sum_{n=1}^{\infty} \frac{(k-1)(3k+(1-\alpha)\cos\lambda)}{(1-\alpha)\cos\lambda}|b_n| \le 1.$$

Ahuja and Jain [1] studied the properties of spiral-likeness of rational functions.

Obradovi´c and Ponnusamy [4] introduced a subclass of rational univalent functions S_{+}, as the subclass of functions of S which can be expressed in the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)} \qquad (2)$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and obtained necessary and sufficient conditions for functions of S to be in S_{+}.

Theorem [4] Let $f \in A$. Then $f \in S_{+}$ if and only if f can take the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1}{n-1} z^n, & \text{for } n = 2,3,\ldots \end{cases}$$

This paper introduces spiral-like subclasses of generalized rational univalent functions for which this type of characterization could be derived.
Now, we define two subclasses of S_{+} by fixing b_1 and obtain characterization for these subclasses similar to that in [4] for S_{+}.

Section 1

Ahuja and Jain [1] obtained a condition on the coefficients $\{b_n\}_{n=1}^{\infty}$ that ensures spiral-likeness of a class of functions of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$. The condition is as follows:

Theorem[1] Let $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n,}$ and α, λ be constants such that $0 \le \alpha < 1$, $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. If the coefficients $\{b_n\}_{n=1}^{\infty}$ satisfy

$\sum_{n=1}^{\infty}\left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right]|b_n| \le 2(1-\alpha)\cos\lambda$

then f is λ – spiral-like of order α in the unit disk U around the origin.

By imposing this condition, this section introduces a subclass $SP_+^{\lambda}(b_1, \alpha)$ of rational univalent functions by fixing b_1 of $g(z)$. And studies coefficient characterization, growth and distortion bounds for the subclass $SP_+^{\lambda}(b_1, \alpha)$.

Definition 1.1

Let$b_1 \in \mathbb{C}$, $0 \le |b_1| \le 1$ be fixed and $0 \le \alpha < 1$.

$$SP_+^{\lambda}(b_1, \alpha) = \{f(z) \in S : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,\ z \in U$$

$$\sum_{n=1}^{\infty}\left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right]b_n \le 2(1-\alpha)\cos\lambda,\ \text{for } b_n \ge 0,$$

$$0 \le \alpha < 1,\ \frac{-\pi}{2} < \lambda < \frac{\pi}{2} \quad \text{for } n \ge 2\}. \tag{3}$$

The following result gives coefficient characterization for the class $SP_+^{\lambda}(b_1, \alpha)$:

Theorem 1.2

Let $f(z) \in S$ for $z \in U$ and $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and $b_1 \in \mathbb{C}$, $0 \le |b_1| \le 1$ be fixed.

Then $f(z) \in SP_+^{\lambda}(b_1, \alpha)$ for $0 \le \alpha < 1$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)} \tag{4}$$

for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & for \quad n = 1 \\ 1 + \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n, & for \quad n = 2,3,\dots \end{cases}$$

Proof : Suppose $f(z) \in S$ has the form $\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$ for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$.

Now, to prove that the function $f(z) \in SP_+^{\lambda}(b_1, \alpha)$ rewrite the function $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n \left[1 + \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right]$$

(by the definition of $\frac{z}{f_n(z)}$)

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n + \sum_{n=2}^{\infty} \mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}} z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n \text{ where}$$

$$b_n = \mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}}\right] \geq 0 \text{ for } n \geq 2.$$

Choosing $\mu_1 \in \mathbb{R}$ such that

$$|b_1| \leq \left[\frac{2(1-\alpha)\cos\lambda}{1 + \{1 - 4(1-\alpha)(\alpha)\cos^2\lambda\}^{1/2}}\right] \mu_1 \leq 1,$$

This gives $\sum_{n=1}^{\infty} \left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right] b_n$

$$= \left[1 + \{1 - 4(1-\alpha)(\alpha)\cos^2\lambda\}^{1/2}\right] |b_1| + \sum_{n=2}^{\infty} \left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right] b_n$$

$$\leq [2(1-\alpha)\cos\lambda]\mu_1 + \sum_{n=2}^{\infty} [2(1-\alpha)\cos\lambda]\mu_n$$

$$= [2(1-\alpha)\cos\lambda]\left(\mu_1 + \sum_{n=2}^{\infty} \mu_n\right) = 2(1-\alpha)\cos\lambda$$

Thus $f(z)$ satisfies (3).

This shows that $f(z) \in SP_+^{\lambda}(b_1, \alpha)$

Conversely, suppose $f(z) \in SP_+^{\lambda}(b_1, \alpha)$.

Then by the definition of $SP_+^{\lambda}(b_1, \alpha)$

$$\sum_{n=1}^{\infty} \left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right] |b_n| \leq 2(1-\alpha)\cos\lambda.$$

Now, taking

$$\mu_n = \frac{n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}}{2(1-\alpha)\cos\lambda} b_n \text{ for } n = 2,3, \dots,$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$

then the function takes the form

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= \mu_1 + \sum_{n=2}^{\infty} \mu_n + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}} z^n$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n \left[1 + \frac{2(1-\alpha)\cos\lambda}{n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

Hence the theorem is proved.

