Nanotechnology Perceptions
ISSN 1660-6795
www.nano-ntp.com

More on Maps and its Application in Pythagorean Fuzzy Topological Spaces

B. Vijayalakshmi[1], M. Ramalakshmi[2], A. Vadivel[3], G. Saravanakumar[4]

[1]Department of Mathematics, Government Arts College, Chidambaram, Tamil Nadu-608 102; Mathematics Section (FEAT), Annamalai University, Annamalai Nagar - 608 002, TamilNadu.
[2]Department of Mathematics, Sri Meenakshi Government Arts College for Women (A), Madurai- 625 002 ^3 Arignar Anna Government Arts College, Namakkal - 637 002, India.
[3]Department of Mathematics, Annamalai University, Annamalai Nagar - 608 002, India.
[4]Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology (Deemed to be University), Avadi, Chennai-600062, India
Email: mathvijaya2006au@gmail.com

In this paper, we introduce and investigate Pythagorean fuzzy M-continuous maps in Pythagorean fuzzy topological spaces and also discuss about some properties and characterization of Pythagorean fuzzy M-irresolute maps. Also one real life applications, we applied entropy measure for decision making problem of calculation of incentive based on the performance of salesman during the seasons.
Keywords: Pythagorean fuzzy M-closed sets, Pythagorean fuzzy M-continuous maps and Pythagorean fuzzy M-irresolute maps, Pythagorean fuzzy entropy..

1. Introduction

Considering the imprecision in decision-making, Zadeh [29] introduced the idea of fuzzy set which has a membership function, μ that assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration. The notion of fuzzy sets generalizes classical sets theory by allowing intermediate situations between the whole and nothing. In a fuzzy set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means belongs, and other values indicate the degree of membership to a class. For fuzzy sets, the membership function replaced the characteristic function in crisp sets. The concept of fuzzy set theory seems to be inconclusive because of the exclusion of nonmembership function and the disregard for the possibility of hesitation margin.

Atanassov critically studied these shortcomings and proposed a concept called intuitionistic fuzzy sets (IFSs) [1, 2, 4, 5]. The construct (that is, IFS's) incorporates both membership function, μ and nonmembership function, ν with hesitation margin, π (that is, neither membership nor non-membership functions), such that $\mu + \nu \leq 1$ and $\mu + \nu + \pi = 1$. Atanassov [3] introduced intuitionistic fuzzy sets of second type (IFSST) with the property that the sum of the square of the membership and non-membership degrees is less than or equal to one. This concept generalizes IFS's in a way. The notion of IFS's provides a flexible framework to elaborate uncertainty and vagueness. The idea of IFS seems to be resourceful in modelling many real-life situations like medical diagnosis [7, 8, 12, 23, 24], career determination [10], selection process [11], and multi-criteria decision-making [15, 16, 17], among others.

There are situations where $\mu + \nu \geq 1$ unlike the cases capture in IFS's. This limitation in IFS naturally led to a construct, called Pythagorean fuzzy sets (pfs's). Pythagorean fuzzy set (pfs) proposed in [26, 27, 28] is a new tool to deal with vagueness considering the membership grade, μ and non-membership grade, ν satisfying the conditions $\mu + \nu \leq 1$ or $\mu + \nu \geq 1$, and also, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where π is the Pythagorean fuzzy set index. In fact, the origin of Pythagorean fuzzy sets emanated from IFSST earlier studied in the literature. As a generalized set, PFS has close relationship with IFS. The construct of PFS's can be used to characterize uncertain information more sufficiently and accurately than IFS. Garg [14] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets (IVPFSs). Based on it, a Pythagorean fuzzy technique for order of preference by similarity to ideal solution (TOPSIS) method by taking the preferences of the experts in the form of interval-valued Pythagorean fuzzy decision matrices was discussed. Other explorations of the theory of PFS's can be found in [6, 9, 13, 18, 19, 21, 22].

Entropy can be viewed as a gauge of the degree of uncertainty present in a set, regardless of how fuzzy, intuitionistic, ambiguous, etc. the set may be. Since the pfs in this case can also handle uncertain data, it follows naturally that we are also interested in determining the entropy of an pfs. In 1965, Zadeh [29] made the firs reference to entropy as a fuzziness metric. More recently, De Luca-Termini [8] axiomatized the entropy that is not probabilistic.

The remainder of this paper is organized as follows. In section 2, some basic definitions of fs's, IFS's and pfs's are briefly reviewed. In section 3, We develop the concept of some Pythagorean fuzzy continuous and irresolute maps in Pythagorean fuzzy topological space and also specialized some of their basic properties with examples. Finally, we presented an entropy measure for pfs's and one real- world scenarios where this entropy measure can be used are mentioned in section 4. The paper is concluded in section 5.

2 Preliminaries

We recall some basic notions of fuzzy sets, IFS's and pfs's .

Definition 2.1 [29] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$. That is:

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if } x \text{ ispartlyin } X. \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle\frac{\mu_A(x)}{x}\right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval [0,1].

Let us consider two examples:

(i) all employees of XYZ who are over 1.8m in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over 1.8m) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2 [1, 2, 4, 5] Let a nonempty set X be fixed. An IFS A in X is an object having the form: $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ or $A = \left\{\left\langle\frac{\mu_A(x), \nu_A(x)}{x}\right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\nu_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. For each A in X: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus: $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$.

Example 2.1 Let $X = \{x, y, z\}$ be a fixed universe of discourse and $A = \left\{\left\langle\frac{0.6,0.1}{x}\right\rangle, \left\langle\frac{0.8,0.1}{y}\right\rangle, \left\langle\frac{0.5,0.3}{z}\right\rangle\right\}$, be the intuitionistic fuzzy set in X. The hesitation margins of the elements x, y, z to A are as follows: $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ and $\pi_A(z) = 0.2$.

