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Coefficient Characterization For Some Spiral-Like Subclasses Of Generalized Rational Univalent Functions

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The aim of this paper is to introduce Spiral-like subclasses $SP_{+}^{\lambda}(b_1, \alpha)$ and $SP_{+}(\lambda, \rho)$ of generalized rational functions and study the geometric properties like coefficient characterization, growth and distortion bounds.

Keywords: univalent, spiral-like, rational univalent

Introduction

A function f(z) which is normalized and analytic in the open unit disk around the origin, non-vanishing outside the origin and is of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ can take the form $\frac{z}{g(z)}$ where g(z) has Taylor coefficientsb_n's in unit disk U.

Sufficient condition for functions of the form

$$\frac{z}{1+b_1z+\cdots+b_nz^n}, \quad b_n \neq 0 \qquad (1)$$

to be univalent in U was obtained by Mitrinovic [2]

Theorem[2] The function $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ is in S if

$\sum_{n=2}^{\infty}(n-1)|b_n| \leq 1$ and $\sum_{n=1}^{\infty}|b_n| \leq 1$.

Different subclasses of univalent rational functions were introduced by Reade et al. [5]. Also obtained sufficient conditions for $f(z) \in S$ to be in those subclasses.

Spacek [8] defined another interesting subclass of univalent functions, the class of spiral-like functions and proved that a function f(z) is λ-spiral-like if it satisfies

$$\mathrm{Re}\left[\frac{e^{i\lambda}\, z f'(z)}{f(z)}\right] > 0 \text{ for some real } \lambda \left(|\lambda| \le \frac{\pi}{2}\right) \text{ and for all } z \text{ in U.}$$

The class of λ-spiral-like functions is denoted by SP^{λ} or S^{λ} .
A function f(z) analytic on open unit disk U with the usual normalization is called λ-spiral-like of order α, if the inequality

$$\mathrm{Re}\left[\frac{e^{i\lambda}\, z f'(z)}{f(z)}\right] > \alpha \cos\lambda$$

holds for some real α, λ $\left(0 \le \alpha < 1,\ |\lambda| \le \frac{\pi}{2}\right)$ and for all z in U.
This class is known as the class of λ-spiral-like functions of order α and is denoted by $SP^{\lambda}(\alpha)$ or $S^{\lambda}(\alpha)$
And note that $SP^{\lambda}(\alpha) \subset SP^{\lambda}$, $SP^{0}(\alpha) = S^{*}(\alpha)$ and $SP^{0}(0) = S^{*}$.

Ahuja and Jain [1] proved the following sufficient condition on $b_n{}'s$ for the function f(z) of the form (1) to be $\lambda -$ Robertson function of order α:
Theorem : Let $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and α, λ be constants $0 \le \alpha < 1$, $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. f(z) is said to be $\lambda -$ Robertson function of order α if the coefficients $b_n{}'s$ satisfy

$$\frac{3+(1-\alpha)\cos\lambda}{(1-\alpha)\cos\lambda}|b_1| + \sum_{n=1}^{\infty} \frac{(k-1)(3k+(1-\alpha)\cos\lambda)}{(1-\alpha)\cos\lambda}|b_n| \le 1.$$

Ahuja and Jain [1] studied the properties of spiral-likeness of rational functions.

Obradovi´c and Ponnusamy [4] introduced a subclass of rational univalent functions S_+, as the subclass of functions of S which can be expressed in the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)} \qquad (2)$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and obtained necessary and sufficient conditions for functions of S to be in S_+.

Theorem [4] Let $f \in A$. Then $f \in S_+$ if and only if f can take the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \lambda_n \frac{z}{f_n(z)}$$

for some sequence $\{\lambda_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \lambda_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & \text{for } n = 1 \\ 1 + \frac{1}{n-1} z^n, & \text{for } n = 2,3,\ldots \end{cases}$$

This paper introduces spiral-like subclasses of generalized rational univalent functions for which this type of characterization could be derived.
Now, we define two subclasses of S_+ by fixing b_1 and obtain characterization for these subclasses similar to that in [4] for S_+.

