Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

Fixed Points From Kannan Type S-Coupled Cyclic Mapping In Complete S-Metric Space

MARY ANUSHIA. S[*†] AND S. N. LEENA NELSON[‡]

[†]Research Scholar, (Reg No.20223282092010), Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India, Email: maryanushia@gmail.com
[‡] Associate Professor, Department of Mathematics, Women's Christian College, Nagercoil, Tamil Nadu, India, Email: leena.wcc@gmail.com
[†‡] Affiliated to Manonmaniam Sundaranar University, Tirunelveli, 627012, Tamil Nadu, India

The aim of this paper is to establish some strong coupled fixed point results in complete S-metric space which is the generalization of G. V. R. Babu and P. Durga Sailaja. Also, we give examples to illustrate our results.

KEYWORDS Coupling, Coupled Fixed Point, Strong Coupled Fixed Point

SUBJECT CLASSIFICATION CODE

47H10, 54H25

1. INTRODUCTION

Coupled fixed point was introduced by Guo et. al [1]. In 2003, Kirk, Srinivasan and veeramani introduced cyclic contractions and proved that such contractions have fixed points [6]. After Bhaskar et. al. proved coupled contraction mapping theorem [2].

In 2017, Binayak S, Choudhury, P. Maity, P. Konar [4] defined the concept of coupling between two non-empty subsets in metric space and they proved that these couplings have strong unique fixed point with the condition that they satisfy Banach type or Chatterjea type contractive inequalities.

Later it was generalized by G. V. R. Babu and P. Durga Sailaja [5]. In this chapter, we generalized the concept of G.V. R. Babu and also derived strong coupled fixed points in the complete S-metric space.

[*] Corresponding Author

2. PRELIMINARIES

Definition 2.1 [9]

Let X be a non empty set. An S-metric on X is a function $S: X \times X \times X \to R^{+}$ that satisfies the following conditions for each $\zeta, \mathcal{v}, \mathcal{w}, t \in X$,

(S1) $S(\zeta, \mathcal{v}, \mathcal{w}) \geq 0$.
(S2) $S(\zeta, \mathcal{v}, \mathcal{w}) = 0$ if and only if $\zeta = \mathcal{v} = \mathcal{w}$.
(S3) $S(\zeta, \mathcal{v}, \mathcal{w}) \leq S(\zeta, \zeta, t) + S(\mathcal{v}, \mathcal{v}, t) + S(\mathcal{w}, \mathcal{w}, t)$

Then the pair (X, S) is called an S-metric space.

Lemma 2.2 [2]

In the S-metric space $S(\zeta, \zeta, \mathcal{v}) = S(\mathcal{v}, \mathcal{v}, \zeta)$.

Lemma 2.3 [2]

In the S-metric space,

$$S(\zeta, \zeta, \mathcal{v}) \leq 2S(\zeta, \zeta, \mathcal{w}) + S(\mathcal{v}, \mathcal{v}, \mathcal{w}) \text{ and}$$

$$S(\zeta, \zeta, \mathcal{v}) \leq 2S(\zeta, \zeta, \mathcal{w}) + S(\mathcal{w}, \mathcal{w}, \mathcal{v})$$

Theorem 2.4 [Kannan] [7]

Let (X, d) be a complete metric space and $C: X \to X$ be any self mapping. Suppose there exist $k \in \left[0, \frac{1}{2}\right)$ satisfies

$$d(C\zeta, C\mathcal{v}) \leq k\big(d(\zeta, C\zeta) + d(\mathcal{v}, C\mathcal{v})\big) \text{ for all } \zeta, \mathcal{v} \in X. \tag{2.1.1}$$

Then C has a unique fixed point in X.

Any mapping satisfying (2.1.1) is called as Kannan mapping.

Definition 2.5 [Kirk, Srinivasan and Veeramani] [7]

Let P and Q be any two nonempty subsets of X. A self mapping C on X is cyclic with respect to P and Q if $C(Q) \subset P$ and $C(P) \subset Q$.

Definition 2.6 [Baskar Lakikantham] [3]

Let X be a non empty set. Let $C: X^2 \to X$ be any mapping. An element $(\zeta, v) \in X \times X$ is said to be coupled fixed pont of C if $C(\zeta, v) = \zeta$ and $C(v, \zeta) = v$

Definition 2.7 [Choudhury Maity] [3]

Let P and Q be any two non empty subsets of X. A mapping $C: X^2 \to X$ is cyclic with respect to P and Q if $C(P, Q) \subset Q$ and $C(Q, P) \subset P$.

Definition 2.8 [3]

Let X be a nonempty set. An element $(\zeta, \zeta) \in X^2$ is said to be a strong coupled fixed point if $C(\zeta, \zeta) = \zeta$.

Definition 2.9 [G. V. R. Babu and P. Durga Sailaja][5]

Let (X, S) be an S-metric space. Let P and Q be any two nonempty subsets of X. A mapping $C: X \times X \to X$ is called a Kannan type cyclic coupled mapping if C is cyclic with respect to P and Q and satisfies the following inequality

$$S\big(C(\zeta, v), C(\mu, u), C(\gamma, w)\big) \leq \alpha\big[\max\{S\big(\zeta, \zeta, C(\zeta, v)\big), S\big(\mu, \mu, S(\mu, u)\big) + S\big(\gamma, \gamma, S(\gamma, w)\big)\}\big]$$

for all $\zeta, \mu, w \in P$ and $v, u, \gamma \in Q$ and $\alpha \in \left(0, \frac{1}{2}\right)$.

Theorem 2.10 [5]

Let (X, S) be a complete S-metric space. Let P and Q be any two nonempty closed subsets of X. Let $C: X \times X \to X$ is called a Kannan type cyclic coupled mapping with respect to P and Q. Then $P \cap Q \neq \phi$ and C has a unique strong coupled fixed point in $P \cap Q$.

3. MAIN RESULTS

Definition 3.1

Let (X, S) be a complete S-metric space. Let P and Q be non-empty subsets of X. The function $C: X \times X \to X$ which is cyclic with respect to P and Q is said to be Kannan type S-coupled cyclic mapping if it satisfies the following inequality.

$$\begin{aligned} S\big(C(\zeta, v), C(\mu, u), C(\gamma, w)\big) &\leq \alpha \Big[max\{S\big(\zeta, \zeta, C(\zeta, v)\big), S\big(\mu, \mu, C(\mu, u)\big)\} \\ &+ \frac{1}{2}\Big(S\big(\gamma, C(\zeta, v), C(\gamma, w)\big) + S\big(\gamma, C(\mu, u), C(\gamma, w)\big)\Big)\Big] \end{aligned} \tag{3.1.1}$$

for all values of $\zeta, \mu, w \in P$ and $v, u, \gamma \in Q$, $\alpha \in \left(0, \frac{1}{3}\right)$.

Example 3.2

Let $X = [0,1]$. Define a mapping $S: X \times X \times X \to R^+$ by

$$S(\zeta, v, w) = \begin{cases} 0 & if\ \zeta = v = w \\ max(\zeta, v, w) & otherwise \end{cases}$$

then (X, S) is a complete S-metric space.

Let $P = \left[0, \frac{1}{8}\right]$, $Q = [0,1]$. Define $C: X \times X \to X$ by

$$C(\zeta, v) = \begin{cases} \frac{4\zeta}{9(v+4)} & if\ \zeta \in P, v \in Q \\ 0 & otherwise \end{cases}$$

Then $C(P, Q) = \left[0, \frac{1}{72}\right] \subset Q$, $C(Q, P) = \left[0, \frac{1}{72}\right] \subset P$. Thus C is cyclic with respect to P and Q.