Definition 2.3 [26, 27, 28] Let X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following: $A = \{< x, \mu_A(x), \nu_A(x) | x \in X\}$ or $A = \left\{\left\langle\frac{\mu_A(x), \nu_A(x)}{x}\right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\nu_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$, $0 \le (\mu_A(x))^2 + (\nu_A(x))^2 \le 1$. Supposing $(\mu_A(x))^2 + (\nu_A(x))^2 \le 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$ and $\pi_A(x) \in [0,1]$. In what follows, $(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\nu_A(x))^2 = 1$. We denote the set of all PFS's over X by pfs(X).

Definition 2.4 [28] Let A and B be pfs's of the forms $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ and $B = \{< a, \lambda_B(a), \mu_B(a) > | a \in X\}$. Then

1. $A \subseteq B$ if and only if $\lambda_A(a) \le \lambda_B(a)$ and $\mu_A(a) \ge \mu_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\overline{A} = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \lambda_A(a) \wedge \lambda_B(a), \mu_A(a) \vee \mu_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \lambda_A(a) \vee \lambda_B(a), \mu_A(a) \wedge \mu_B(a) > | a \in X\}$.
6. $\phi = \{< a, \phi, X > | a \in X\}$ and $X = \{< a, X, \phi > | a \in X\}$.
7. $\overline{X} = \phi$ and $\overline{\phi} = X$.

Definition 2.5 [20] An Pythagorean fuzzy topology by subsets of a non-empty set X is a family τ of pfs's satisfying the following axioms.

1. $\phi, X \in \tau$.
2. $G_1 \cap G_2 \in \tau$ for every $G_1, G_2 \in \tau$ and
3. $\bigcup G_i \in \tau$ for any arbitrary family $\{G_i | i \in j\} \subseteq \tau$. The pair (X, τ) is called an Pythagorean fuzzy topological space (pfts in short) and any pfs G in τ is called an Pythagorean fuzzy open set (pfos in short) in X. The complement $\overline{A}$ of an Pythagorean fuzzy open set A in an pfts(X, τ) is called an Pythagorean fuzzy closed set (pfcs in short).

Definition 2.6 [20] Let (X, τ) be an pfts and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ be an pfs in X. Then the interior and the closure of A are denoted by pfint(A) and pfcl(A) and are defined as follows: $\mathrm{pfcl}(A) = \cap \{K | K \text{ isan pfcs and } A \subseteq K\}$ and $\mathrm{pfint}(A) = \cup \{G | G \text{ isan pfos and } G \subseteq A\}$. Also, it can be established that pfcl(A) is an pfcs and pfint(A) is an pfos, A is an pfcs if and only if $\mathrm{pfcl}(A) = A$ and A is an pfos if and only if $\mathrm{pfint}(A) = A$. We say that A is pf-dense if $\mathrm{pfcl}(A) = X$.

Lemma 2.1 [25] For any Pythagorean fuzzy set A in (X, τ), we have $X - \mathrm{pfint}(A) = \mathrm{pfcl}(X - A)$ and $X - \mathrm{pfcl}(A) = \mathrm{pfint}(X - A)$.

Definition 2.7 [25] Let (X, τ) be an $pfts$ and A be an pfs. Then A is said to be an Pythagorean fuzzy (i) regular open set ($pfros$ in short) if $A = pfint(pfcl(A))$. (ii) regular closed set ($pfrcs$ in short) if $A = pfcl(pfint(A))$. By Lemma 2.1, it follows that A is an $pfros$ iff $\bar{A}$ is an $pfrcs$.

3 Pythagorean fuzzy M-continuous maps

Definition 3.1 Let (X_1, Γ_P) (or X_1) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. Then the δ-interior and the δ-closure of A are denoted by $pf\delta int(A)$ and $pf\delta cl(A)$ and are defined as follows. $pf\delta cl(A) = \cap \{K | K$ is an $pfrcs$ and $A \subseteq K\}$, $(pf\delta int(A) = \cup \{G | G$ is an $pfros$ and $G \subseteq A\}$.

Definition 3.2 Let (X_1, Γ_P) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. A set A is said to be pf

1. δ-open set (briefly, $pf\delta os$) if $A = pf\delta int(A)$,
2. δ-pre open set (briefly, $pf\delta\mathcal{P}os$) if $A \subseteq pfint(pf\delta cl(A))$.
3. δ-semi open set (briefly, $pf\delta\mathcal{S}os$) if $A \subseteq pfcl(pf\delta int(A))$.
4. e open set (briefly, $pfeos$) if $A \subseteq pfcl(pf\delta int(A)) \cup pfint(pf\delta cl(A))$.
5. δ (resp. δ-pre, δ-semi and e) dense if $pf\delta cl(A)$ (resp. $pf\delta\mathcal{P}cl(A), pf\delta\mathcal{S}cl(A)$ and $pfecl(A)) = X_1$.

The complement of an $pf\delta os$ (resp. $pf\delta\mathcal{P}os, pf\delta\mathcal{S}os$ and $pfeos$) is called an $pf\delta$ (resp. $pf\delta\mathcal{P}, pf\delta\mathcal{S}$ and pfe) closed set (briefly, $pf\delta cs$ (resp. $pf\delta\mathcal{P}cs, pf\delta\mathcal{S}cs$ and $pfecs$ in X_1.