Next results give growth and distortion bounds for the functions of subclass $SP_+^{\lambda}(b_1, \alpha)$

Theorem 1.3

If $f(z) \in SP_+^{\lambda}(b_1, \alpha), z \in U$, for $0 \leq \alpha < 1$ and $0 \leq |b_1| \leq 1$, $|z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1| r - \frac{(1-\alpha)}{1 + \{1 - (1-\alpha^2)\cos^2\lambda\}^{1/2}} r^2\right\} \leq \left|\frac{z}{f(z)}\right|$$

$$\leq 1 + |b_1| r + \frac{(1-\alpha)}{1 + \{1 - (1-\alpha^2)\cos^2\lambda\}^{1/2}} r^2$$

Proof: Since $f(z) \in SP_+^{\lambda}(b_1, \alpha)$, $z \in U$,

by Theorem 1.2,

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

(5)

Thus $\left|\frac{z}{f(z)}\right| \le 1 + |b_1 z| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right|$

$$\le 1 + |b_1||z| + |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\le 1 + |b_1| r + \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2 \text{ for } |z| = r < 1.$$

And $\left|\frac{z}{f(z)}\right| \ge 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right|$

$$\ge 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\ge 1 - |b_1| r - \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2 \text{ for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1| r - \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2\right\} \le \left|\frac{z}{f(z)}\right|$$

$$\le 1 + |b_1| r + \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2$$

Theorem 1.4

If $f(z) \in SP_+^{\lambda}(b_1, \alpha), z \in U,$ for $0 \le \alpha < 1,\ 0 \le |b_1| \le 1$ and $|z| = r < 1,$ then $\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r\right\} \le \left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r$

Proof: Since $f(z) \in SP_+^{\lambda}(b_1, \alpha)$, from (5) the function $\frac{z}{f(z)}$ can be expressed as

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}$$

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}\right|$$

$$\le |b_1| + |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\le |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r \quad \text{for } |z| = r < 1$$

and

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \ge |b_1| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}\right|$$

$$\ge |b_1| - |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\ge |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r\right\} \le \left|\left\{\frac{z}{f(z)}\right\}'\right| \le |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r$$

Section 2

Ahuja and Jain [1] obtained coefficient bounds for Taylor coefficients $b_n's$ of $f(z)$ for $f(z)$ to be in $SP(\lambda,\rho)$.

Proposition [1]

If $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ for z in U, $b_n's$ are complex, λ is real with $|\lambda| < \pi/2$ and

$$\sum_{n=1}^{\infty}\left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right)|\,b_n| \leq 2(1-\rho)\cos\lambda$$

Then $f(z) \in SP(\lambda,\rho)$.

Imposing this condition, next section introduces a subclass $SP_+(\lambda,\rho)$ of rational univalent functions by fixing b_1 and gives coefficient characterization, growth and distortion bounds for the subclass $SP_+(\lambda,\rho)$.

Definition 2.1

Let $b_1 \in \mathbb{C}$, $0 \leq |b_1| \leq 1$ be fixed and $0 \leq \alpha < 1$.

Define

$SP_+(\lambda,\rho) = \{f(z) \in S : f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n},\ z \in U$ and $b_n \geq 0\ for\ n \geq 2$, λ is real with $|\lambda| < \pi/2$ and $\sum_{n=1}^{\infty}\left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right) b_n \leq 2(1-\rho)\cos\lambda\ \}$

(6)

Coefficient characterization for the subclass $SP_+(\lambda,\rho)$ of S_+ is given by the following result:

Theorem 2.2

Let $f(z) \in S$ for $z \in U$ be of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and $b_1 \in \mathbb{C}$, $|b_1| \leq 1$ be fixed.

Then $f(z) \in SP_+(\lambda,\rho)$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

(7)

For some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & for\ \ n = 1 \\ 1 + \frac{2(1-\rho)\cos\lambda}{n+\left|n\, e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n, & for\ \ n = 2,3,\ldots \end{cases}$$

Proof : Suppose $f(z) \in S$ has the form $\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$ for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$.

To prove that the function $f(z) \in SP_+(\lambda,\rho)$, rewrite $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n \left[1 + \frac{2(1-\rho)\cos\lambda}{n+\left|n\, e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n\right]$$

(by the definition of $\frac{z}{f_n(z)}$)

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+\left|n\, e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

where $b_n = \mu_n \left[\frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|}\right] \geq 0$ for $n \geq 2.$

Taking $b_1 \leq \frac{2(1-\rho)\cos\lambda}{1 + |\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} \mu_1,$

$$\sum_{n=1}^{\infty} \left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right) b_n$$

$$= \left[1 + \left|\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right] b_1 + \sum_{n=2}^{\infty} \left[n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right] b_n$$

$$\leq [2(1-\rho)\cos\lambda]\mu_1 + \sum_{n=2}^{\infty} [2(1-\rho)\cos\lambda]\mu_n$$

$$= [2(1-\rho)\cos\lambda]\left(\mu_1 + \sum_{n=2}^{\infty} \mu_n\right) = 2(1-\rho)\cos\lambda$$

Thus $f(z) \in SP_+(\lambda,\rho)$, by the definition of $SP_+(\lambda,\rho)$

Conversely, suppose $(z) \in SP_+(\lambda,\rho)$.