Case (i)

Suppose $\zeta, \mu, w \in P$ and $v, u, \gamma \in Q$ and $\zeta, \mu, w, v, u, \gamma \in \left[0, \frac{1}{8}\right]$,

Let $\zeta = \mu = w = \frac{1}{8} = 0.125$

$v = u = \gamma = \frac{1}{10} = 0.1$

$$C(\zeta, v) = C\left(\frac{1}{8}, \frac{1}{10}\right) = 0.01355$$

$$C(\mu, v) = C\left(\frac{1}{8}, \frac{1}{10}\right) = 0.01355$$

$$C(\gamma, w) = C\left(\frac{1}{10}, \frac{1}{8}\right) = 0.0054$$

$$\begin{aligned} S\big(C(\zeta, v), C(\mu, v), C(\gamma, w)\big) &= S(0.01355, 0.01355, 0.0054) \\ &= max\ \{0.01355, 0.01355, 0.0054\} \\ &= 0.01355 \end{aligned}$$

$$max\{S(\zeta, \zeta, C(\zeta, v)), S(\mu, \mu, C(\mu, u))\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, v), C(\gamma, w)\big) + S\big(\gamma, C(\mu, u), C(\gamma, w)\big)\Big)$$

$$= max\{S(0.125,0.125,0.01355), S(0.125,0.125,0.01355)\} + \frac{1}{2}[S(0.1, 0.01355, 0.0054) + S(0.1, 0.01355, 0.0054)]$$

$$= 0.125 + 0.1$$

$$= 0.2225$$

$$\therefore S\big(C(\zeta, \mathcal{v}), C(\mu, \mathcal{v}), C(\gamma, \mathcal{w})\big) \le \alpha \Big[max\{S\big(\zeta, \zeta, C(\zeta, \mathcal{v})\big), S\big(\mu, \mu, C(\mu, \mathcal{u})\big)\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, \mathcal{v}), C(\gamma, \mathcal{w})\big) + S\big(\gamma, C(\mu, \mathcal{u}), C(\gamma, \mathcal{w})\big)\Big)\Big]$$

where $\alpha = \frac{1}{5} \in \left(0, \frac{1}{3}\right)$.

Case (ii)

Suppose $\zeta \ne \mathcal{v} \ne \mu \ne \mathcal{u} \ne \gamma \ne \mathcal{w} \in \left[0, \frac{1}{8}\right]$

Let $\zeta = \frac{1}{8} = 0.125, \mu = \frac{1}{9} = 0.111$

$\gamma = \frac{1}{11} = 0.0909, \mathcal{u} = \frac{1}{12} = 0.083$

$\mathcal{v} = \frac{1}{10} = 0.1, \mathcal{w} = \frac{1}{13} = 0.076$

$$C(\zeta, \mathcal{v}) = C\left(\frac{1}{8}, \frac{1}{10}\right) = 0.01355$$

$$C(\mu, \mathcal{u}) = C\left(\frac{1}{9}, \frac{1}{12}\right) = 0.012$$

$$C(\gamma, \mathcal{w}) = C\left(\frac{1}{11}, \frac{1}{13}\right) = 0.0099$$

$$S\big(C(\zeta, \mathcal{v}), C(\mu, \mathcal{u}), C(\gamma, \mathcal{w})\big) = S(0.01355, 0.012, 0.0099)$$

$$= 0.01355$$

$$max\{S\big(\zeta, \zeta, C(\zeta, \mathcal{v})\big), S\big(\mu, \mu, C(\mu, \mathcal{u})\big)\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, \mathcal{v}), C(\gamma, \mathcal{w})\big) + S\big(\gamma, C(\mu, \mathcal{u}), C(\gamma, \mathcal{w})\big)\Big)$$

$$= max\{S(0.125,0.125,0.01355), S(0.111,0.111,0.012)\} + \frac{1}{2}[S(0.0909,0.0355,0.0099) + S(0.0909,0.012,0.0099)]$$

$$= max\{0.125,0.111\} + \frac{1}{2}[0.0909 + 0.0909]$$

$$= 0.125 + 0.0909$$

$$= 0.2159$$

Inequality (3.2.1) is satisfied with $\alpha = \frac{1}{5}$.

Case (iii)

Suppose $\zeta, \mu, w \in \left[0, \frac{1}{8}\right], v, u, \gamma \in \left(\frac{1}{8}, 1\right]$

Let $\zeta = \mu = w = \frac{1}{10} = 0.1,\ v = u = \gamma = \frac{1}{6} = .167$

$$C(\zeta, v) = C(\mu, u) = C\left(\frac{1}{10}, \frac{1}{6}\right) = 0.0107$$

$$C(\gamma, w) = C\left(\frac{1}{6}, \frac{1}{10}\right) = 0$$

$$S\big(C(\zeta, v), C(\mu, u), C(\gamma, w)\big) = S(0.0107, 0.0107, 0)$$

$$= 0.0107$$

$$max\{S\big(\zeta, \zeta, C(\zeta, v)\big), S\big(\mu, \mu, C(\mu, u)\big)\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, v), C(\gamma, w)\big) + S\big(\gamma, C(\mu, u), C(\gamma, w)\big)\Big)$$

$$= max\{S(0.1,0.1,0.0107), S(0.1,0.1,0.0107)\} + \frac{1}{2}[S(0.167,0.0107,0) + S(0.167,0.0107,0)]$$

$$= 0.1 + 0.167$$

$$= 0.267$$

Inequality (3.1.1) is satisfied with $\alpha = \frac{1}{5}$

Case (iv)

Suppose $\zeta, \mu, w \in \left[0, \frac{1}{8}\right], v, u, \gamma \in \left(\frac{1}{8}, 1\right]$

$\zeta \neq \mu \neq w$ and $v \neq u \neq \gamma$

Let $\zeta = \frac{1}{8} = 0.125, \mu = \frac{1}{10} = 0.1, w = \frac{1}{9} = 0.111$

$\mathcal{v} = \frac{1}{4} = 0.25, \mathcal{u} = \frac{1}{2} = 0.5, \gamma = \frac{1}{5} = 0.2$

$$C(\zeta, \mathcal{v}) = C\left(\frac{1}{8}, \frac{1}{4}\right) = 0.0131$$

$$C(\mu, \mathcal{u}) = C\left(\frac{1}{10}, \frac{1}{2}\right) = 0.0099$$

$$C(\gamma, \mathcal{w}) = C\left(\frac{1}{5}, \frac{1}{9}\right) = 0$$

$$S\big(C(\zeta, \mathcal{v}), C(\mu, \mathcal{u}), C(\gamma, \mathcal{w})\big) = S(0.0131, 0.0099, 0)$$
$$= 0.0131$$

$$max\{S\big(\zeta, \zeta, C(\zeta, \mathcal{v})\big), S\big(\mu, \mu, C(\mu, \mathcal{u})\big)\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, \mathcal{v}), C(\gamma, \mathcal{w})\big) + S\big(\gamma, C(\mu, \mathcal{u}), C(\gamma, \mathcal{w})\big)\Big)$$
$$= max\{S(0.125, 0.125, 0.0131), S(0.1, 0.1, 0.0099)\} + \frac{1}{2}[S(0.2, 0.0131, 0) + S(0.2, 0.0099, 0)]$$
$$= 0.125 + 0.2$$
$$= 0.325$$

Inequality (3.1.1) is satisfied with $\alpha = \frac{1}{5}$.