The family of all $pf\delta os$ (resp. $pf\delta cs, pf\delta\mathcal{P}os, pf\delta\mathcal{P}cs, pf\delta\mathcal{S}os, pf\delta\mathcal{S}cs, pfeos$ and $pfecs$) of X_1 is denoted by $pf\delta OS(X_1)$, (resp. $pf\delta CS(X_1), pf\delta\mathcal{P}OS(X_1), pf\delta\mathcal{P}CS(X_1), pf\delta\mathcal{S}OS(X_1), pf\delta\mathcal{S}CS(X_1)$, $pfeOS(X_1)$ and $pfeCS(X_1)$).

Definition 3.3 Let (X, τ) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. Then the $pf\delta$-pre (resp. $pf\delta$-semi and $pf\delta\beta$)-interior and the $pf\delta$-pre (resp. $pf\delta$-semi and pfe)-closure of A are denoted by $pf\delta\mathcal{P}int(A)$ (resp. $pf\delta\mathcal{S}int(A)$ and $pfeint(A)$) and the $pfecl(A)$ (resp. $pf\delta\mathcal{S}cl(A)$ and $pfecl(A)$ and are defined as follows:

$pf\delta\mathcal{P}int(A)$ (resp. $pf\delta\mathcal{S}int(A)$ and $pfeint(A) = \cup \{G | G$ in a $pf\delta\mathcal{P}os$ (resp. $pf\delta\mathcal{S}os$ and $pfeos$)

and $G \subseteq A\}$ and $pf\delta\mathcal{P}cl(A)$ (resp. $pf\delta\mathcal{S}cl(A)$ and $pfecl(A)) = \cap \{K | K$ is an $pf\delta\mathcal{P}cs$ (resp. $pf\delta\mathcal{S}cs, pfecs$) and $A \subseteq K\}$.

Definition 3.4 Let (X_1, Γ_P) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. A set A is said to be pf

1. θ-interior of A (briefly, $pf\theta int(A)$) is defined by $pf\theta int(A) = \bigcup \{pfint(B): B \subseteq A$ & B isa $pfcsinX_1\}$.
2. θ-open set (briefly, $pf\theta os$) if $A = pf\theta int(A)$.
3. θ -semi open set (briefly, $pf\theta\mathcal{S}os$) if $A \subseteq pfcl(pf\theta int(A))$.

4. M-open set (briefly, $pfMos$) if $A \subseteq pfcl(pf\theta int(A)) \cup pfint(pf\delta cl(A))$.

The complement of a $pfMos$ (resp. $pf\theta os$ & $pf\theta\mathcal{S}os$) is called an pfM (resp. $pf\theta$ & $pf\theta\mathcal{S}$) closed set (briefly, $pfMcs$ (resp. $pf\theta cs$ & $pf\theta\mathcal{S}cs$)) in X_1.

The family of all $pf\theta os$ (resp. $pf\theta cs, pf\theta\mathcal{S}os, pf\theta\mathcal{S}cs, pfMos$ *and* $pfMcs$) of X_1 is denoted by $pf\theta OS(X_1)$, (resp. $pf\theta CS(X_1), pf\theta\mathcal{S}OS(X_1), pf\theta\mathcal{S}CS(X_1), pfMOS(X_1)$ and $pfMCS(X_1)$).

Definition 3.5 Let (X_1, Γ_P) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X_1\}$ be an pfs in X_1. Then the pf

1. M-interior (resp. $pf\theta$-interior and $pf\theta$-semi interior) of A (briefly, $pfMint(A)$ (resp. $pf\theta int(A), pf\theta\mathcal{S}int(A)$) is defined by $pfMint(A)$ (resp. $pf\theta int(A)$ and $pf\theta\mathcal{S}int(A)$) $=\cup \{B: B \subseteq A$ and B is a $pfMos$ (resp. $pf\theta os, pf\theta\mathcal{S}os$)$\}$ in X_1.

2. M-closure (resp. θ-closure and θ-semi closure) of A (briefly, $pfMcl(A)$ (resp. $pf\theta cl(A)$&$pf\theta\mathcal{S}cl(A)$) is defined by $pfMcl(A)$ (resp. $pf\theta cl(A)$ and $pf\theta\mathcal{S}cl(A)$) $=\cap \{B: A \subseteq B$ and A is a $pfMcs$ (resp. $pf\theta cs, pf\theta\mathcal{S}cs$)$\}$ in X_1.

Definition 3.6 Let (X_1, Γ_P) and (X_2, Ψ_P) be any two $pfts$'s. A mapping $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ is said to be a Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, e, θ, $\theta\mathcal{S}$ and M)-continuous (briefly, $pfCts$ (resp. $pf\delta Cts$, $pf\delta\mathcal{P}Cts$, $pf\delta\mathcal{S}Cts$, $pfeCts$, $pf\theta Cts$, $pf\theta\mathcal{S}Cts$ and $pfMCts$)) if the inverse image of every $pfos$ in (X_2, Ψ_P) is a $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pfeos$, $pf\theta os$, $pf\theta\mathcal{S}os$ and $pfMos$) in (X_1, Γ_P).

Proposition 3.1 Let (X_1, Γ_P) & (X_2, Ψ_P) be a $pfts$'s. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a mapping. Then the following statements are hold for $pfts$, but not conversely.

1. Every $pf\theta Cts$ is a $pfCts$.
2. Every $pf\theta Cts$ is a $pf\theta\mathcal{S}Cts$.
3. Every $pf\theta\mathcal{S}Cts$ is a $pfMCts$.
4. Every $pf\delta Cts$ is a $pf\delta\mathcal{S}Cts$.
5. Every $pf\delta Cts$ is a $pf\delta\mathcal{P}Cts$.
6. Every $pf\delta\mathcal{S}Cts$ is a $pfeCts$.
7. Every $pf\delta\mathcal{P}Cts$ is a $pfMCts$.
8. Every $pfMCts$ is a $pfeCts$.
9. Every $pf\delta Cts$ is a $pfCts$.