Then by the definition of $SP_+(\lambda,\rho)$,

$$\sum_{n=1}^{\infty} \left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right) b_n \leq 2(1-\rho)\cos\lambda$$

Taking

$$\mu_n = \frac{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|}{2(1-\rho)\cos\lambda} b_n, \quad \text{for } n \geq 2 \quad \text{and} \quad \mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n$$

the function $\frac{z}{f(z)}$ takes the form

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= \mu_1 + \sum_{n=2}^{\infty} \mu_n + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n \left[1 + \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

Thus the theorem is proved.

The next two results give growth and distortion bounds for the subclass $SP_+(\lambda,\rho)$

Theorem 2.3

If $f(z) \in SP_+(\lambda,\rho)$ for $z \in U$, $0 \leq \alpha < 1$ and $0 \leq |b_1| \leq 1$, for $|z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1| r - \frac{(1-\rho)}{1 + |\, e^{i\lambda} - (1-\rho)\cos\lambda|} r^2\right\} \leq \left|\frac{z}{f(z)}\right|$$

$$\leq 1 + |b_1| r + \frac{(1-\rho)}{1 + |\, e^{i\lambda} - (1-\rho)\cos\lambda|} r^2$$

Proof: Since $f(z) \in SP_+(\lambda,\rho)$, by Theorem 2.2, the function $\frac{z}{f(z)}$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n$$

(8)

$$\left|\frac{z}{f(z)}\right| \leq 1 + |b_1 z| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n\right|$$

$$\leq 1 + |b_1||z| + |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|}\right|$$
$$\leq 1 + |b_1| r + \frac{(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r^2 \text{ for } |z| = r < 1$$

Also from (8), it can be shown that

$$\left|\frac{z}{f(z)}\right| \geq 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n\right|$$
$$\geq 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|}\right|$$
$$\geq 1 - |b_1| r - \frac{(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r^2 \text{ for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1| r - \frac{(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r^2\right\} \leq \left|\frac{z}{f(z)}\right|$$
$$\leq 1 + |b_1| r + \frac{(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r^2$$

Theorem 2.4

If $f(z) \in SP_+(\lambda, \rho)$ for $0 \leq \alpha < 1$ and $0 \leq |b_1| \leq 1$, for $|z| = r < 1$, then

$$\max\left\{0,\ |b_1| - \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r.$$

Proof: Since $f(z) \in SP_+(\lambda, \rho)$, from (8), the function $\frac{z}{f(z)}$ can be expressed as

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n$$
$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n z^{n-1}$$
$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n z^{n-1}\right|$$
$$\leq |b_1| + |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n\right|$$
$$\leq |b_1| + \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r \text{ for } |z| = r < 1$$

and also

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \geq |b_1| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n z^{n-1}\right|$$
$$\geq |b_1| - |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n\right|$$
$$\geq |b_1| - \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r \text{ for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ |b_1| - \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right|$$
$$\leq |b_1| + \frac{2(1-\rho)}{1 + |e^{i\lambda} - (1-\rho)\cos\lambda|} r$$

References:

[1] Ahuja. O. P, Pawan, K, Jain - On the spiral-likeness of rational functions, Rendiconti del Circolo Matematico di . Palermo Serie II, Tomo XXXV(1986), pp.376-385
[2] Mitrinovic´.D.S.- On the univalence of rational functions, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 634–677 (1979) 221– 227.
[3] Mogra. M.L., and Ahuja. O.P.- On spiral-like functions of order α and type β-Yokohama Mathematica Journal, Vol.29, 1981, pp145-156.
[4] Obradovi´c. M., Ponnusamy.S., Coefficient characterization for certain classes of univalent functions, Bull. Belg. Math. Soc. Simon Stevin 16 (2009), 251–263
[5] Reade M.O, Silverman.H, and Todorov.P.G. - Classes of rational functions- Contemporary Mathematics, Volume 38 (1985), 99- 103.
[6] Silvia E.M.: On a subclass of spiral-like functions, Proc. Amer. Math. Soc. 44(2)(1974), pp.411-420.
[7] Silvia, E. M.: Subclasses of spiral-like functions, Tamkang J. Maths, 14(1983),pp.161-169.
[8] Spacek L. - Contribition de la theorie des functions univalent. Casop-pest, Math- Phys, 62(1932), 12-19.
[9] Umarani. G. Prakash : On a subclass of spiral-like functions, Indian J. pure appl. Math.,10(10), Oct 1979, pp.1292-1297.