Hence, in all cases, we get C is the Kannan type S-coupled cyclic mapping with $\alpha = \frac{1}{5} \in \left(0, \frac{1}{3}\right)$.

Note 3.3

Here $C\left(\frac{1}{6}, \frac{1}{8}\right) = 0$. But $C\left(\frac{1}{8}, \frac{1}{6}\right) = 0.013$. Thus $C(\zeta, \mathcal{v}) \neq C(\mathcal{v}, \zeta)$ for all values of ζ and $\mathcal{v}$.

Theorem 3.4

Let (X, S) be a complete S-metric space. Let P and Q be any two non-empty closed subsets of X. Let $C: X \times X \to X$ be a Kannan type S-coupled cyclic mapping with respect to P and Q. Then $P \cap Q \neq \phi$ and C has a strong coupled fixed point in $P \cap Q$ and the fixed point is unique.

Proof.

Let $\zeta_0 \in P$ and $\mathcal{v}_0 \in Q$ be any two arbitrary point.

Define the sequences $\{\zeta_m\}$ and $\{\upsilon_m\}$ by $\zeta_{m+1} = C(\upsilon_m, \zeta_m)$, $\upsilon_{m+1} = C(\zeta_m, \upsilon_m), m$ is a non-negative integer.

Since C is cyclic, $\zeta_m \in P$ and $\upsilon_m \in Q$.

Now,

$$S(\zeta_1, \zeta_1, \upsilon_2) = S(\upsilon_2, \upsilon_2, \zeta_1)$$

$$= S\big(C(\zeta_1, \upsilon_1), C(\zeta_1, \upsilon_1), C(\upsilon_0, \zeta_0)\big)$$

$$\leq \alpha\Bigg(max\{S\big(\zeta_1, \zeta_1, C(\zeta_1, \upsilon_1)\big), S\big(\zeta_1, \zeta_1, C(\zeta_1, \upsilon_1)\big)\} + \frac{1}{2}\Big(S\big(\upsilon_0, C(\zeta_1, \upsilon_1), C(\upsilon_0, \zeta_0)\big) + S\big(\upsilon_0, C(\zeta_1, \upsilon_1), C(\upsilon_0, \zeta_0)\big)\Big)\Bigg)$$

[by using 3.1.1]

$$= \alpha\Bigg(max\{S(\zeta_1, \zeta_1, \upsilon_2), S(\zeta_1, \zeta_1, \upsilon_2)\} + \frac{1}{2}\big(S(\upsilon_0, \upsilon_2, \zeta_1) + S(\upsilon_0, \upsilon_2, \zeta_1)\big)\Bigg)$$

$$= \alpha\big(S(\zeta_1, \zeta_1, \upsilon_2) + S(\upsilon_0, \upsilon_2, \zeta_1)\big)$$

$$\leq \alpha\big(S(\zeta_1, \zeta_1, \upsilon_2) + S(\upsilon_0, \upsilon_0, \zeta_1) + S(\zeta_1, \zeta_1, \upsilon_2) + S(\zeta_1, \zeta_1, \zeta_1)\big) \quad [by\ (S3)]$$

$$= \alpha\big(2S(\zeta_1, \zeta_1, \upsilon_2) + S(\upsilon_0, \upsilon_0, \zeta_1)\big)$$

$$\Rightarrow S(\zeta_1, \zeta_1, \upsilon_2) - 2\alpha S(\zeta_1, \zeta_1, \upsilon_2) \leq \alpha S(\upsilon_0, \upsilon_0, \zeta_1)$$

$$\Rightarrow (1 - 2\alpha) S(\zeta_1, \zeta_1, \upsilon_2) \leq \alpha S(\upsilon_0, \upsilon_0, \zeta_1)$$

$$\Rightarrow S(\zeta_1, \zeta_1, \upsilon_2) \leq \left(\frac{\alpha}{1 - 2\alpha}\right) S(\upsilon_0, \upsilon_0, \zeta_1)$$

Let $\beta = \frac{\alpha}{1-2\alpha}$

Since $\alpha \in \left(0, \frac{1}{3}\right)$, we get $0 < \beta < 1$

$$S(\zeta_1, \zeta_1, \upsilon_2) \leq \beta S(\upsilon_0, \upsilon_0, \zeta_1) \tag{3.4.1}$$

Consider,

$$S(\upsilon_1,\upsilon_1,\zeta_2) = S\big(C(\zeta_0,\upsilon_0),C(\zeta_0,\upsilon_0),C(\upsilon_1,\zeta_1)\big)$$

$$\leq \alpha\left(max\{S\big(\zeta_0,\zeta_0,C(\zeta_0,\upsilon_0)\big),S\big(\zeta_0,\zeta_0,C(\zeta_0,\upsilon_0)\big)\}\right.$$
$$\left.+\frac{1}{2}\Big(S\big(\upsilon_1,C(\zeta_0,\upsilon_0),C(\upsilon_1,\zeta_1)\big)+S\big(\upsilon_1,C(\zeta_0,\upsilon_0),C(\upsilon_1,\zeta_1)\big)\Big)\right)$$

$$= \alpha\left(S(\zeta_0,\zeta_0,\upsilon_1) \;+\frac{1}{2}\big(S(\upsilon_1,\upsilon_1,\zeta_2)+S(\upsilon_1,\upsilon_1,\zeta_2)\big)\right)$$

$$= \alpha\big(S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_1,\upsilon_1,\zeta_2)\big)$$

$$\Rightarrow S(\upsilon_1,\upsilon_1,\zeta_2)-\alpha S(\upsilon_1,\upsilon_1,\zeta_2) \leq \alpha S(\zeta_0,\zeta_0,\upsilon_1)$$

$$\Rightarrow (1-\alpha)S(\upsilon_1,\upsilon_1,\zeta_2) \leq \alpha S(\zeta_0,\zeta_0,\upsilon_1)$$

$$\Rightarrow S(\upsilon_1,\upsilon_1,\zeta_2) \leq \left(\frac{\alpha}{1-\alpha}\right)S(\zeta_0,\zeta_0,\upsilon_1)$$

$$\leq \frac{\alpha}{1-2\alpha}S(\zeta_0,\zeta_0,\upsilon_1)$$

$$= \beta S(\zeta_0,\zeta_0,\upsilon_1)$$

Thus, $S(\upsilon_1,\upsilon_1,\zeta_2) \leq \beta S(\zeta_0,\zeta_0,\upsilon_1)$ (3.4.2)

Similarly,

$$S(\zeta_2,\zeta_2,\upsilon_3) = S(\upsilon_3,\upsilon_3,\zeta_2)$$

$$= S\big(C(\zeta_2,\upsilon_2),C(\zeta_2,\upsilon_2),C(\upsilon_1,\zeta_1)\big)$$

$$\leq \alpha\left(max\{S\big(\zeta_2,\zeta_2,C(\zeta_2,\upsilon_2)\big),S\big(\zeta_2,\zeta_2,C(\zeta_2,\upsilon_2)\big)\}\right.$$
$$\left.+\frac{1}{2}\Big(S\big(\upsilon_1,C(\zeta_2,\upsilon_2),C(\upsilon_1,\zeta_1)\big)+S\big(\upsilon_1,C(\zeta_2,\upsilon_2),C(\upsilon_1,\zeta_1)\big)\Big)\right)$$

$$= \alpha\big(S(\zeta_2,\zeta_2,\upsilon_3)+S(\upsilon_1,\upsilon_3,\zeta_2)\big)$$

$$\leq \alpha\big(S(\zeta_2,\zeta_2,\upsilon_3)+S(\upsilon_1,\upsilon_1,\zeta_2)+S(\zeta_2,\zeta_2,\upsilon_3)\big)$$

