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Subclasses of Starlike and Convex Functions Involving Pascal Distribution Series

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The main object of this paper is to establish connections between various subclasses of normalized analytic univalent functions and Pascal distribution series by applying certain convolution operator involving Pascal distribution series.
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1. Introduction

Let $\mathcal{A}$ be the class of functions f(z) of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z: z \in \mathbb{C} \text{ and } |z| < 1\}$$

As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions which are normalized by $f(0) = 0 = f'(0) - 1$ and are also univalent in $\mathbb{U}$. Further, let $\mathcal{T}$ be a subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{U} \tag{1.2}$$

Also let $\mathcal{S}^*(\alpha)$ and $\mathcal{C}(\alpha)$ denote the subclasses of $\mathcal{A}$ consisting of functions which are, respectively, starlike and convex of order $\alpha (0 \leq \alpha < 1)$ in $\mathbb{U}$. Thus, we have (see, for details, [9,13,14,21])

$$\mathcal{S}^*(\alpha) = \left\{ f\colon f \in \mathcal{A} \text{ and } \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\} \tag{1.3}$$

$$\mathcal{C}(\alpha) = \left\{ f\colon f \in \mathcal{A} \text{ and } \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, (z \in \mathbb{U}; 0 \leq \alpha < 1) \right\} \tag{1.4}$$

where, for convenience,

$$\mathcal{S}^*(0) = \mathcal{S}^*,\ \mathcal{C}(0) = \mathcal{C}$$

It is a well-established fact that

$$f(z) \in \mathcal{C}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha)$$

Kanas and Wisniowska [15] introduced the class $\beta - \mathrm{UCV}$ which includes geometric aspect in connection with conic domains. The family $\beta - \mathrm{UCV}$ is of special interest for it contains many well-known, as well as new, classes of analytic univalent functions. The class $\beta - \mathrm{UCV}$ map each circular arc contained in the unit disk $\mathbb{U}$ with a center $\xi, |\xi| \leq \beta (0 \leq \beta < 1)$, onto a convex arc. The notion of β-uniformly convex function is simple extension of classical convexity. For various choices of β, these classes reduce to well-known subclasses of uniformly starlike and convex functions. In particular, when $\beta = 0$, the center ξ is the origin and the class β-uniformly convex function reduces to the familiar classes of convex univalent functions $\mathcal{K}$. Moreover for $\beta = 1$ the class reduces to uniformly convex functions UCV introduce by Goodman [12], and studied extensively by Rønning [20]. We recall the following subclass of uniformly convex functions and corresponding subclass of starlike functions due to [20].

For $-1 < \alpha \leq 1$ and $\beta \geq 0$, a function $f \in \mathcal{A}$ is said to in the class
i) β-uniformly starlike functions of order α, denoted by $S_p(\alpha, \beta)$ if it satisfies analytic criterion

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} > \beta \left|\frac{zf'(z)}{f(z)} - 1\right|, z \in \mathbb{U} \tag{1.5}$$

and
ii) β-uniformly convex functions of order α, denoted by $\mathrm{UCV}(\alpha, \beta)$ if it satisfies analytic criterion

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} > \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|, z \in \mathbb{U} \tag{1.6}$$

It follows from (1.5) and (1.6) that

$$f \in \mathrm{UCV}(\alpha, \beta) \Leftrightarrow zf' \in S_p(\alpha, \beta)$$

We recall the following subclass of uniformly convex functions and corresponding subclass of starlike functions due to [17].
For $0 \leq \lambda < 1, 0 \leq \alpha < 1$ and $\beta \geq 0$, we let $\mathcal{P}_\lambda(\alpha, \beta)$ be the subclass of A consisting of functions of the form (1.1) and satisfying the analytic criterion

$$\operatorname{Re}\left\{\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - \alpha\right\} > \beta\left|\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda zf'(z)} - 1\right| \tag{1.7}$$

also let $\mathcal{Q}_\lambda(\alpha, \beta)$ be the subclass of $\mathcal{S}$ satisfying the analytic criteria

$$\operatorname{Re}\left\{\frac{\lambda z^2 f'''(z) + (1+2\lambda)zf''(z) + f'(z)}{f'(z) + \lambda zf''(z)} - \alpha\right\} > \beta\left|\frac{\lambda z^2 f'''(z) + (1+\lambda)zf''(z)}{f'(z) + \lambda zf''(z)}\right| \tag{1.8}$$

