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A Novel Geometric Modification To The Regula Falsi Method To Achieve Cubic Convergence Order

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Chen and Li (2007) presented a family of quadratically convergent Regula Falsi iterative methods for solving nonlinear equations $f(x) = 0$. Additionally, it is demonstrated there that the iterative point sequence converges to zero. This work aims to accelerate the method convergence from quadratic to cubic forms. This can be achieved by substituting an appropriately specified function, $p(x)$, for the parameter p in Chen and Li's iteration. A convergence theorem is used to determine the cubic convergence of the iterate sequence to the root. The numerical examples show that the proposed method is more efficient and required fewer number of iterations in comparison to Newton's method, Steffensen's method, Regula Falsi method, and those provided in Chen and Li.

KEYWORDS:

Nonlinear equations, Convergence order, Newton method, Regula Falsi methods, Steffensen's method.

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INTRODUCTION:

An important and challenging problems in numerical analysis is finding approximations for the solutions to the nonlinear equations $f(x) = 0...(1)$, where $f:[a,b] \rightarrow R$ is a real value function. Many applications exist that, depending on one or more parameters, result in thousands of such equations. For example, solving nonlinear equations is the solution to boundary value problems that arise in kinetic theory of gases, elasticity, and other fields. These equations also arise in many optimization problems. Frequently, one or more initial guesses are used to solve these equations iteratively to find the desired root. These methods fulfil a variety of requirements, including having strong convergence properties, being effective, and being numerically stable. The globally convergent iterative Bisection and Regula Falsi techniques repeatedly use linear interpolation between the bracketing estimations to identify a simple root of the nonlinear equation (1). Nevertheless, the iterative sequence $\{(x_n - \alpha)\}$, where α represents a simple root, has a linear asymptotic convergence rate. These methods clearly have a drawback in that they repeatedly maintain one endpoint when they reach a concave or convex region of $f(x)$. Several changes to address these challenges have been covered in [3, 6, 7]. Dowell and Jarratt [6,7] proposed modifications to increase the asymptotic rate of convergence of the Regula Falsi method from linear to super-linear.

To find a root of (1), iterative formulas like the well-known quadratically convergent Newton's approach and its variations [9, 10, 12,] are usually used. However, these approaches might not converge if the derivative vanishes close to the root or if the initial point is far from the root. For solving nonlinear equations, several third order techniques are also used [4, 5, 8]. Although the computational cost of these methods is higher, they are beneficial in situations when rapid convergence is needed, like in stiff systems of equations.

A collection of third-order techniques is given by

$$x_{n+1} = x_n - \left(1 + \frac{1}{2} \frac{L_f(x_n)}{1 - \beta L_f(x_n)}\right) \frac{f(x_n)}{f'(x_n)} \ldots (2)$$

Where $L_f(x_n) = \frac{f''(x_n) f(x_n)}{f'(x_n)^2}$

If $\beta = 1$ then equation (2) reduced to super – Halley method, if $\beta = \frac{1}{2}$ then equation (2) reduced to Halley's method and if $\beta = 0$ then equation (2) reduced to Chebyshev's method [5,4,8]. However, all these methods have the drawback of requiring the computation of the second order derivative of f, which can be difficult. Additionally, [14] describes a derivative-free technique for solving nonlinear equations.

But none of these techniques discusses the asymptotic convergence properties of an interval sequence $\{(a_n, b_n)\}$ that enclosing the root α. From an analytical perspective, the sequence's convergence is crucial to enclose the root α. Ostrowski [10] proved that the diameter sequence is Q-quadratically convergent to the root by using a technique for solving (1). Alefeld and Potra [1] and Schmidt [11] simultaneously presented iterative methods for solving nonlinear equations that provide both the asymptotic convergence of the iterate sequence and the sequence of diameters of enclosing intervals. The method developed by Wu et al. [15] coupled Regula Falsi type methods with Steffensen's method to determine the convergence of a point and diameter sequence. Similarly, by combining the bisection approach with several exponential iteration methods, Chen [2] got the quadratic convergence of the point-diameter sequence. Additionally, some work in this approach is being done in [13, 16] for derivative-free algorithms.