[by using (S3)]

$$= \alpha\big(2S(\zeta_2,\zeta_2,\upsilon_3)+S(\upsilon_1,\upsilon_1,\zeta_2)\big)$$

$$\Rightarrow (1-2\alpha)S(\zeta_2,\zeta_2,\upsilon_3) \leq \alpha S(\upsilon_1,\upsilon_1,\zeta_2)$$

$$\Rightarrow S(\zeta_2,\zeta_2,\upsilon_3) \le \left(\frac{\alpha}{1-2\alpha}\right) S(\upsilon_1,\upsilon_1,\zeta_2)$$
$$= \beta S(\upsilon_1,\upsilon_1,\zeta_2)$$
$$\le \beta\beta S(\zeta_0,\zeta_0,\upsilon_1) \qquad [by\ using\ 3.4.2]$$
$$S(\zeta_2,\zeta_2,\upsilon_3) \le \beta^2\, S(\zeta_0,\zeta_0,\upsilon_1) \tag{3.4.3}$$

Also,

$$S(\upsilon_2,\upsilon_2,\zeta_3) = S\big(C(\zeta_1,\upsilon_1),C(\zeta_1,\upsilon_1),C(\upsilon_2,\zeta_2)\big)$$
$$\le \alpha\Bigg(max\{S\big(\zeta_1,\zeta_1,C(\zeta_1,\upsilon_1)\big),S\big(\zeta_1,\zeta_1,C(\zeta_1,\upsilon_1)\big)\} + \frac{1}{2}\Big(S\big(\upsilon_2,C(\zeta_1,\upsilon_1),C(\upsilon_2,\zeta_2)\big)+S\big(\upsilon_2,C(\zeta_1,\upsilon_1),C(\upsilon_2,\zeta_2)\big)\Big)\Bigg)$$
$$= \alpha\big(S(\zeta_1,\zeta_1,\upsilon_2)+S(\upsilon_2,\upsilon_2,\zeta_3)\big)$$
$$\Rightarrow (1-\alpha)S(\upsilon_2,\upsilon_2,\zeta_3) \le \alpha S(\zeta_1,\zeta_1,\upsilon_2)$$
$$\Rightarrow S(\upsilon_2,\upsilon_2,\zeta_3) \le \left(\frac{\alpha}{1-\alpha}\right) S(\zeta_1,\zeta_1,\upsilon_2)$$
$$\le \left(\frac{\alpha}{1-2\alpha}\right) S(\zeta_1,\zeta_1,\upsilon_2)$$
$$= \beta S(\zeta_1,\zeta_1,\upsilon_2)$$
$$\le \beta\beta S(\upsilon_0,\upsilon_0,\zeta_1) \qquad [by\ using\ 3.4.1]$$
$$= \beta^2 S(\upsilon_0,\upsilon_0,\zeta_1)$$
$$(i.e), S(\upsilon_2,\upsilon_2,\zeta_3) \le \beta^2 S(\upsilon_0,\upsilon_0,\zeta_1) \tag{3.4.4}$$

Now, we have to prove the following inequalities are satisfied.

$$S(\zeta_{2m+1},\zeta_{2m+1},\upsilon_{2m+2}) \le \beta^{2m+1} S(\upsilon_0,\upsilon_0,\zeta_1)$$
$$S(\upsilon_{2m+1},\upsilon_{2m+1},\zeta_{2m+2}) \le \beta^{2m+1} S(\zeta_0,\zeta_0,\upsilon_1)$$
$$S(\zeta_{2m},\zeta_{2m},\upsilon_{2m+1}) \le \beta^{2m}(\zeta_0,\zeta_0,\upsilon_1)$$
$$S(\upsilon_{2m},\upsilon_{2m},\zeta_{2m+1}) \le \beta^{2m} S(\upsilon_0,\upsilon_0,\zeta_1)$$

for all non-negative integers m.

For this, it is enough to prove that the following inequalities are satisfied.

for all odd integers m

(i) $S(\zeta_m, \zeta_m, \upsilon_{m+1}) \leq \beta^m S(\upsilon_0, \upsilon_0, \zeta_1)$ (3.4.5)

(ii) $S(\upsilon_m, \upsilon_m, \zeta_{m+1}) \leq \beta^m S(\zeta_0, \zeta_0, \upsilon_1)$ (3.4.6)

(iii) $S(\zeta_m, \zeta_m, \upsilon_{m+1}) \leq \beta^m (\zeta_0, \zeta_0, \upsilon_1)$ (3.4.7)

(iv) $S(\upsilon_m, \upsilon_m, \zeta_{m+1}) \leq \beta^m S(\upsilon_0, \upsilon_0, \zeta_1)$ (3.4.8)

} for all even integers m

(i) and (ii) are true for $m = 1$ by (3.4.1) and (3.4.2).

(iii) and (iv) are true for $m = 2$ by (3.4.3) and (3.4.4).

Let k be a positive integer.

Assume the inequalities (3.4.5) and (3.4.6) are true for all $m \leq k$ where k is odd and the inequalities (3.4.7) and (3.4.8) are true for all $m \leq k$ where k is even.

Let k be even.

Now,

$$S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2})$$
$$= S(\upsilon_{k+2}, \upsilon_{k+2}, \zeta_{k+1})$$
$$= S\big(C(\zeta_{k+1}, \upsilon_{k+1}), C(\zeta_{k+1}, \upsilon_{k+1}), C(\upsilon_k, \zeta_k)\big)$$
$$\leq \alpha\Bigg(max\{S\big(\zeta_{k+1}, \zeta_{k+1}, C(\zeta_{k+1}, \upsilon_{k+1})\big), S\big(\zeta_{k+1}, \zeta_{k+1}, C(\zeta_{k+1}, \upsilon_{k+1})\big)\} + \frac{1}{2}\Big(S\big(\upsilon_k, C(\zeta_{k+1}, \upsilon_{k+1}), C(\upsilon_k, \zeta_k)\big) + S\big(\upsilon_k, C(\zeta_{k+1}, \upsilon_{k+1}), C(\upsilon_k, \zeta_k)\big)\Big)\Bigg)$$
$$= \alpha\big(S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2}) + S(\upsilon_k, \upsilon_{k+2}, \zeta_{k+1})\big)$$
$$\leq \alpha\big(S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2}) + S(\upsilon_k, \upsilon_k, \zeta_{k+1}) + S(\upsilon_{k+2}, \upsilon_{k+2}, \zeta_{k+1})\big)$$

[by (S3)]

$$= \alpha\big(S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2}) + S(\upsilon_k, \upsilon_k, \zeta_{k+1}) + S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2})\big)$$

[by Lemma 2.2]

$$= \alpha\big(2S(\zeta_{k+1},\zeta_{k+1},\upsilon_{k+2}) + S(\upsilon_k,\upsilon_k,\zeta_{k+1})\big)$$

$$\Longrightarrow S(\zeta_{k+1},\zeta_{k+1},\upsilon_{k+2}) \le \frac{\alpha}{1-2\alpha} S(\upsilon_k,\upsilon_k,\zeta_{k+1})$$

$$= \beta S(\upsilon_k,\upsilon_k,\zeta_{k+1})$$

(i.e) $S(\zeta_{k+1},\zeta_{k+1},\upsilon_{k+2}) \le \beta\beta^k S(\upsilon_0,\upsilon_0,\zeta_1)$

Since, by induction hypothesis, the result is true for k.

$$\Longrightarrow S(\zeta_{k+1},\zeta_{k+1},\upsilon_{k+2}) \le \beta^{k+1} S(\upsilon_0,\upsilon_0,\zeta_1)$$

That is the result is true for all k.