Section 1
Ahuja and Jain [1] obtained a condition on the coefficients $\{b_n\}_{n=1}^{\infty}$ that ensures spiral-likeness of a class of functions of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$.
The condition is as follows:
Theorem[1] Let $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n,}$ and α, λ be constants such that $0 \leq \alpha < 1$, $\frac{-\pi}{2} < \lambda < \frac{\pi}{2}$. If the coefficients $\{b_n\}_{n=1}^{\infty}$ satisfy
$\sum_{n=1}^{\infty}\left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right]|b_n| \leq 2(1-\alpha)\cos\lambda$
then f is λ – spiral-like of order α in the unit disk U around the origin.

By imposing this condition, this section introduces a subclass $SP_+^{\lambda}(b_1, \alpha)$ of rational univalent functions by fixing b_1 of $g(z)$. And studies coefficient characterization, growth and distortion bounds for the subclass $SP_+^{\lambda}(b_1, \alpha)$.

Definition 1.1
Let$b_1 \in \mathbb{C}$, $0 \leq |b_1| \leq 1$ be fixed and $0 \leq \alpha < 1$.
$SP_+^{\lambda}(b_1, \alpha) = \{f(z) \in S : \frac{z}{f(z)} = 1 + \sum_{n=1}^{\infty} b_n z^n,\ z \in U$
$\sum_{n=1}^{\infty}\left[n + \{n^2 - 4(1-\alpha)(n+\alpha-1)\cos^2\lambda\}^{1/2}\right] b_n \leq 2(1-\alpha)\cos\lambda$, for $b_n \geq 0$,

$$0 \leq \alpha < 1,\ \frac{-\pi}{2} < \lambda < \frac{\pi}{2} \quad \text{for } n \geq 2\}.$$

(3) The following result gives coefficient characterization for the class $SP_+^{\lambda}(b_1, \alpha)$:
Theorem 1.2
Let $f(z) \in S$ for $z \in U$ and $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and $b_1 \in \mathbb{C}$, $0 \leq |b_1| \leq 1$ be fixed.
Then $f(z) \in SP_+^{\lambda}(b_1, \alpha)$ for $0 \leq \alpha < 1$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)} \quad (4)$$

for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty}\mu_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & for \quad n=1 \\ 1+\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}z^n, & for \quad n=2,3,\ldots \end{cases}$$

Proof : Suppose $f(z) \in S$ has the form $\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\mu_n \frac{z}{f_n(z)}$ for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty}\mu_n = 1$.
Now, to prove that the function $f(z) \in SP_+^{\lambda}(b_1,\alpha)$ rewrite the function $\frac{z}{f(z)}$ as

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\mu_n \frac{z}{f_n(z)}$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty}\mu_n \left[1+\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}z^n\right]$$

(by the definition of $\frac{z}{f_n(z)}$)

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty}\mu_n + \sum_{n=2}^{\infty}\mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty}\mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}z^n\right]$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n \text{ where}$$

$$b_n = \mu_n \left[\frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right] \geq 0 \text{ for } n \geq 2.$$

Choosing $\mu_1 \in \mathbb{R}$ such that

$$|b_1| \leq \left[\frac{2(1-\alpha)\cos\lambda}{1+\{1-4(1-\alpha)(\alpha)cos^2\lambda\}^{1/2}}\right]\mu_1 \leq 1,$$

This gives $\sum_{n=1}^{\infty}\left[n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}\right]b_n$

$$= \left[1+\{1-4(1-\alpha)(\alpha)cos^2\lambda\}^{1/2}\right]|b_1| + \sum_{n=2}^{\infty}\left[n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}\right]b_n$$

$$\leq [2(1-\alpha)\cos\lambda]\mu_1 + \sum_{n=2}^{\infty}[2(1-\alpha)\cos\lambda]\mu_n$$

$$= [2(1-\alpha)\cos\lambda](\mu_1 + \sum_{n=2}^{\infty}\mu_n) = 2(1-\alpha)\cos\lambda$$