This study aims to accelerate the convergence of Chen and Li's [3] approaches from quadratic to cubic, as well as the convergence of the iterate sequences to zero. The theoretical analysis and numerical experiments provided demonstrate the effectiveness of our proposed method and their comparability to the methods presented in [3], Newton's method, Steffensen's method, and Regula Falsi method.

The paper is designed as follows: section 2 presents proposed technique derived from the method of Chen and Li [3] by replacing p by p(x). As done in [3], Sect. 3 presents a convergence theorem that proves the cubic convergence of iterate sequence. In Sect. 4, numerical examples are computed using our technique, Newton's method, Steffensen's method, Chen, and Li's method [3], and Regula Falsi method. A table summarizing the results demonstrates the superior efficacy of modified approach over the other methods under comparison. Finally, conclusions from the Section 5.

PROPOSED METHODOLOGY:

Assume that α is a root of $f(x) = 0$. Assuming $f(a)f(b) < 0$, α is ensured to be a simple root of $f(x) = 0$ in $[a, b]$. Under Chen and Li's [3] analysis, the class of iteration formulae with a parameter is given by

$$x_{n+1} = x_n \exp\left\{-\frac{[f(x_n)]^2}{x_n[p\{f(x_n\}^2+f(x_n) - f(x_n-f(x_n))]}\right\} \ldots (3)$$

For $e_n = x_n - \alpha$ and $f'(\alpha) \neq 0$, they proved that

$$\lim_{n\to\infty} e_{n+1} = e_n{}^2[p + \frac{f''(\alpha)}{2f'(\alpha)} - \frac{f''(\alpha)}{2} + \frac{1}{2\alpha}]$$

Hence, order of the convergence of the iterative technique (3) is 2.

Now we define a function $p(x)$ as

$$p(x) = -\frac{f(x-f(x))[f(x-f(x)+f(x+f(x))-2f(x)]}{2[f(x)-f(x-f(x))]f(x)^2} - \frac{1}{2x} \ldots (4)$$

So that $\lim_{x\to a} p(x) = \frac{f''(\alpha)}{2} - \frac{f''(\alpha)}{2f'(\alpha)} - \frac{1}{2\alpha}$

This results in the iteration (3) modify that follows

$$x_{n+1} = x_n \exp\left\{-\frac{[f(x_n)]^2}{x_n[p(x_n)\{f(x_n\}^2+f(x_n) - f(x_n-f(x_n))]}\right\}, n = 0,1,2 \ \ldots (5)$$

CONVERGENCE ANALYSIS:

Theorem:

Assume that $f: [a, b] \to R$ be continuously differentiable function with $f(\alpha) = 0$. Suppose that $f'(\alpha) \neq 0$ and $k(\alpha)$ be a sufficiently small neighborhood of α. Then order of convergence of the sequence of iterates generated by the equation (5) with (4) is three.

Proof:

Let α be the simple zero of $f(x)$ and $e_n = x_n - \alpha$. Using Taylor expansion around $x = \alpha$ and considering $f(\alpha) = 0$, we get

$$f(x_n) = e_n f'(\alpha) + \frac{e_n^2}{2} f''(\alpha) + \frac{e_n^3}{6} f'''(\alpha) + o(e_n^4)$$

$$f\big(x_n - f(x_n)\big) = \big(1 - f'(\alpha)\big)f'(\alpha)e_n + (1 - 3f'(\alpha)f'(\alpha)^2 \frac{f'(\alpha)f''(\alpha)}{2} e_n^2 + o(e_n^3)$$

$$f\big(x_n + f(x_n)\big) = \big(1 + f'(\alpha)\big)f'(\alpha)e_n + (1 + 3f'(\alpha)f'(\alpha)^2 \frac{f'(\alpha)f''(\alpha)}{2} e_n^2 + o(e_n^3)$$

$$f(x_n)^2 = f'(\alpha)^2 e_n^2 + f'(\alpha)f''(\alpha)e_n^2 + o(e_n^3)$$

$$f(x_n) - f\big(x_n - f(x_n)\big) = f'(\alpha)^2 e_n^2 + 3[f'(\alpha) - f'(\alpha)^2] \frac{f''(\alpha)}{2} e_n^2 + o(e_n^3)$$