Hence this result is true for all even integer k.

Also,

$$S(\upsilon_{k+1},\upsilon_{k+1},\zeta_{k+2}) = S\big(C(\zeta_k,\upsilon_k), C(\zeta_k,\upsilon_k), C(\upsilon_{k+1},\zeta_{k+1})\big)$$

$$\le \alpha\Bigg(max\big\{S\big(\zeta_k,\zeta_k,C(\zeta_k,\upsilon_k)\big), S\big(\zeta_k,\zeta_k,C(\zeta_k,\upsilon_k)\big)\big\} + \frac{1}{2}\Big(S\big(\upsilon_{k+1},C(\zeta_k,\upsilon_k),C(\upsilon_{k+1},\zeta_{k+1})\big) + S\big(\upsilon_{k+1},C(\zeta_k,\upsilon_k),C(\upsilon_{k+1},\zeta_{k+1})\big)\Big)\Bigg)$$

$$= \alpha\big(S(\zeta_k,\zeta_k,\upsilon_{k+1}) + S(\upsilon_{k+1},\upsilon_{k+1},\zeta_{k+2})\big)$$

$$S(\upsilon_{k+1},\upsilon_{k+1},\zeta_{k+2}) \le \frac{\alpha}{1-\alpha} S(\zeta_k,\zeta_k,\upsilon_{k+1})$$

$$\le \frac{\alpha}{1-2\alpha} S(\zeta_k,\zeta_k,\upsilon_{k+1})$$

$$\le \beta S(\zeta_k,\zeta_k,\upsilon_{k+1})$$

$$\le \beta\beta^k S(\zeta_0,\zeta_0,\upsilon_1)$$

$$= \beta^{k+1} S(\zeta_0,\zeta_0,\upsilon_1)$$

(i.e) the result is true for all integer $m = k + 1$, where k is even.

Now, let us show that the (3.4.5) and (3.4.6) are true for $m = k + 1$, where k is an odd integer.

$$S(\zeta_{k+1}, \zeta_{k+1}, \upsilon_{k+2}) \leq \beta S(\upsilon_k, \upsilon_k, \zeta_{k+1})$$

[by repeating the process, when k is even]

$$\leq \beta\beta^k S(\zeta_0, \zeta_0, \upsilon_1)$$
$$= \beta^{k+1} S(\zeta_0, \zeta_0, \upsilon_1)$$

Similarly,

$$S(\upsilon_{k+1}, \upsilon_{k+1}, \zeta_{k+2}) \leq \beta S(\zeta_k, \zeta_k, \upsilon_{k+1})$$
$$\leq \beta^{k+1} S(\upsilon_0, \upsilon_0, \zeta_1)$$

Hence, all the four inequalities (3.4.5)-(3.4.8) are true when $m = k + 1$.

Hence, by induction, these are true for all m.

Suppose $m = 2q + 2$, q is a positive integer.

Then,

$$S(\zeta_{2q+2}, \zeta_{2q+2}, \upsilon_{2q+2}) = S(\upsilon_{2q+2}, \upsilon_{2q+2}, \zeta_{2q+2})$$
$$= S\left(C(\zeta_{2q+1}, \upsilon_{2q+1}), C(\zeta_{2q+1}, \upsilon_{2q+1}), C(\upsilon_{2q+1}, \zeta_{2q+1})\right)$$
$$\leq \alpha\Bigg(S\left(\zeta_{2q+1}, \zeta_{2q+1}, C(\zeta_{2q+1}, \upsilon_{2q+1})\right) + \frac{1}{2}\Big(S\left(\upsilon_{2q+1}, C(\zeta_{2q+1}, \upsilon_{2q+1}), C(\upsilon_{2q+1}, \zeta_{2q+1})\right) + S\left(\upsilon_{2q+1}, C(\zeta_{2q+1}, \upsilon_{2q+1}), C(\upsilon_{2q+1}, \zeta_{2q+1})\right)\Big)\Bigg)$$
$$= \alpha\left(S(\zeta_{2q+1}, \zeta_{2q+1}, \upsilon_{2q+2}) + S(\upsilon_{2q+1}, \upsilon_{2q+2}, \zeta_{2q+2})\right)$$
$$\leq \alpha\left(S(\zeta_{2q+1}, \zeta_{2q+1}, \upsilon_{2q+2}) + S(\upsilon_{2q+1}, \upsilon_{2q+1}, \zeta_{2q+2}) + S(\upsilon_{2q+2}, \upsilon_{2q+2}, \zeta_{2q+2})\right) \qquad [by\ (S3)]$$
$$= \alpha[\beta^{2q+1} S(\upsilon_0, \upsilon_0, \zeta_1) + \beta^{2q+1} S(\zeta_0, \zeta_0, \upsilon_1) + S(\upsilon_{2q+2}, \upsilon_{2q+2}, \zeta_{2q+2})]$$
$$\Rightarrow S(\zeta_{2q+2}, \zeta_{2q+2}, \upsilon_{2q+2})(1 - \alpha) \leq \alpha\beta^{2q+1}[S(\upsilon_0, \upsilon_0, \zeta_1) + S(\zeta_0, \zeta_0, \upsilon_1)]$$
$$\Rightarrow S(\zeta_{2q+2}, \zeta_{2q+2}, \upsilon_{2q+2}) \leq \frac{\alpha}{1 - \alpha}\beta^{2q+1}[S(\upsilon_0, \upsilon_0, \zeta_1) + S(\zeta_0, \zeta_0, \upsilon_1)]$$

$$\leq \frac{\alpha}{1-2\alpha}\beta^{2q+1}[S(\upsilon_0,\upsilon_0,\zeta_1)+S(\zeta_0,\zeta_0,\upsilon_1)]$$
$$=\beta\beta^{2q+1}[S(\upsilon_0,\upsilon_0,\zeta_1)+S(\zeta_0,\zeta_0,\upsilon_1)]$$

$$S(\zeta_{2q+2},\zeta_{2q+2},\upsilon_{2q+2})\leq \beta^{2q+2}[S(\upsilon_0,\upsilon_0,\zeta_1)+S(\zeta_0,\zeta_0,\upsilon_1)] \quad (3.4.9)$$

If m is an odd integer is of the form $m=2q+1$ where q is a positive integer.