Proof. [(i)]

1. Let B be a $pfos$ in (X_2, Ψ_P). Since h_P is $pf\theta Cts$, $h_P^{-1}(B)$ is $pf\theta os$ in (X_1, Γ_P). Since every $pf\theta os$ is a $pfos$, $h_P^{-1}(B)$ is a $pfos$ in (X_1, Γ_P). Hence, h_P is a $pfCts$.

2. Let B be a $pfos$ in (X_2, Ψ_P). Since h_P is $pf\theta Cts$, $h_P^{-1}(B)$ is $pf\theta os$ in (X_1, Γ_P). Since every $pf\theta os$ is a $pf\theta\mathcal{S}os$, $h_P^{-1}(B)$ is a $pfos$ in (X_1, Γ_P). Hence, h_P is a $pf\theta\mathcal{S}Cts$.

3. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\theta\mathcal{S}Cts$, $h_P^{-1}(B)$ is $pf\theta\mathcal{S}os$ in (X_1,Γ_P). Since every $pf\theta\mathcal{S}os$ is a $pfMos$, $h_P^{-1}(B)$ is a $pfMos$ in (X_1,Γ_P). Hence, h_P is a $pfMCts$.

4. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\delta Cts$, $h_P^{-1}(B)$ is $pf\delta os$ in (X_1,Γ_P). Since every $pf\delta os$ is a $pf\delta\mathcal{S}os$, $h_P^{-1}(B)$ is a $pf\delta\mathcal{S}os$ in (X_1,Γ_P). Hence, h_P is a $pf\delta\mathcal{S}Cts$.

5. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\delta Cts$, $h_P^{-1}(B)$ is $pf\delta os$ in (X_1,Γ_P). Since every $pf\delta os$ is a $pf\delta\mathcal{P}os$, $h_P^{-1}(B)$ is a $pf\delta\mathcal{P}os$ in (X_1,Γ_P). Hence, h_P is a $pf\delta\mathcal{P}Cts$.

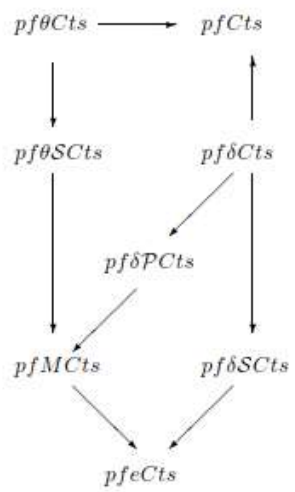
6. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\delta\mathcal{S}Cts$, $h_P^{-1}(B)$ is $pf\delta\mathcal{S}os$ in (X_1,Γ_P). Since every $pf\delta\mathcal{S}os$ is a $pfeos$, $h_P^{-1}(B)$ is a $pfeos$ in (X_1,Γ_P). Hence, h_P is a $pfeCts$.

7. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\delta\mathcal{P}Cts$, $h_P^{-1}(B)$ is $pf\delta\mathcal{P}os$ in (X_1,Γ_P). Since every $pf\delta\mathcal{P}os$ is a $pfMos$, $h_P^{-1}(B)$ is a $pfMos$ in (X_1,Γ_P). Hence, h_P is a $pfMCts$.

8. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pfMCts$, $h_P^{-1}(B)$ is $pfMos$ in (X_1,Γ_P). Since every $pfMos$ is a $pfeos$, $h_P^{-1}(B)$ is a $pfeos$ in (X_1,Γ_P). Hence, h_P is a $pfeCts$.

9. Let B be a $pfos$ in (X_2,Ψ_P). Since h_P is $pf\delta Cts$, $h_P^{-1}(B)$ is $pf\delta os$ in (X_1,Γ_P). Since every $pf\delta os$ is a $pfos$, $h_P^{-1}(B)$ is a $pfos$ in (X_1,Γ_P). Hence, h_P is a $pfCts$.

Remark 3.1 We obtain the following diagram from the results are discussed above.



Note: $A \rightarrow B$ denotes A implies B, but not conversely.

Example 3.1 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3 & A_4 in X_1 are defined as,

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Then, we have $\Gamma_P = \Psi_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfCts$ but not $pf\theta Cts$, because the set A_1 is $pfos$ in X_2 but $h_P^{-1}(A_1) = A_1$ is not $pf\theta os$ in X_1.

Example 3.2 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3, A_4 in X_1 & B_1 in X_2 are defined as,

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.40 >\}$$

Then, we have $\Gamma_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\Psi_P = \{0_X, 1_X, B_1\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pf\theta\mathcal{S}Cts$ (resp. $pf\delta\mathcal{S}Cts$) but not $pf\theta Cts$ (resp. $pf\delta Cts$), because the set B_1 is $pfos$ in X_2 but $h_P^{-1}(B_1) = B_1$ is not $pf\theta os$ (resp. $pf\delta os$) in X_1.

Example 3.3 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3, A_4 in X_1 & B_1 in X_2 are defined as,

$$A_1 = B_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Then, we have $\Gamma_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\Psi_P = \{0_X, 1_X, B_1\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMCts$ but not $pf\theta\mathcal{S}Cts$, because the set B_1 is $pfos$ in X_2 but $h_P^{-1}(B_1) = B_1$ is not $pf\theta\mathcal{S}os$ in X_1.