$$\frac{f(x_n) - f\big(x_n - f(x_n)\big)}{f(x_n)} = f'(\alpha) - [f'(\alpha) - 2]\frac{f''(\alpha)}{2} e_n + [2f'(\alpha)^2 f'''(\alpha) + 6f'''(\alpha) - 3f''(\alpha)^2 - 6f'(\alpha)f'''(\alpha)] \frac{e_n^2}{12} + o(e_n^3)$$

$$p(x_n) = -\frac{1}{2}\left[\big(1 - f'(\alpha)\big)\frac{f''(\alpha)}{f'(\alpha)} + \frac{1}{x_n}\right] + \left[\frac{f''(\alpha)^2}{2f'(\alpha)^2} + \frac{f'''(\alpha)}{2} - \frac{f'''(\alpha)}{2f'(\alpha)} - \frac{f''(\alpha)^2}{4f'(\alpha)}\right] e_n + o(e_n^2)$$

and

$$\frac{f(x_n)}{p(x_n)f(x_n) + \frac{f(x_n) - f(x_n - f(x_n))}{f(x_n)}} = e_n + \frac{1}{2x_n} e_n^2 + \left[\frac{f''(\alpha)^2}{4f'(\alpha)} - \frac{f'(\alpha)f'''(\alpha)}{6} + \frac{1}{4x_n^2} - \frac{f''(\alpha)^2}{4f'(\alpha)^2} + \frac{f'''(\alpha)}{6f'(\alpha)}\right] e_n^3 + o(e_n^4)$$

Now expanding the exponential function in the equation (5), we get

$$x_{n+1} = x_n \exp\left\{-\frac{[f(x_n)]^2}{x_n[p(x_n)\{f(x_n\}^2 + f(x_n) - f(x_n - f(x_n))]}\right\}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{p(x_n)f(x_n) + \frac{f(x_n) - f(x_n - f(x_n))}{f(x_n)}} + \frac{f(x_n)^2}{2x_n p(x_n)f(x_n)[\frac{f(x_n) - f(x_n - f(x_n))}{f(x_n)}]^2} + o\Big(\frac{f(x_n)^3}{2x_n p(x_n)f(x_n)[\frac{f(x_n) - f(x_n - f(x_n))}{f(x_n)}]^3}$$

After solving, we get

$$\lim_{n\to\infty} e_{n+1} = e_n^3 [\frac{f'''(\alpha)}{6f'(\alpha)}[f'(\alpha)^2 - 1] + \frac{f''(\alpha)^2}{4f'(\alpha)^2}[1 - f'(\alpha)] + \frac{1}{12\alpha^2}$$

This prove that the order of convergence of the sequence of iterates generated by the equation (5) with (4) is three.

NUMERICAL EXAMPLES:

Numerical Experiments to evaluate the performance of the proposed modified method, we conducted numerical experiments on a range of benchmark problems and compared the results with other established numerical methods. The experiments were implemented in MATLAB,

and the following methods were considered Regula Falsi (RF), Modified Regula Falsi (MRF), Newton Raphson method (NRM), and Steffensen method (SM)
The benchmark problems were selected to cover a variety of nonlinear equations with different characteristics, such as polynomial equations, transcendental equations, and equations with multiple roots. The following benchmark problems were considered:

Problem1: $f(x) = ln\,x$ in the interval $[0.5, 5]$
Problem 2: $f(x) = 10x - e^x$ in the interval $[0, 1]$
Problem 3: $f(x) = x + 1 - e^{sin\,x}$ in the interval [1,4]
Problem 4: $f(x) = x^{11} - 0.091$ in the interval $[0.1, 1]$
Problem 5: $f(x) = e^{sin\,x} - x^2 - 1$ in the interval $[1, 4]$
The tolerance for convergence was set to $\varepsilon = 10^{-15}$, the performance of each method was evaluated based on the following metrics:

1. Number of iterations required for convergence (n)
2. The iterate obtained after the iterations (x_n) .

The numerical results for each problem and method are summarized in Table 1.