Also denote $\mathcal{P}_\lambda^*(\alpha, \beta) = \mathcal{P}_\lambda(\alpha, \beta) \cap \mathcal{T}$ and $\mathcal{Q}_\lambda^*(\alpha, \beta) = \mathcal{Q}_\lambda(\alpha, \beta) \cap \mathcal{T}$, the subclasses of $\mathcal{T}$. In particular, we note that $\mathcal{P}_0^*(\alpha, \beta) = \mathcal{TS}_P(\alpha, \beta)$ and $\mathcal{P}_1^*(\alpha, \beta) = \mathcal{UCT}(\alpha, \beta)$, the classes of uniformly starlike and uniformly convex were introduced by Bharati et al.[5]. For $\beta = 0$, the classes $\mathcal{TS}_P(\alpha, \beta)$ and $\mathcal{UC}(\alpha, \beta)$ respectively, reduces to the classes $\mathcal{T}^*(\alpha)$ and $\mathcal{C}(\alpha)$ introduced and studied by Silverman [23]. Further by taking $\alpha = 0$ we have $\mathcal{P}_0^*(\beta) = \mathcal{TS}_P(\beta)$ and $\mathcal{P}_1^*(\beta) = \mathcal{UCT}(\beta)$ introduced and studied by Subramanian et al.[26]. For more interesting developments of some related subclasses of uniformly starlike and uniformly convex functions, the readers may be referred to the works of Frasin et al.[11], Subramanian et al.[25] and Srivastava et al.[24].

A variable x is said to be Pascal distribution if it takes the values 0,1,2,3, ... with probabilities $(1-q)^m, \frac{qm(1-q)^m}{1!}, \frac{q^2 m(m+1)(1-q)^m}{2!}, \frac{q^3 m(m+1)(m+2)(1-q)^m}{3!}, \ldots$ respectively, where q and m called parameters, and thus

$$P(X = k) = \binom{k+m-1}{m-1} q^k (1-q)^m, k = 0,1,2,3, \cdots \tag{1.9}$$

Recently, El-Deeb et al.[8] introduced a power series whose coefficients are probabilities of Pascal distribution

$$\Psi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m z^n,\ z \in \mathbb{U} \tag{1.10}$$

where $m \geq 1; 0 \leq q \leq 1$ and we note that, by ratio test the radius of convergence of above series is infinity. We also define the series

$$\Phi_q^m(z) = 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m a_n z^n,\ z \in \mathbb{U} \tag{1.11}$$

Now, we considered the linear operator $\mathcal{J}_q^m(z): \mathcal{A} \to \mathcal{A}$ defined by Hadamard product (convolution) as

$$\mathcal{J}_q^m(z) = \Psi_q^m(z) * f(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m a_n z^n,\ z \in \mathbb{U} \tag{1.12}$$

To establish our main results, we need the following Lemmas.
Lemma 1.1. [17] A function f of the form (1.2) is in $\mathcal{P}_\lambda^*(\alpha,\beta)$ if and only if it satisies

$$\sum_{n=2}^{\infty} (1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]|a_n| \leq 1-\alpha \tag{1.13}$$

Lemma 1.2. [17] A function f of the form (1.2) is in $\mathcal{Q}_\lambda^*(\alpha,\beta)$ if and only if it satisfies

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]|a_n| \leq 1-\alpha \tag{1.14}$$

Lemma 1.3. [7] If $f \in \mathcal{R}^\tau(C,D)$ is of the form (1.1) then

$$|a_n| = \frac{(C-D)|\nu|}{n}, n \in N \setminus \{1\} \tag{1.15}$$

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions by using hypergeometric functions [6, 17, 18, 19, 22] and by the recent investigation in generalized Bessel functions and Struve functions [1, 2, 3, 10, 24, 27, 28] in the present paper we establish a number of connections between the classes $\mathcal{P}_\lambda^*(\alpha,\beta)$ and $\mathcal{Q}_\lambda^*(\alpha,\beta)$ by applying the convolution operator given by (1.12). We also determine conditions for the integral operator $\mathcal{H}(m,z) = \int_0^z \frac{\mathcal{I}_q^m f(t)}{t} dt$ belonging to the class $\mathcal{Q}_\lambda^*(\alpha,\beta)$.