Then,

$$S(\zeta_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})=S(\upsilon_{2q+1},\upsilon_{2q+1},\zeta_{2q+1})$$
$$=S\left(C(\zeta_{2q},\upsilon_{2q}),C(\zeta_{2q},\upsilon_{2q}),C(\upsilon_{2q},\zeta_{2q})\right)$$
$$\leq \alpha\left(max\left\{S\left(\zeta_{2q},\zeta_{2q},C(\zeta_{2q},\upsilon_{2q})\right),S\left(\zeta_{2q},\zeta_{2q},C(\zeta_{2q},\upsilon_{2q})\right)\right\}\right.$$
$$\left.+S\left(\upsilon_{2q},C(\zeta_{2q},\upsilon_{2q}),C(\upsilon_{2q},\zeta_{2q})\right)\right)$$
$$=\alpha\left(S(\zeta_{2q},\zeta_{2q},\upsilon_{2q+1})+S(\upsilon_{2q},\upsilon_{2q+1},\zeta_{2q+1})\right)$$
$$\leq \alpha\left(S(\zeta_{2q},\zeta_{2q},\upsilon_{2q+1})+S(\upsilon_{2q},\upsilon_{2q},\zeta_{2q+1})\right.$$
$$\left.+S(\upsilon_{2q+1},\upsilon_{2q+1},\zeta_{2q+1})\right) \qquad [by\ S3]$$
$$=\alpha(S(\zeta_{2q},\zeta_{2q},\upsilon_{2q+1})+S(\upsilon_{2q},\upsilon_{2q},\zeta_{2q+1})$$
$$+S(\upsilon_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})) \qquad (by\ Lemma\ 2.2)$$

$$\Rightarrow (1-\alpha)S(\zeta_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})\leq \alpha\left(S(\zeta_{2q},\zeta_{2q},\upsilon_{2q+1})+S(\upsilon_{2q},\upsilon_{2q},\zeta_{2q+1})\right)$$
$$\Rightarrow S(\zeta_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})\leq \beta\left[S(\zeta_{2q},\zeta_{2q},\upsilon_{2q+1})+S(\upsilon_{2q},\upsilon_{2q},\zeta_{2q+1})\right]$$
$$\Rightarrow S(\zeta_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})\leq \beta(\beta^{2q}[S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)])$$
$$=\beta^{2q+1}[S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)]$$

$$S(\zeta_{2q+1},\zeta_{2q+1},\upsilon_{2q+1})\leq \beta^{2q+1}[S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)] \quad (3.4.10)$$

Hence by inequalities (3.4.9) and (3.4.10),

$$S(\zeta_m,\zeta_m,\upsilon_m) \leq \beta^m[S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)] \text{ for all } m. \tag{3.4.11}$$

Consider,

$$\begin{aligned} S(\zeta_m,\zeta_m,\zeta_{m+1})+S(\upsilon_m,\upsilon_m,\upsilon_{m+1}) &\leq 2S(\zeta_m,\zeta_m,\upsilon_m)+S(\upsilon_m,\upsilon_m,\zeta_{m+1}) \\ &\quad +2S(\upsilon_m,\upsilon_m,\zeta_m)+S(\zeta_m,\zeta_m,\upsilon_{m+1}) \\ &= 4S(\zeta_m,\zeta_m,\upsilon_m)+S(\upsilon_m,\upsilon_m,\zeta_{m+1}) \\ &\quad +S(\zeta_m,\zeta_m,\upsilon_{m+1}) \\ &\leq 4\big[\beta^m\big(S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)\big)\big] \\ &\quad +\beta^m S(\zeta_0,\zeta_0,\upsilon_1)+\beta^m S(\upsilon_0,\upsilon_0,\zeta_1) \end{aligned}$$

(by 3.4.11)

$$\begin{aligned} &= \beta^m\big[5\big(S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)\big)\big] \\ &= \beta^m T, \quad where\ T = 5\big(S(\zeta_0,\zeta_0,\upsilon_1)+S(\upsilon_0,\upsilon_0,\zeta_1)\big) \end{aligned}$$

Let $n > m$ where m and n are positive integers.

Now,

$$\begin{aligned} S(\zeta_m,\zeta_m,\zeta_n)+S(\upsilon_m,\upsilon_m,\upsilon_n) &\leq 2S(\zeta_m,\zeta_m,\zeta_{m+1})+(\zeta_{m+1},\zeta_{m+1},\zeta_n) \\ &\quad +2S(\upsilon_m,\upsilon_m,\upsilon_{m+1})+S(\upsilon_{m+1},\upsilon_{m+1},\upsilon_n) \\ &\leq 2S(\zeta_m,\zeta_m,\zeta_{m+1})+2S(\zeta_{m+1},\zeta_{m+1},\zeta_{m+2}) \\ &\quad +(\zeta_{m+2},\zeta_{m+2},\zeta_n)+2S(\upsilon_m,\upsilon_m,\upsilon_{m+1}) \\ &\quad +2S(\upsilon_{m+1},\upsilon_{m+1},\upsilon_{m+2})+S(\upsilon_{m+2},\upsilon_{m+2},\upsilon_n) \end{aligned}$$

Repeating the above process we get,

$$\begin{aligned} S(\zeta_m,\zeta_m,\zeta_n)+S(\upsilon_m,\upsilon_m,\upsilon_n) &\leq 2S(\zeta_m,\zeta_m,\zeta_{m+1})+2S(\zeta_{m+1},\zeta_{m+1},\zeta_{m+2}) \\ &\quad +\cdots+S(\zeta_{n-1},\zeta_{n-1},\zeta_n)+2S(\upsilon_m,\upsilon_m,\upsilon_{m+1}) \\ &\quad +2S(\upsilon_{m+1},\upsilon_{m+1},\upsilon_{m+2})+\cdots+S(\upsilon_{n-1},\upsilon_{n-1},\upsilon_n) \\ &\leq 2S(\zeta_m,\zeta_m,\zeta_{m+1})+2S(\upsilon_m,\upsilon_m,\upsilon_{m+1}) \\ &\quad +2S(\zeta_{m+1},\zeta_{m+1},\zeta_{m+2})+2S(\upsilon_{m+1},\upsilon_{m+1},\upsilon_{m+2}) \end{aligned}$$

$$+\cdots+2S(\zeta_{n-1},\zeta_{n-1},\zeta_n)+2S(\upsilon_{n-1},\upsilon_{n-1},\upsilon_n)$$

$$\leq 2\beta^m T+2\beta^{m+1}T+\cdots+2\beta^{n-1}T$$

$$=2T\beta^m[1+\beta+\cdots+\beta^{n-m+1}]$$

$$\leq 2T\beta^m[1+\beta+\beta^2+\cdots]$$

$$\leq 2T\left(\frac{\beta^m}{1-\beta}\right)$$

Since, $0<\beta<1, \frac{\beta^m}{1-\beta}\to 0$ as $m\to\infty$

$$\therefore \lim_{m,n\to\infty} S(\zeta_m,\zeta_m,\zeta_n)+S(\upsilon_m,\upsilon_m,\upsilon_n)=0$$

This shows that $\{\zeta_n\}$ and $\{\upsilon_n\}$ are Cauchy sequences in X and hence convergent in X. (Since X is complete).

Let $\{\zeta_n\}\to\zeta$ and $\{\upsilon_n\}\to\upsilon$.

Since $\{\zeta_n\}\subset P$ and $\{\upsilon_n\}\subset Q$ and P and Q are closed subsets of X, $\zeta\in P$ and $\upsilon\in Q$.

Also, by inequality (3.4.11), $\lim_{n\to\infty} S(\zeta_n,\zeta_n,\upsilon_n)=0$.

This implies $S(\zeta,\zeta,\upsilon)=0$.

This is possible only if $\upsilon=\zeta$.

Hence $P\cap Q\neq\phi$ and the limit $\zeta\in P\cap Q$.