Table 1: Numerical Results for Benchmark Problems

Problem	**Method**	**n**	$\mathbf{x_n}$
1	RF	27	1
	MRF	7	1
	NR	divergent	-
	SM	failure	-
2	RF	15	111833×10^{-6}
	MRF	6	111833×10^{-6}
	NR	failure	-
	SM	failure	-
3	RF	32	169681×10^{-5}
	MRF	11	169681×10^{-5}
	NR	failure	-
	SM	failure	-
4	RF	101	804133×10^{-6}
	MRF	21	804133×10^{-6}
	NR	200	804133×10^{-6}
	SM	divergent	-
5	RF	100	126203×10^{-5}
	MRF	19	126203×10^{-5}
	NR	divergent	-
	SM	divergent	-

CONCLUSION:

An exponential iterative approach combined with the Regula Falsi method presents a class of third order Regula Falsi methods for solving nonlinear equations. This is slightly different as described in [3]. It is established that the sequence of iterates $x_n - \alpha$ has a cubic asymptotic convergence rate. Following several numerical examples, the algorithm is tested, and the results up to the required precision, $\varepsilon = 10^{-15}$ are compared with our approach, Regula Falsi (RF), Steffensen's and Newton's techniques. It has been noted that the modified approach requires fewer iterations and more efficient as compared to these techniques.

Authors Contribution:

Inderjeet: Methodology, Software, Writing-Original Draf.

Rashmi Bhardwaj: Conceptualization, Supervision, Investigation, Writing-Review and Editing.

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REFERENCES:

1. Alefeld, G., Potra, F.A., Shi, Y.X.: On enclosing simple roots nonlinear equations. Math. Comput. 61, 733–744 (1993)
2. Chen, J.: A new family of exponential iteration methods with quadratic convergence of both diameters and points for enclosing zeros of nonlinear equations. Math. Comput. Modell. 45, 544–552 (2007)
3. Chen, J., Li, W.: An improved exponential regula falsi methods with quadratic convergence of both diameter and point for solving nonlinear equations. Appl. Numer. Math. 57, 80–88 (2007)
4. Chen, D., Argyros, I.K., Qian, Q.S.: A note on Halley method in Banach spaces. Appl. Math. Comput. 58, 215–224 (1993)
5. Chen, D., Argyros, I.K., Qian, Q.S.: A local convergence theorem for Super-Halley method in Banach space. Appl. Math. Lett. 7, 49–52 (1994)
6. Dowel, M., Jarratt, D.: A modified regula falsi method for computing the root of an equation. BIT 12, 168–174 (1971)
7. Dowell, M., Jarrett, P.: The “Pegasus” method for computing the root of an equation. BIT 12, 503–508 (1972)

8. Gutiérrez, J.M., Hernández, M.A.: A family of Chebyshev-Halley type methods in Banach space. Bull. Aust. Math. Soc. 55, 113–130 (1997)
9. Ortega, J.M., Rheinboldt, W.C.: Iterative Solution of Nonlinear Equations in Several Variables. Academic Press, San Diego (1970)
10. Ostrowski, A.M.: Solution of Equations in Euclidean and Banach Space, 3rd edn. Academic Press, San Diego (1973)
11. Schmidt, J.W.: On the R-order of coupled sequences. Computing 26, 333–342 (1981)
12. Traub, J.F.: Iterative Methods for the Solution of Equations. Prentice Hall, New York (1964) 13. Wu, X., Fu, D.: New higher-order convergence iteration methods without employing derivatives for solving nonlinear equations. Comput. Math. Appl. 41, 489–495 (2001)
14. Wu, X., Wu, H.: On a class of quadratic convergence iteration formulae without derivatives. Appl. Math. Comput. 107, 77–80 (2000)
15. Wu, X.Y., Shen, Z.H., Xia, J.L.: An improved regula falsi method with quadratic convergence of both diameter and point for enclosing simple zeros of nonlinear equations. Appl. Math. Comput. 144, 381–388 (2003)
16. Zhu, Y., Wu, X.: A free-derivative iteration method of order three having convergence of both point and interval for nonlinear equations. Appl. Math. Comput. 137, 49–55 (2003).