2. Coefficient Estimates

Theorem 2.1. If $m \geq 1$, then $\phi_q^m(z)$ is in $\mathcal{P}_\lambda^*(\alpha,\beta)$ if and only if

$$(1+\beta)\lambda\frac{q^2 m(m+1)}{(1-q)^2} + [\,(1+\beta)(1+2\lambda)-\lambda(\alpha+\beta)]\,\frac{qm}{(1-q)}$$

Proof. Since

$$\Phi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m a_n z^n$$

in view of Lemma 1.1, it is sufficient to show that

$$\sum_{n=2}^{\infty} (1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m \leq 1-\alpha \tag{2.2}$$

In the proofs of all our results we will use the relation

$$\sum_{n=0}^{\infty} \binom{n+m-2}{m-1} q^n = \frac{1}{(1-q)^m}, \ (0 \leq q \leq 1)$$

and the corresponding ones obtained by replacing the value of m with $m-1, m+1$, and $m+$

2.

Writing $\boldsymbol{n^2 = (n-2)(n-1) + 3(n-1) + 1}$ and $\boldsymbol{n = (n-1) + 1}$ in $\boldsymbol{(2.2)}$

$$\sum_{n=2}^{\infty} (1 + n\lambda - \lambda)[n(1+\beta) - (\alpha+\beta)] \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m$$

$$= (1+\beta)\lambda \sum_{n=2}^{\infty} [(n-2)(n-1) + 3(n-1) + 1] \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m$$

$$+ [(1+\beta)(1-\lambda) - \lambda(\alpha+\beta)] \sum_{n=2}^{\infty} [(n-1)+1] \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m$$

$$+ (1-\alpha) \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} \cdot q^{n-1}(1-q)^m$$

$$= (1+\beta)\lambda q^2 m(m+1)(1-q)^m \sum_{n=0}^{\infty} \binom{n+m+1}{m+1} \cdot q^n$$

$$+ [(1+\beta)(1-\lambda) - \lambda(\alpha+\beta)] qm(1-q)^m \sum_{n=0}^{\infty} \binom{n+m}{m} \cdot q^n$$

$$+ (1-\alpha)(1-q)^m \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \cdot q^{n-1}$$

$$= (1+\beta)\lambda q^2 m(m+1)(1-q)^m \frac{1}{(1-q)^{m+2}} + [(1+\beta)(1-\lambda) - \lambda(\alpha+\beta)] qm$$

$$. (1-q)^m \frac{1}{(1-q)^{m+1}} + (1-\alpha)(1-q)^m \left[\frac{1}{(1-q)^m} - 1\right]$$

$$= (1+\beta)\lambda q^2 m(m+1) \frac{1}{(1-q)^2} + [(1+\beta)(1+2\lambda) - \lambda(\alpha+\beta)] \frac{qm}{(1-q)}$$

$$. (1-\alpha)[1 - (1-q)^m]$$

But this last expression is bounded above by $\boldsymbol{1-\alpha}$ if and only if (2.1) holds.

Theorem 2.2. If $\boldsymbol{m \geq 1}$, then $\boldsymbol{\phi_q^m(z)}$ is in $\boldsymbol{\mathcal{Q}_\lambda^*(\alpha, \beta)}$ if and only if

$$= (1+\beta)\lambda \frac{q^3 m(m+1)(m+2)}{(1-q)^3} + [(1+\beta)(1+5\lambda) - \lambda(\alpha+\beta)] \frac{q^2 m(m+1)}{(1-q)^2}$$

Proof. The proof of Theorem 2.2 is lines similar to the proof of Theorem 2.1, so we omitted the proof.

By taking $\boldsymbol{\lambda = 0}$ and $\boldsymbol{\lambda = 1}$ we state the following Corollaries.

Corollary 2.1. If $\boldsymbol{m \geq 1}$ then necessary and sufficient condition for $\boldsymbol{\phi_q^m(z) \in \mathcal{P}_0^*(\alpha, \beta) \equiv \mathcal{TS}_P(\alpha, \beta)}$ is

$$(\mathbf{1}+\boldsymbol{\beta})\frac{\boldsymbol{qm}}{(\mathbf{1}-\boldsymbol{q})}+(\mathbf{1}-\boldsymbol{\alpha})[\mathbf{1}-(\mathbf{1}-\boldsymbol{q})^{\boldsymbol{m}}]\leq \mathbf{1}-\boldsymbol{\alpha} \qquad (\mathbf{2.4})$$

Corollary 2.2. If $\boldsymbol{m}\geq \mathbf{1}$ then necessary and sufficient condition for $\boldsymbol{\phi}_{\boldsymbol{q}}^{\boldsymbol{m}}(\boldsymbol{z})\in \mathcal{P}_{\mathbf{1}}^{*}(\boldsymbol{\alpha},\boldsymbol{\beta})\equiv \mathcal{UCT}(\boldsymbol{\alpha},\boldsymbol{\beta})$ is

$$(\mathbf{1}+\boldsymbol{\beta})\frac{\boldsymbol{q}^{\mathbf{2}}\boldsymbol{m}(\boldsymbol{m}+\mathbf{1})}{(\mathbf{1}-\boldsymbol{q})^{\mathbf{2}}}+[\mathbf{3}(\mathbf{1}+\boldsymbol{\beta})-(\boldsymbol{\alpha}+\boldsymbol{\beta})]\frac{\boldsymbol{qm}}{(\mathbf{1}-\boldsymbol{q})}$$