Thus $f(z)$ satisfies (3).
This shows that $f(z) \in SP_+^{\lambda}(b_1,\alpha)$
Conversely, suppose $f(z) \in SP_+^{\lambda}(b_1,\alpha)$.
Then by the definition of $SP_+^{\lambda}(b_1,\alpha)$

$$\sum_{n=1}^{\infty}\left[n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}\right]|b_n| \leq 2(1-\alpha)\cos\lambda.$$

Now, taking

$$\mu_n = \frac{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}{2(1-\alpha)\cos\lambda}\ b_n \text{ for } n = 2,3,\dots,$$

and $\mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n$

then the function takes the form

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= \mu_1 + \sum_{n=2}^{\infty}\mu_n + b_1 z + \sum_{n=2}^{\infty}\mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty}\mu_n \left[1 + \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty}\mu_n \frac{z}{f_n(z)}$$

Hence the theorem is proved.

Next results give growth and distortion bounds for the functions of subclass $SP_+^{\lambda}(b_1,\alpha)$

Theorem 1.3

If $f(z) \in SP_+^{\lambda}(b_1,\alpha), z \in U,$ for $0 \le \alpha < 1$ and $0 \le |b_1| \le 1, |z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1|r - \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2\right\} \le \left|\frac{z}{f(z)}\right| \le 1 + |b_1|r + \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2$$

Proof: Since $f(z) \in SP_+^{\lambda}(b_1,\alpha),\ z \in U,$

by Theorem 1.2,

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty}\mu_n \frac{z}{f_n(z)}$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty}\mu_n + \sum_{n=2}^{\infty}\mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty}\mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n \qquad (5)$$

Thus
$$\left|\frac{z}{f(z)}\right| \le 1 + |b_1 z| + \left|\sum_{n=2}^{\infty}\mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right|$$

$$\le 1 + |b_1||z| + |z|^2\left|\sum_{n=2}^{\infty}\mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\le 1 + |b_1|r + \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2 \text{ for } |z| = r < 1.$$

And $\left|\frac{z}{f(z)}\right| \geq 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n\right|$

$$\geq 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\geq 1 - |b_1|r - \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2 \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1|r - \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2\right\} \leq \left|\frac{z}{f(z)}\right| \leq 1 + |b_1|r + \frac{(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r^2$$

Theorem 1.4

If $f(z) \in SP_+^{\lambda}(b_1, \alpha), z \in U$, for $0 \leq \alpha < 1$, $0 \leq |b_1| \leq 1$ and $|z| = r < 1$, then

$$\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}}\ r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r$$

Proof: Since $f(z) \in SP_+^{\lambda}(b_1, \alpha)$, from (5) the function $\frac{z}{f(z)}$ can be expressed as

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^n$$

$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}$$

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}\right|$$

$$\leq |b_1| + |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\leq |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r \quad \text{for } |z| = r < 1$$

and

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \geq |b_1| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}} z^{n-1}\right|$$

$$\geq |b_1| - |z| \left|\sum_{n=2}^{\infty} \mu_n \frac{2n(1-\alpha)\cos\lambda}{n+\{n^2-4(1-\alpha)(n+\alpha-1)cos^2\lambda\}^{1/2}}\right|$$

$$\geq |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ |b_1| - \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}}\ r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\alpha)}{1+\{1-(1-\alpha^2)cos^2\lambda\}^{1/2}} r$$

Section 2

Ahuja and Jain [1] obtained coefficient bounds for Taylor coefficients $b_n's$ of $f(z)$ for $f(z)$ to be in $SP(\lambda,\rho)$.

Proposition [1]

If $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ for z in U, $b_n's$ are complex, λ is real with $|\lambda| < \pi/2$ and

$$\sum_{n=1}^{\infty}\left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right)|\, b_n| \le 2(1-\rho)\cos\lambda$$

Then $f(z) \in SP(\lambda,\rho)$.

Imposing this condition, next section introduces a subclass $SP_+(\lambda,\rho)$ of rational univalent functions by fixing b_1 and gives coefficient characterization, growth and distortion bounds for the subclass $SP_+(\lambda,\rho)$.

Definition 2.1

Let $b_1 \in \mathbb{C}$, $0 \le |b_1| \le 1$ be fixed and $0 \le \alpha < 1$.