Example 3.4 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3, A_4 in X_1 & B_1 in X_2 are defined as,

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.40, 0.20 >, < x_2, 0.40, 0.40 >\}$$

Then, we have $\Gamma_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\Psi_P = \{0_X, 1_X, B_1\}$. Let $h_P: (X_1, \Gamma_P) \rightarrow (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfeCts$ but not $pfMCts$, because the set B_1 is $pfos$ in X_2 but $h_P^{-1}(B_1) = B_1$ is not $pfMos$ in X_1.

Example 3.5 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3, A_4 in X_1 & B_1 in X_2 are defined as,

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$B_1 = A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

Then, we have $\Gamma_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\Psi_P = \{0_X, 1_X, B_1\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfCts$ (resp. $pfeCts$ and $pf\delta\mathcal{P}Cts$) but not $pf\delta Cts$ (resp. $pf\delta\mathcal{S}Cts$ and $pf\delta Cts$), because the set B_1 is $pfos$ in X_2 but $h_P^{-1}(B_1) = B_1$ is not $pf\delta os$ (resp. $pf\delta\mathcal{S}os$ and $pf\delta os$) in X_1.

Example 3.6 Let $X_1 = X_2 = \{x_1, x_2\}$ and pfs's A_1, A_2, A_3, A_4 in X_1 & B_1 in X_2 are defined as,

$$A_1 = \{< x_1, 0.20, 0.80 >, < x_2, 0.40, 0.60 >\}$$

$$A_2 = \{< x_1, 0.10, 0.90 >, < x_2, 0.30, 0.70 >\}$$

$$A_3 = \{< x_1, 0.90, 0.10 >, < x_2, 0.70, 0.30 >\}$$

$$A_4 = \{< x_1, 0.20, 0.80 >, < x_2, 0.30, 0.70 >\}$$

$$B_1 = \{< x_1, 0.80, 0.20 >, < x_2, 0.60, 0.30 >\}$$

Then, we have $\Gamma_P = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$ and $\Psi_P = \{0_X, 1_X, B_1\}$. Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be an identity mapping. Then, h_P is $pfMCts$ but not $pf\delta\mathcal{P}Cts$, because the set B_1 is $pfos$ in X_2 but $h_P^{-1}(B_1) = B_1$ is not $pf\delta\mathcal{P}os$ in X_1.

Theorem 3.1 A map $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ is $pfMCts$ (resp. $pfCts$, $pf\delta Cts$, $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pf\theta Cts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$) iff the inverse image of each $pfcs$ in (X_2, Ψ_P) is $pfMcs$ (resp. $pfcs$, $pf\delta cs$, $pf\delta\mathcal{S}cs$, $pf\delta\mathcal{P}cs$, $pf\theta cs$, $pfecs$ and $pf\theta\mathcal{S}cs$) in (X_1, Γ_P).

Proof. Let B be a $pfcs$ in (X_2, Ψ_P). This implies B^c is $pfos$ in (X_2, Ψ_P). Since h_P is $pfMCts$, $h_P^{-1}(B^c)$ is $pfMos$ in (X_1, Γ_P). Since, $h_P^{-1}(B^c) = h_P^{-1}(B)^c$, $h_P^{-1}(B)$ is a $pfMcs$ in (X_1, Γ_P).

Conversely, let B be a $pfcs$ in (X_2, Ψ_P). Then, B^c is a $pfos$ in (X_2, Ψ_P). By hypothesis $h_P^{-1}(B^c)$ is $pfMos$ in (X_1, Γ_P). Since, $h_P^{-1}(B^c) = (h_P^{-1}(B))^c, (h_P^{-1}(B))^c$ is a $pfMos$ in (X_1, Γ_P). Therefore, $h_P^{-1}(B)$ is a $pfMos$ in (X_1, Γ_P). Hence, h_P is $pfMCts$. The proof of other cases are similar.

Definition 3.7 A $pfts$ (X_1, Γ_P) is said to be a Pythagorean fuzzy $MU_{1/2}$ (resp. $pf\delta\mathcal{S}U_{1/2}$, $pf\delta\mathcal{P}U_{1/2}$, $pf\theta U_{1/2}$, $pfeU_{1/2}$ and $pf\theta\mathcal{S}U_{1/2}$)-space, if every $pfMos$ (resp. $pf\delta\mathcal{S}os$, $pf\delta\mathcal{P}os$, $pf\theta os$, $pfeos$ and $pf\theta\mathcal{S}os$) in X_1 is a $pfos$ in X_1.

Theorem 3.2 Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a $pfMCts$ (resp. $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pf\theta Cts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$), then h_P is a $pfCts$ if X_1 is a $pfMU_{1/2}$) (resp. $pf\delta\mathcal{S}U_{1/2}$, $pf\delta\mathcal{P}U_{1/2}$, $pf\theta U_{1/2}$, $pfeU_{1/2}$ and $pf\theta\mathcal{S}U_{1/2}$)-space.

Proof. Let B be a $Pfos$ in X_2. Then, $h_P^{-1}(B)$ is a $pfMos$ in X_1, by hypothesis. Since X_1 is a $pfMU_{1/2}$)-space, $h_P^{-1}(B)$ is a $pfos$ in X_1. Hence, h_P is a $pfMCts$. The proof of other cases are similar.

Theorem 3.3 Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a $pfMCts$ map and $g_P: (X_2, \Psi_P) \to (X_3, \Phi_P)$ be a $pfCts$, then $g_P \circ h_P: (X_1, \Gamma_P) \to (X_3, \Phi_P)$ is a $pfMCts$.

Proof. Let K be a $pfos$ in X_3. Then, $g_P^{-1}(K)$ is a $pfos$ in X_2, by hypothesis. Since h_P is a $pfMCts$ map, $h_P^{-1}(g_P^{-1}(K))$ is a $pfMos$ in X_1. Hence $g_P \circ f_P$ is a $pfMCts$ map. The proof of other cases are similar.