Now,

$$S\big(\zeta,\zeta,C(\zeta,\zeta)\big)\leq 2S(\zeta,\zeta,\upsilon_{m+1})+S\big(\upsilon_{m+1},\upsilon_{m+1},C(\zeta,\zeta)\big)$$

$$=2S(\zeta,\zeta,\upsilon_{m+1})+S\big(C(\zeta_m,\upsilon_m),C(\zeta_m,\upsilon_m),C(\zeta,\zeta)\big)$$

$$\leq 2S(\zeta,\zeta,\upsilon_{m+1}) + \alpha\Big(max\{S\big(\zeta_m,\zeta_m,C(\zeta_m,\upsilon_m)\big),S\big(\zeta_m,\zeta_m,C(\zeta_m,\upsilon_m)\big)\} + \frac{1}{2}\big[S\big(\zeta,C(\zeta_m,\upsilon_m),C(\zeta,\zeta)\big)+S\big(\zeta,C(\zeta_m,\upsilon_m),C(\zeta,\zeta)\big)\big]\Big)$$

$$=2S(\zeta,\zeta,\upsilon_{m+1}) + \alpha\Big(S(\zeta_m,\zeta_m,\upsilon_{m+1})+\frac{1}{2}\big[S\big(\zeta,\upsilon_{m+1},C(\zeta,\zeta)\big)+S\big(\zeta,\upsilon_{m+1},C(\zeta,\zeta)\big)\big]\Big)$$

$$= 2S(\zeta,\zeta,\upsilon_{m+1}) + \alpha\big[S(\zeta_m,\zeta_m,\upsilon_{m+1}) + S\big(\zeta,\upsilon_{m+1},C(\zeta,\zeta)\big)\big]$$

As, $m \to \infty$, this inequality becomes,

$$S\big(\zeta,\zeta,C(\zeta,\zeta)\big) = 0$$

This shows that $\zeta = C(\zeta,\zeta)$

(i.e) $C(\zeta,\zeta)$ is a strong coupled fixed point of C.

Now we have to prove the uniqueness of this strong coupled fixed point of C.

Suppose there exist two strong coupled fixed point of C.

Let it be (ζ,ζ) and (υ,υ)

Hence, $\zeta,\upsilon \in P \cap Q$.

Consider,

$$S(\zeta,\zeta,\upsilon) = S\big(C(\zeta,\zeta),C(\zeta,\zeta),C(\upsilon,\upsilon)\big)$$

$$\le \alpha\Big(max\{S\big(\zeta,\zeta,C(\zeta,\zeta)\big),S\big(\zeta,\zeta,C(\zeta,\zeta)\big)\} + \frac{1}{2}\big[S\big(\upsilon,C(\zeta,\zeta),C(\upsilon,\upsilon)\big) + S\big(\upsilon,C(\zeta,\zeta),C(\upsilon,\upsilon)\big)\big]\Big)$$

$$= \alpha\Big(max\{S(\zeta,\zeta,\zeta),S(\zeta,\zeta,\zeta)\} + S\big(\upsilon,C(\zeta,\zeta),C(\upsilon,\upsilon)\big)\Big)$$

$$= \alpha S(\upsilon,\zeta,\upsilon)$$

$$\le \alpha\big(S(\upsilon,\upsilon,\zeta) + S(\zeta,\zeta,\upsilon)\big)$$

$$= \alpha\big(S(\zeta,\zeta,\upsilon) + S(\zeta,\zeta,\upsilon)\big)$$

$$= \alpha\big(2S(\zeta,\zeta,\upsilon)\big)$$

$\Rightarrow (1-2\alpha)S(\zeta,\zeta,\upsilon) \le 0$

$\Rightarrow S(\zeta,\zeta,\upsilon) \le 0$

$\Rightarrow S(\zeta,\zeta,\upsilon) = 0$ $\quad$ $[\because S(\zeta,\upsilon,w) \ge 0 \ \forall \zeta,\upsilon,w \in X]$

$\Rightarrow \zeta = \upsilon$.

Hence proved.

Corollary 3.5

Let (X,S) be a complete S-metric space. Let $C: X \times X \to X$ be a Kannan type S-coupled cyclic mapping. Then C has a unique strong coupled fixed point in X.

Proof.

Let us take $P = Q = X$ in the above theorem, we get the result.

Corollary 3.6

Let (X,S) be a complete S-metric space. Let $C: X \times X \to X$ be a mapping satisfying the following inequality with $\alpha \in \left(0, \frac{1}{3}\right)$,

$$S\big(C(\zeta,v), C(\mu,u), C(\gamma,w)\big) \leq \alpha\left[max\{S(\zeta,\zeta,C(\zeta,v)), S(\mu,\mu,C(\mu,u))\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta,v), C(\gamma,w)\big) + S\big(\gamma, C(\mu,u), C(\gamma,w)\big)\Big)\right]$$

for all $\zeta, \mu, w, v, u, \gamma \in X$.

Then C has a unique strong coupled fixed point in X.

Corollary 3.7

Let (X,S) be a complete S-metric space. Let $C: X \times X \to X$ be a mapping satisfying the inequality

$$S\big(C(\zeta,v), C(\mu,u), C(\gamma,w)\big) \leq \alpha\left[max\left\{S(\zeta,\zeta,C(\zeta,v)), S(\mu,\mu,C(\mu,u)), \frac{1}{2}\left[S(\zeta,\zeta,C(\mu,u)), S(\mu,\mu,C(\zeta,v))\right]\right\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta,v), C(\gamma,w)\big) + S\big(\gamma, C(\mu,u), C(\gamma,w)\big)\Big)\right]$$

for all $\zeta, \mu, w, v, u, \gamma \in X$ with $\alpha \in \left(0, \frac{1}{3}\right)$.

Then C has a strong coupled fixed point in X.

Note 3.8

Here α must be less than $\frac{1}{3}$. Otherwise $\beta < 0$ which leads to our theorem fails.

Example 3.9

Let $X = [0,1]$. Define $S: X^3 \to R^+$ by

$$S(\zeta,\vartheta,w)=\begin{cases}0 & if\ \zeta=\vartheta=w\\ max\{\zeta,\vartheta,w\} & Otherwise\end{cases}$$

Then (X,S) is a complete S-metric space.

Let $P=\left[0,\frac{1}{5}\right], Q=[0,1]$. Define $C: X\times X\to X$ by

$$C(\zeta,\vartheta)=\begin{cases}\dfrac{\zeta}{5(\zeta+\vartheta+1)} & if\ \zeta\in P,\vartheta\in Q\\ 0 & Otherwise\end{cases}$$

Then $C(P,Q)=\left[0,\frac{1}{30}\right]\subset Q$
$C(Q,P)=\left[0,\frac{1}{30}\right]\subset P$
This shows that C is cyclic with respect to P and Q.
Let $\zeta,\mu,w\in P$ and $\vartheta,u,\gamma\in Q$

Case (i)

Let $\zeta,\mu,w,\vartheta,u,\gamma\in\left[0,\frac{1}{5}\right]$. Then

$$S\big(C(\zeta,\vartheta),C(\mu,u),C(\gamma,w)\big)=S\left(\frac{\zeta}{5(\zeta+\vartheta+1)},\frac{\mu}{5(\mu+u+1)},\frac{\gamma}{5(\gamma+w+1)}\right)$$

$$=max\left\{\frac{\zeta}{5(\zeta+\vartheta+1)},\frac{\mu}{5(\mu+u+1)},\frac{\gamma}{5(\gamma+w+1)}\right\}$$

$$\le max\left\{\frac{\zeta}{5},\frac{\mu}{5},\frac{\gamma}{5}\right\}$$

$$=\frac{1}{5}max\{\zeta,\mu,\gamma\}$$

$$\le\frac{1}{5}(max\{\zeta,\mu\}+\gamma)$$

$$\le\alpha\left[max\{S(\zeta,\zeta,C(\zeta,\vartheta)),S(\mu,\mu,C(\mu,u))\}+\frac{1}{2}\Big(S\big(\gamma,C(\zeta,\vartheta),C(\gamma,w)\big)+S\big(\gamma,C(\mu,u),C(\gamma,w)\big)\Big)\right]$$

where $\alpha=\frac{1}{5}$.