Define

$SP_+(\lambda,\rho) = \{f(z) \in S :\ f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n},\ z \in U$ and $b_n \ge 0\ \ for\ \ n \ge 2,\ \lambda$ is real with $|\lambda| < \pi/2$ and $\sum_{n=1}^{\infty}\left(n + \left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right) b_n \le 2(1-\rho)\cos\lambda\ \}$ (6)

Coefficient characterization for the subclass $SP_+(\lambda,\rho)$ of S_+ is given by the following result:

Theorem 2.2

Let $f(z) \in S$ for $z \in U$ be of the form $f(z) = \frac{z}{1+\sum_{n=1}^{\infty} b_n z^n}$ and $b_1 \in \mathbb{C}$, $|b_1| \le 1$ be fixed.

Then $f(z) \in SP_+(\lambda,\rho)$ if and only if $f(z)$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)} \qquad (7)$$

For some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$ and

$$\frac{z}{f_n(z)} = \begin{cases} 1, & for\ \ n = 1 \\ 1 + \frac{2(1-\rho)\cos\lambda}{n+\left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|} z^n, & for\ \ n = 2,3,\ldots \end{cases}$$

Proof : Suppose $f(z) \in S$ has the form $\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$ for some sequence $\{\mu_n\}_{n=1}^{\infty}$ of non-negative real numbers with $\sum_{n=1}^{\infty} \mu_n = 1$.

To prove that the function $f(z) \in SP_+(\lambda,\rho)$, rewrite $\frac{z}{f(z)}$ as

$$\begin{aligned} \frac{z}{f(z)} &= b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)} \\ &= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n \left[1 + \frac{2(1-\rho)\cos\lambda}{n+\left|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda\right|} z^n\right] \end{aligned}$$

(by the definition of $\frac{z}{f_n(z)}$)

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty} \mu_n + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

where $b_n = \mu_n \left[\frac{2(1-\rho)\cos\lambda}{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|}\right] \geq 0$

for $n \geq 2$.

Taking $b_1 \leq \frac{2(1-\rho)\cos\lambda}{1+\left|e^{i\lambda}-2(1-\rho)\cos\lambda\right|}\mu_1$,

$$\sum_{n=1}^{\infty}\left(n + \left|n\,e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right)b_n$$

$$= \left[1 + \left|e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right]b_1 + \sum_{n=2}^{\infty}\left[n + \left|n\,e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right]b_n$$

$$\leq [2(1-\rho)\cos\lambda]\mu_1 + \sum_{n=2}^{\infty}[2(1-\rho)\cos\lambda]\mu_n$$

$$= [2(1-\rho)\cos\lambda]\left(\mu_1 + \sum_{n=2}^{\infty}\mu_n\right) = 2(1-\rho)\cos\lambda$$

Thus $f(z) \in SP_+(\lambda,\rho)$, by the definition of $SP_+(\lambda,\rho)$

Conversely, suppose $(z) \in SP_+(\lambda,\rho)$.

Then by the definition of $SP_+(\lambda,\rho)$,

$$\sum_{n=1}^{\infty}\left(n + \left|n\,e^{i\lambda} - 2(1-\rho)\cos\lambda\right|\right)b_n \leq 2(1-\rho)\cos\lambda$$

Taking

$$\mu_n = \frac{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|}{2(1-\rho)\cos\lambda}\, b_n, \quad \text{for } n \geq 2 \quad \text{and} \quad \mu_1 = 1 - \sum_{n=2}^{\infty}\mu_n$$

the function $\frac{z}{f(z)}$ takes the form

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

$$= \mu_1 + \sum_{n=2}^{\infty}\mu_n + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n$$

$$= b_1 z + \mu_1 + \sum_{n=2}^{\infty}\mu_n \left[1 + \frac{2(1-\rho)\cos\lambda}{n+\left|n\,e^{i\lambda}-2(1-\rho)\cos\lambda\right|} z^n\right]$$

$$= b_1 z + \sum_{n=1}^{\infty}\mu_n \frac{z}{f_n(z)}$$

Thus the theorem is proved.