Theorem 3.4 Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a $pfMCts$ map. Then, the following conditions are hold.

1. $h_P(pfMcl(A)) \leq pfcl(h_P(A))$, for all $pfcs(A)$ in X_1.
2. $pfMcl(h_P^{-1}(B)) \leq h_P^{-1}(pfcl(B))$, for all $pfcs(B)$ in X_2.

Proof. [(i)]

1. Since $pfMcl(f(A))$ is a $pfMcs$ in X_2 and h_P is $pfMCts$, then $h_P^{-1}(pfMcl(h_P(A)))$ is $pfMcs$ in X. Now, since $A \leq h_P^{-1}(pfcl(h_P(A))), pfMcl(A) \leq h_P^{-1}(pfMcl(h_P(A)))$. Therefore, $h_P(pfMcl(A)) \leq pfcl(h_P(A)) \leq pfcl(h_P(A))$.
2. By replacing A with B in (i), we obtain $h_P(pfMcl(h_P^{-1}(B))) \leq pfcl(h_P(h_P^{-1}(B))) \leq pfcl(B)$. Hence, $pfMcl(h_P^{-1}(B)) \leq h_P^{-1}(pfcl(B))$.

Theorem 3.5 For h_P is $pfMCts$ iff $h_P(pfint(B)) \leq pfMint(h_P^{-1}(B))$, for all $pfcs$ B in X_2.

Proof. Let h_P be $pfMCts$ and $B \in X_2$. $pfint(B)$ is $pfos$ in X_2 and hence, $h_P^{-1}(pfint(B))$ is $pfMos$ in X_1. Therefore, $pfMint(h_P^{-1}(pfMint(B))) = h_P^{-1}(pfint(B))$. Also, $pfint(B) \leq B$ implies that $h_P^{-1}(pfint(B)) \leq h_P^{-1}(B)$. Therefore, $pfMint(h_P^{-1}(pfint(B))) \leq pfMint(h_P^{-1}(B))$. That is, $h_P^{-1}(pfint(B)) \leq pfMint(h_P^{-1}(B))$.

Conversely, let $h_P^{-1}(pfint(B)) \leq pfMint(h_P^{-1}(B))$, for all subset B of X_2. If B is $pfos$ in X_2, then $pfint(B) = B$. By assumption, $h_P^{-1}(pfint(B)) \leq pfMint(h_P^{-1}(B))$. Thus, $h_P^{-1}(B) \leq pfMint(h_P^{-1}(B))$. But $pfMint(h_P^{-1}(B)) \leq h_P^{-1}(B)$. Therefore, $pfMint(h_P^{-1}(B)) = h_P^{-1}(B)$. That is, $h_P^{-1}(B)$ is $pfMos$ in X_1, for all $pfos(B)$ in X_2. Therefore, h_P is $pfMCts$ on X_1.

Remark 3.2 Theorems 3.3, 3.4 and 3.5 are true for $pf\delta Cts$, $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pf\theta Cts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$.

4 Pythagorean fuzzy M -irresolute maps in $pfts$

In this section, we introduce $pfMIrr$ (resp. $pfIrr$, $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$) maps and study some of its characterizations.

Definition 4.1 A map $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ is called a Pythagorean fuzzy M (resp. pf, $pf\delta$, $pf\delta\mathcal{S}$, $pf\delta\mathcal{P}$, $pf\theta$, pfe and $pf\theta\mathcal{S}$) -irresolute (briefly, $pfIrr$, $pfMIrr$ (resp. $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$)) map if $h_P^{-1}(B)$ is a $pfMos$ (resp. $pf\mathcal{S}os$, $pf\delta os$, $pf\delta\mathcal{S}os$, $pf\delta\mathcal{P}os$, $pf\theta os$, $pfeos$ and $pf\theta\mathcal{S}os$) in (X_1, Γ_P) for every $pfMos$ (resp. $pf\mathcal{S}os$, $pf\delta os$, $pf\delta\mathcal{S}os$, $pf\delta\mathcal{P}os$, $pf\theta os$, $pfeos$ and $pf\theta\mathcal{S}os$) B of (X_2, Ψ_P).

Theorem 4.1 Let $h_P: (X_1, \Gamma_P) \to (X_2, \Psi_P)$ be a $pfIrr$ (resp. $pfMIrr$, $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$), then h_P is a $pf\mathcal{S}Cts$ (resp. $pfMCts$, $pfCts$, $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pfCts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$) map. But not conversely.

Proof. (i) Let h_P be a $pfIrr$ map. Let B be any $pfos$ in X_2. Since every $pfos$ is a $pf\mathcal{S}os$, B is a $pf\mathcal{S}os$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pf\mathcal{S}os$ in X_1. Hence, h_P is a $pf\mathcal{S}Cts$ map.

(ii) Let h_P be a $pfMIrr$ map. Let B be any $pfos$ in X_2. Since every $pfos$ is a $pfMos$, B is a $pfMos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pfMos$ in X_1. Hence, h_P is a $pfMCts$ map.

(iii) Let h_P be a $pf\delta Irr$ map. Let B be any $pf\delta os$ in X_2. Since every $pf\delta os$ is a $pfos$, B is a $pfos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pfos$ in X_1. Hence, h_P is a $pfCts$ map.

(iv) Let h_P be a $pf\delta\mathcal{S}Irr$ map. Let B be any $pf\delta os$ in X_2. Since every $pf\delta os$ is a $pfos$, B is a $pfos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pf\delta\mathcal{S}os$ in X_1. Hence, h_P is a $pf\delta\mathcal{S}Cts$ map.