3. Inclusion Properties

Theorem 3.1. If $\boldsymbol{f}\in \mathcal{R}^{\tau}(\boldsymbol{C},\boldsymbol{D})$ is of the form (1.1) and the operator $\mathcal{J}_{\mathbf{q}}^{\mathbf{m}}(\mathbf{z})$, defined by (1.12) in the class $\mathcal{P}_{\boldsymbol{\lambda}}^{*}(\boldsymbol{\alpha},\boldsymbol{\beta})$, if

$$(\mathbf{C}-\mathbf{D})|\boldsymbol{\nu}|(\mathbf{1}-\mathbf{q})^{\mathbf{m}}\left[(\mathbf{1}+\boldsymbol{\beta})\boldsymbol{\lambda}\left(\frac{\mathbf{qm}}{(\mathbf{1}-\mathbf{q})}\right)+[(\mathbf{1}+\boldsymbol{\beta})-\boldsymbol{\lambda}(\boldsymbol{\alpha}+\boldsymbol{\beta})](\mathbf{1}-(\mathbf{1}-\mathbf{q})^{\mathbf{m}})\right.$$

Proof. In view of Lemma 1.1, it is sufficient to show that

$$\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}|a_n|(1-q)^m\leq 1-\alpha \qquad (3.2)$$

Since $f\in \mathcal{R}^{\tau}(C,D)$, then by Lemma 1.3 we have

$$|a_n|=\frac{(C-D)|\nu|}{n}.$$

Thus, we have

$$\begin{aligned}
&=\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}|a_n|(1-q)^m\\
&\leq\sum_{n=2}^{\infty}\frac{(C-D)|\nu|}{n}(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}(1-q)^m\\
&=(C-D)|\nu|\left[\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}(1-q)^m\right]\\
&=(C-D)|\nu|(1-q)^m\left[\left((1+\beta)\lambda\sum_{n=2}^{\infty}n\binom{n+m-2}{m-1}\cdot q^{n-1}+[(1+\beta)-(\alpha+\beta)]\right.\right.\\
&\left.\left.\times\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}-(1-\lambda)(\alpha+\beta)\sum_{n=2}^{\infty}\frac{1}{n}\cdot\binom{n+m-2}{m-1}\cdot q^{n-1}\right)\right]
\end{aligned}$$

writing $n=(n-1)+1$, upon simple computation

$$= (C-D)|\nu|(1-q)^m \Bigg[(1+\beta)\lambda\left(qm\sum_{n=0}^{\infty}\binom{n+m}{m}\cdot q^n + \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}\cdot q^n - 1\right)$$
$$+ [(1+\beta)(1-\lambda)-\lambda(\alpha+\beta)]\left(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}\cdot q^n - 1\right)$$
$$- \frac{(1-\lambda)(\alpha+\beta)}{q(m-1)}\left(\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}\cdot q^n - 1 - (m-1)q\right)\Bigg]$$
$$= (C-D)|\nu|(1-q)^m\Bigg[(1+\beta)\lambda\left(\frac{qm}{(1-q)} + 1 - (1-q)^m\right) + [(1+\beta)(1-\lambda)-\lambda(\alpha+\beta)]$$
$$\times(1-(1-q)^m) - \frac{(1-\lambda)(\alpha+\beta)}{q(m-1)}((1-q)-(1-q)^m - q(m-1)(1-q)^m)\Bigg]$$
$$= (C-D)|\nu|(1-q)^m\Bigg[(1+\beta)\lambda\left(\frac{qm}{(1-q)}\right) + [(1+\beta)-\lambda(\alpha+\beta)]$$
$$\times(1-(1-q)^m) - \frac{(1-\lambda)(\alpha+\beta)}{q(m-1)}((1-q)-(1-q)^m - q(m-1)(1-q)^m)\Bigg]$$

Theorem 3.2. If $f \in \mathcal{R}^\tau(C,D)$ is of the form (1.1) and the operator $\mathcal{J}_q^m(z)$, defined by (1.12) in the class $\mathcal{Q}_\lambda^*(\alpha,\beta)$, if

$$(C-D)|\nu|\left[(1+\beta)\lambda\frac{q^2m(m+1)}{(1-q)^2} + [(1+\beta)(1+2\lambda)-\lambda(\alpha+\beta)]\frac{qm}{(1-q)}\right.$$