Therefore, in this case inequality (3.1.1) is satisfied

Case (ii)

Let $\zeta, \mu, w \in \left[0, \frac{1}{5}\right], v, u, \gamma \in \left(\frac{1}{5}, 1\right]$. Then

$$S\big(C(\zeta, v), C(\mu, u), C(\gamma, w)\big) = S\left(\frac{\zeta}{5(\zeta + v + 1)}, \frac{\mu}{5(\mu + u + 1)}, 0\right)$$

$$= max\left\{\frac{\zeta}{5(\zeta + v + 1)}, \frac{\mu}{5(\mu + u + 1)}\right\}$$

$$\leq max\left\{\frac{\zeta}{5}, \frac{\mu}{5}\right\}$$

$$= \frac{1}{5} max\{\zeta, \mu\}$$

$$\leq \alpha\left[max\{S(\zeta, \zeta, C(\zeta, v)), S(\mu, \mu, C(\mu, u))\} + \frac{1}{2}\Big(S\big(\gamma, C(\zeta, v), C(\gamma, w)\big) + S\big(\gamma, C(\mu, u), C(\gamma, w)\big)\Big)\right]$$

Thus in both the cases, we get the mapping C is the Kannan type S-coupled cyclic mapping with respect to P and Q with $\alpha = \frac{1}{5}$.

(i.e) C satisfies all the conditions of the above theorem.

Hence by theorem (3.4), C has unique strong coupled fixed point and that point is (0,0).

Example 3.10

Define $C(\zeta, v) = \begin{cases} \zeta & \text{if } \zeta \in P \text{ and } v \in Q \\ 0 & \text{Otherwise} \end{cases}$ and the metric S and the space X and the closed subsets P and Q are as in example 2.8.

Let $\zeta, \mu, w, v, u, \gamma \in P$

Put $\zeta = \frac{1}{5}, \mu = \frac{1}{6}, w = \frac{1}{7}, v = \frac{1}{8}, u = \frac{1}{9}, \gamma = \frac{1}{10}$

$$S\big(C(\zeta, v), C(\mu, u), C(\gamma, w)\big) = S\left(C\left(\frac{1}{5}, \frac{1}{8}\right), C\left(\frac{1}{6}, \frac{1}{9}\right), C\left(\frac{1}{10}, \frac{1}{7}\right)\right)$$

$$= S\left(\frac{1}{5}, \frac{1}{6}, \frac{1}{10}\right)$$

$$= \max\left\{\frac{1}{5}, \frac{1}{6}, \frac{1}{10}\right\}$$

$$= \frac{1}{5}$$

$$\text{Now, } \alpha \left[\max\{S(\zeta,\zeta,C(\zeta,\upsilon)), S(\mu,\mu,C(\mu,u))\} + \frac{1}{2}\left(S(\gamma,C(\zeta,\upsilon),C(\gamma,w)) + S(\gamma,C(\mu,u),C(\gamma,w))\right)\right]$$

$$= \alpha \max\left\{S\left(\frac{1}{5},\frac{1}{5},C\left(\frac{1}{5},\frac{1}{8}\right)\right), S\left(\frac{1}{6},\frac{1}{6},C\left(\frac{1}{6},\frac{1}{9}\right)\right)\right\} + \frac{1}{2}\left[S\left(\frac{1}{10},\frac{1}{5},C\left(\frac{1}{10},\frac{1}{7}\right) + S\left(\frac{1}{6},\frac{1}{6},C\left(\frac{1}{10},\frac{1}{7}\right)\right)\right)\right]$$

$$= \alpha \max\left\{S\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right), S\left(\frac{1}{6},\frac{1}{6},\frac{1}{6}\right)\right\} + \frac{1}{2}\left[S\left(\frac{1}{10},\frac{1}{5},\frac{1}{10}\right) + S\left(\frac{1}{6},\frac{1}{6},\frac{1}{10}\right)\right]$$

$$= \alpha\left[\frac{1}{2}\left(\frac{1}{5}+\frac{1}{6}\right)\right]$$

$$= \alpha\left[\frac{1}{2}\times\frac{11}{30}\right]$$

$$= \alpha \times \frac{11}{60}$$

Since $0 < \alpha < \frac{1}{3}$.

$$S(C(\zeta,\upsilon), C(\mu,u), C(\gamma,w)) > \alpha\left[\max\{S(\zeta,\zeta,C(\zeta,\upsilon)), S(\mu,\mu,C(\mu,u))\} + \frac{1}{2}\left(S(\gamma,C(\zeta,\upsilon),C(\gamma,w)) + S(\gamma,C(\mu,u),C(\gamma,w))\right)\right]$$

Hence C is not Kannan type S-coupled cyclic mapping.

Hence by the theorem (3.4), C has no unique strong coupled fixed point.

Since, $C(\zeta,\upsilon) = \zeta$ is satisfied by all the points of Q.

In this example, the uniqueness of fixed point is failed.

Example 3.11

Define $C(\zeta,\upsilon) = \begin{cases} \zeta + 1, & \zeta \in P, \upsilon \in Q \\ 0 & \text{Otherwise} \end{cases}$

The space X, metric S are as in example 3.2.

Repeat the process as in Example (3.10), we get C is not a Kannan type S- coupled cyclic mapping.

Hence by theorem (3.4), C has no unique strong coupled fixed point.

In this example no point is a strong coupled point of C.

REFERENCE

[1] Guo, Dajun, and Vangipuram Lakshmikantham. "Coupled fixed points of nonlinear operators with applications." Nonlinear analysis: theory, methods & applications 11.5 (1987): 623-632.
[2] Gnana, T. "Bhaskar and V. Lakshmikanthan, Fixed point theorem in partially ordered metric spaces and applications." Nonlinear Anal. TMA 65 (2006).
[3] Choudhury, Binayak S., and Pranati Maity. "Cyclic coupled fixed point result using Kannan type contractions." Journal of Operators 2014.1 (2014): 876749.
[4] Choudhury, Binayak S., P. Maity, and P. Konar. "Fixed point results for couplings on metric spaces." UPB Scientific Bulletin, Series A: Applied Mathematics and Physics 79.1 (2017): 1-12.
[5] Babu, G. V. R., Pericherla Durga Saılaja, and Gadhavajjala Srıchandana. "Strong coupled fixed points of Chatterjea type (ψ; ϕ)-weakly cyclic coupled mappings in S-metric spaces." Proceedings of International Mathematical Sciences 2.1 (2020): 60-78.
[6] Kirk, W. A., Pinchi S. Srinivasan, and Panimalar Veeramani. "Fixed points for mappings satisfying cyclical contractive conditions." Fixed point theory 4.1 (2003): 79-89.
[7] Kannan, Rangachary. "Some results on fixed points." Bull. Cal. Math. Soc. 60 (1968): 71-76.
[8] Nguyen, D.V, Nguyen, H.T., Radojavic, S., "Fixed point theorems foe G-Monotone Maps on partially Ordered S-Metric Spaces", Fillomat. 28(2014)1885-1898 https://doi.org/10.2298/fi/1409885d.
[9] Sedghi, S, Shobe, N., Aliouche, A," A Generalization of fixed point Theorem in S-metric spaces" ,Math,Vesnik,64(2012),258-266.