(v) Let h_P be a $pf\delta\mathcal{P}Irr$ map. Let B be any $pf\delta os$ in X_2. Since every $pf\delta os$ is a $pfos$, B is a $pfos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pf\delta\mathcal{P}os$ in X_1. Hence, h_P is a $pf\delta\mathcal{P}Cts$ map.

(vi) Let h_P be a $pf\theta Irr$ map. Let B be any $pf\theta os$ in X_2. Since every $pf\theta os$ is a $pfos$, B is a $pfos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pf\theta os$ in X_1. Hence, h_P is a $pf\theta Cts$ map.

(vii) Let h_P be a $pfeIrr$ map. Let B be any $pfos$ in X_2. Since every $pfos$ is a $pfeos$, B is a $pfeos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pfeos$ in X_1. Hence, h_P is a $pfeCts$ map.

(viii) Let h_P be a $pf\theta\mathcal{S}Irr$ map. Let B be any $pf\theta os$ in X_2. Since every $pf\theta os$ is a $pfos$, B is a $pfos$ in X_2. By hypothesis $h_P^{-1}(B)$ is a $pf\theta\mathcal{S}os$ in X_1. Hence, h_P is a $pf\theta\mathcal{S}Cts$ map.

Example 4.1 Let $X = \{x_1, x_2\} = Y = \{y_1, y_2\}$ and pfs's A_1, A_2, A_3, A_4, A_5 & A_6 in X and B_1 &B_2 in Y are defined as,

$$A_1 = \{< x_1, 0.00020, 0.00080 >, < x_2, 0.00040, 0.00060 >\}$$

$$A_2 = \{< x_1, 0.00010, 0.00090 >, < x_2, 0.00030, 0.00070 >\}$$

$$A_3 = \{< x_1, 0.00090, 0.00010 >, < x_2, 0.00070, 0.00030 >\}$$

$$A_4 = \{< x_1, 0.00020, 0.00080 >, < x_2, 0.00030, 0.00070 >\}$$

$$B_1 = \{< y_1, 0.00090, 0.00010 >, < y_2, 0.00070, 0.00030 >\}$$

$$B_2 = \{< y_1, 0.00040, 0.00020 >, < y_2, 0.00040, 0.00040 >\}.$$

Here, $\tau_1 = \{0_X, 1_X, A_1, A_2, A_3, A_4\}$, $\tau_2 = \{0_Y, 1_Y, B_1\}$, $\tau_3 = \{0_Y, 1_Y, A_2\}$, $\tau_4 = \{0_Y, 1_Y, A_4\}$ and $\tau_5 = \{0_Y, 1_Y, A_2, A_3\}$.

(i) Let $f_1: (X, \tau_1) \to (Y, \tau_2)$ be an identity mapping. Then, f_1 is $pfMCts$ but not $pfMIrr$, because the set B_2 is a $pfMos$ in Y but $f^{-1}(B_2)$ is not $pfMos$ in X. (ii) Let $f_2: (X, \tau_1) \to$

(Y,τ_3) be an identity mapping. Then, f_2 is $pf\mathcal{S}Cts$ but not $pfIrr$, because the set A_2 is a $pf\mathcal{S}os$ in Y but $f_2^{-1}(A_2)$ is not $pf\mathcal{S}os$ in X. (iii) Let $f_3:(X,\tau_1)\to(Y,\tau_4)$ be an identity mapping. Then, f_3 is $pfCts$ (resp. $pf\delta\mathcal{S}Cts$) but not $pf\delta Irr$ (resp. $pf\delta\mathcal{S}Irr$), because the set A_4 is a $pf\delta os$ (resp. $pf\delta\mathcal{S}os$) in Y but $f_3^{-1}(A_4)$ is not $pf\delta os$ (resp. $pf\delta\mathcal{S}os$) in X.

Theorem 4.2 Let $h_P:(X_1,\Gamma_P)\to(X_2,\Psi_P)$ be a $pfMIrr$ (resp. $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$), then h_P is a $pfCts$ map if X_1 is a $pfMU_{1/2}$ (resp. $pf\delta U_{1/2}$, $pf\delta\mathcal{S}U_{1/2}$, $pf\delta\mathcal{P}U_{1/2}$, $pf\theta U_{1/2}$, $pfeU_{1/2}$ and $pf\theta\mathcal{S}U_{1/2}$)-space.

Proof. Let B be a $pfos$ in X_2. Then, B is a $pfMos$ in X_2. Therefore $h_P^{-1}(B)$ is a $pfMos$ in X_1, by hypothesis. Since X_1 is a $pfMU_{1/2}$ -space, $h_P^{-1}(B)$ is a $pfos$ in X_1. Hence, h_P is a $pfCts$ map. The proof of other cases is similar.

Theorem 4.3 Let $h_P:(X_1,\Gamma_P)\to(X_2,\Psi_P)$ and $g_P:(X_2,\Psi_P)\to(X_3,\Phi_P)$ be a $pfMIrr$ (resp. $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$) maps, then $g_P\circ h_P:(X_1,\Gamma_P)\to(X_3,\Phi_P)$ is a $pfMIrr$ (resp. $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$) map.

Proof. Let K be a $pfMos$ in X_3. Then, $g_P^{-1}(K)$ is a $pfMos$ in X_2. Since h_P is a $pfMIrr$ map, $h_P^{-1}(g_P^{-1}(K))$ is a $pfMos$ in X_1. Hence $g_P\circ f_P$ is a $pfMIrr$ map. The proof of other cases is similar.