Proof. In view of Lemma 1.2, it is sufficient to show that

$$\sum_{n=2}^{\infty} n(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}|a_n|(1-q)^m \le 1-\alpha \quad (3.4)$$

Since $f \in \mathcal{R}^\tau(C,D)$, then by Lemma 1.3 we have

$$|a_n| = \frac{(C-D)|\nu|}{n}$$

Thus we have

$$\Rightarrow \sum_{n=2}^{\infty}\frac{(C-D)|\nu|}{n} n(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}(1-q)^m$$
$$\Rightarrow \le (C-D)|\nu|(1-q)^m\sum_{n=2}^{\infty}(1+n\lambda-\lambda)[n(1+\beta)-(\alpha+\beta)]\binom{n+m-2}{m-1}\cdot q^{n-1}$$

Further, proceeding as in Theorem 2.1, we get

$$(C-D)|\nu|(1+\beta)\lambda\frac{q^2m(m+1)}{(1-q)^2} + [(1+\beta)(1+2\lambda)-\lambda(\alpha+\beta)]\frac{qm}{(1-q)}$$
$$+ (1-\alpha)[1-(1-q)^m] \le 1-\alpha$$

Corollary 3.1. Let $m \geq 1$. If $f \in \mathcal{R}^{\tau}(C, D)$ is of the form (1.1) and the operator $\mathcal{J}_q^m(z)$, defined by (1.12) in the class $\mathcal{P}_0^*(\alpha, \beta) \equiv \mathcal{TS}_P(\alpha, \beta)$, if

$$\Rightarrow (C - D)|\nu|(1 - q)^m \left[(1 + \beta)(1 - (1 - q)^m) \frac{(\alpha + \beta)}{q(m - 1)}\right.$$

Corollary 3.2. Let $m \geq 1$. If $f \in \mathcal{R}^{\tau}(C, D)$ is of the form (1.1) and the operator $\mathcal{J}_q^m(z)$, defined by (1.12) in the class $\mathcal{P}_0^*(\alpha, \beta) \equiv \mathcal{TS}_P(\alpha, \beta)$, if

$$(C - D)|\nu| \left[(1 + \beta) \frac{q^2 m(m + 1)}{(1 - q)^2} + [3(1 + \beta) - (\alpha + \beta)] \frac{qm}{(1 - q)}\right.$$

4. An Integral Operator

In this section we introduce an integral operator $\mathcal{H}(m, z)$ as follows:

$$\mathcal{H}(m, z) = \int_0^z \frac{\mathcal{J}_q^m f(t)}{t} dt \tag{4.1}$$

and we obtain a necessary and sufficient condition for $\mathcal{H}(m, z)$ belonging to the class $\mathcal{Q}_z^*(\alpha, \beta)$.

Theorem 4.1. If $m \geq 1$, then the integral $\mathcal{H}(m, z)$ defined by (4.1) is in the class $\mathcal{Q}_\lambda^*(\alpha, \beta)$, if and only if inequality (2.1) is satisfied.

Proof. Since

$$\mathcal{H}(z) = z + \sum_{n=2}^{\infty} \binom{n + m - 2}{m - 1} \cdot q^{n-1}(1 - q)^m \frac{z^n}{n}$$

in view of Lemma (1.2), we need only to show that

$$\sum_{n=2}^{\infty} n(1 + n\lambda - \lambda)[n(1 + \beta) - (\alpha + \beta)] \times \frac{1}{n} \binom{n + m - 2}{m - 1} \cdot q^{n-1}(1 - q)^m \leq 1 - \alpha$$

or, equivalently

$$\sum_{n=2}^{\infty} (1 + n\lambda - \lambda)[n(1 + \beta) - (\alpha + \beta)] \binom{n + m - 2}{m - 1} \cdot q^{n-1}(1 - q)^m \leq 1 - \alpha$$

The remaining part of the proof of Theorem 4.1 is similar to that of Theorem 2.1, and so we omit the details.

5. Conclusion

In this investigation, we obtain sufficient condition and inclusion results for function $f \in \mathcal{A}$ to be in the classes of starlike and convex functions $\mathcal{P}_\lambda^*(\alpha, \beta)$ and $\mathcal{Q}_\lambda^*(\alpha, \beta)$ associated with Pascal distribution series in the open unit disc $\mathbb{U}$. Further, we derive the result of integral operator related to Pascal distribution series.

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