The next two results give growth and distortion bounds for the subclass $SP_+(\lambda,\rho)$

Theorem 2.3

If $f(z) \in SP_+(\lambda,\rho)$ for $z \in U$, $0 \leq \alpha < 1$ and $0 \leq |b_1| \leq 1$, for $|z| = r < 1$, then

$$\max\left\{0,\ 1 - |b_1|r - \frac{(1-\rho)}{1+\left|e^{i\lambda}-(1-\rho)\cos\lambda\right|}r^2\right\} \leq \left|\frac{z}{f(z)}\right| \leq 1 + |b_1|r + \frac{(1-\rho)}{1+\left|e^{i\lambda}-(1-\rho)\cos\lambda\right|}r^2$$

Proof: Since $f(z) \in SP_{+}(\lambda, \rho)$, by Theorem 2.2, the function $\frac{z}{f(z)}$ has the form

$$\frac{z}{f(z)} = b_1 z + \sum_{n=1}^{\infty} \mu_n \frac{z}{f_n(z)}$$

$$= 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n \quad (8)$$

$$\left|\frac{z}{f(z)}\right| \leq 1 + |b_1 z| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n\right|$$

$$\leq 1 + |b_1||z| + |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|}\right|$$

$$\leq 1 + |b_1| r + \frac{(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r^2 \quad \text{for } |z| = r < 1$$

Also from (8), it can be shown that

$$\left|\frac{z}{f(z)}\right| \geq 1 - |b_1 z| - \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n\right|$$

$$\geq 1 - |b_1||z| - |z|^2 \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|}\right|$$

$$\geq 1 - |b_1| r - \frac{(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r^2 \quad \text{for } |z| = r < 1$$

Therefore

$$\max\left\{0,\ 1 - |b_1| r - \frac{(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r^2\right\} \leq \left|\frac{z}{f(z)}\right| \leq 1 + |b_1| r + \frac{(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r^2$$

Theorem 2.4

If $f(z) \in SP_{+}(\lambda, \rho)$ for $0 \leq \alpha < 1$ and $0 \leq |b_1| \leq 1$, for $|z| = r < 1$, then

$$\max\left\{0,\ |b_1| - \frac{2(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r\right\} \leq \left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \frac{2(1-\rho)}{1 + | e^{i\lambda} - (1-\rho)\cos\lambda|} r.$$

Proof: Since $f(z) \in SP_{+}(\lambda, \rho)$, from (8), the function $\frac{z}{f(z)}$ can be expressed as

$$\frac{z}{f(z)} = 1 + b_1 z + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} z^n$$

$$\left\{\frac{z}{f(z)}\right\}' = b_1 + \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} n z^{n-1}$$

$$\left|\left\{\frac{z}{f(z)}\right\}'\right| \leq |b_1| + \left|\sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n + |n e^{i\lambda} - 2(1-\rho)\cos\lambda|} n z^{n-1}\right|$$

$$\leq |b_1| + |z| \left| \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n \right|$$

$$\leq |b_1| + \frac{2(1-\rho)}{1+|\, e^{i\lambda} - (1-\rho)\cos\lambda|} r \quad \text{for} \quad |z| = r < 1$$

and also

$$\left| \left\{ \frac{z}{f(z)} \right\}' \right| \geq |b_1| - \left| \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n\, z^{n-1} \right|$$

$$\geq |b_1| - |z| \left| \sum_{n=2}^{\infty} \mu_n \frac{2(1-\rho)\cos\lambda}{n+|n\, e^{i\lambda} - 2(1-\rho)\cos\lambda|} n \right|$$

$$\geq |b_1| - \frac{2(1-\rho)}{1+|\, e^{i\lambda} - (1-\rho)\cos\lambda|} r \quad \text{for} \quad |z| = r < 1$$

Therefore

$$\max\left\{0,\ |b_1| - \frac{2(1-\rho)}{1+|\, e^{i\lambda} - (1-\rho)\cos\lambda|} r\right\} \leq \left| \left\{ \frac{z}{f(z)} \right\}' \right|$$

$$\leq |b_1| + \frac{2(1-\rho)}{1+|\, e^{i\lambda} - (1-\rho)\cos\lambda|} r$$

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