Theorem 4.4 Let $h_P:(X_1,\Gamma_P)\to(X_2,\Psi_P)$ be a $pfMIrr$ (resp. $pf\delta Irr$, $pf\delta\mathcal{S}Irr$, $pf\delta\mathcal{P}Irr$, $pf\theta Irr$, $pfeIrr$ and $pf\theta\mathcal{S}Irr$) map and $g_P:(X_2,\Psi_P)\to(X_3,\Phi_P)$ be a $pfMCts$ (resp. $pf\delta Cts$, $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pf\theta Cts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$) map, then $g\circ f:(X,\tau_1)\to(Z,\tau_3)$ is a $pfMCts$ (resp. $pf\delta Cts$, $pf\delta\mathcal{S}Cts$, $pf\delta\mathcal{P}Cts$, $pf\theta Cts$, $pfeCts$ and $pf\theta\mathcal{S}Cts$) map.

Proof. Let K be a $pfos$ in X_3. Then, $g_P^{-1}(K)$ is a $pfMos$ in X_2. Since, h_P is a $pfMIrr, h_P^{-1}(g_P^{-1}(K))$ is a $pfMos$ in X_1. Hence, $g_P\circ h_P$ is a $pfMCts$ map. The proof of other cases is similar.

Theorem 4.5 Let a map $h_P:(X_1,\Gamma_P)\to(X_2,\Psi_P)$. Then the following conditions are equivalent if X_1 and X_2 are $pfMU_{1/2}$ -spaces.

1. h_P is a $pfMIrr$ map.
2. $h_P^{-1}(B)$ is a $pfMos$ in X_1, for each $pfMos(B)$ in X_2.
3. $pf(h_P^{-1}(B))\subseteq h_P^{-1}(pfcl(B))$, for each pfs B of X_2.

Proof. (i) → (ii): Let B be any $pfMos$ in X_2. Then, B^c is a $pfMcs$ in X_2. Since h_P is $pfMIrr$, $h_P^{-1}(B^c)$ is a $pfMcs$ in X_1. But $h_P^{-1}(B^c)=(h_P^{-1}(B))^c$. Therefore, $h_P^{-1}(B)$ is a $pfMos$ in X_1.

(ii) → (iii) : Let B be any pfs in X_2 and $B\le pfcl(B)$. Then, $h_P^{-1}(B)\le h_P^{-1}(pfcl(B))$. Since $pfcl(B)$ is a $pfcs$ in X_2, $pfcl(B)$ is a $pfMcs$ in X_2. Therefore, $(pfcl(B))^c$ is a $pfMos$ in X_2. By hypothesis, $h_P^{-1}((pfcl(B))^c)$ is a $pfMos$ in X_1. Since, $h_P^{-1}((pfcl(B))^c)=(h_P^{-1}(pfcl(B)))^c, h_P^{-1}(pfcl(B))$ is a $pfMcs$ in X_1. Since, X_1 is a $pfMU_{1/2}$ -space, $h_P^{-1}(pfcl(B))$ is a $pfcs$ in X_1. Hence, $pfcl(h_P^{-1}(B))\subseteq$

$pfcl(h_P^{-1}(pfcl(B))) = h_P^{-1}(pfcl(B))$. That is, $pfcl(h_P^{-1}(B)) \subseteq h_P^{-1}(pfcl(B))$.

(iii) → (i) : Let B be any pfMcs in X_2. Since X_2 is a $pfMU_{1/2}$-space, B is a pfcs in X_2 and $pfcl(B) = B$. Hence, $h_P^{-1}(B) = h_P^{-1}(pfMcl(B)) \subseteq pfMcl(h_P^{-1}(B))$. But clearly, $h_P^{-1}(B) \subseteq pfcl(h_P^{-1}(B))$. Therefore, $pfcl(h_P^{-1}(B)) = h_P^{-1}(B)$. This implies, $h_P^{-1}(B)$ is a pfcs and hence, it is a pfMcs in X_1. Thus, h_P is a pfMIrr map.

Remark 4.1 Theorem 4.5 is true for pfδIrr, pfδ$\mathcal{S}$Irr, pfδ$\mathcal{P}$Irr, pfθIrr, pfeIrr and pfθ$\mathcal{S}$Irr.

5 Application

Entropy as a measure of fuzziness was first proposed by Zadeh [29]. Later many mathematicians defined several entropy measures. In this section, we focus on defining an entropy measure for pfs that connects the degree of membership and non-membership. As an example, we have applied the proposed entropy measure in the field of seasons.

Definition 5.1 Let $A = \{< x, \mu_A(x), \nu_A(x) | x \in X\}$ be a pfs in X. The new entropy measure for A denoted by $\varepsilon_{pfs}(A)$, is a function, $\varepsilon_{pfs}: \tau_{pfs}(X) \to [0,1]$ and is defined as $\varepsilon_{pfs}(A) = 1 - \frac{1}{n}\sum_{i=1}^{n} (\alpha_A - \gamma_A)^2;$ forevery`$x_i \in A$, where $\tau_{pfs}(X)$ denote the family of all pfs's on X.

Example 5.1 Consider an example of a incentive calculation based on performance of the salesman in different seasons. Now the company wants to identify the work efficiency on each salesman among each seasons. A company appoints 4 salesman S_1, S_2, S_3 and S_4 and observes their sales in three seasons: summer, winter and monsoon. The observed data's were represented at pfs as follows.

Table 1. Sales of each salesman in different seasons.

	Salesman 1 (S_1)	Salesman 2 (S_2)	Salesman 3 (S_3)	Salesman 4 (S_4)
Summer (S)	$< S, S_1; 0.45, 0.55 >$	$< S, S_2; 0.40, 0.68 >$	$< S, S_3; 0.38, 0.82 >$	$< S, S_4; 0.37, 0.55 >$
Winter (W)	$<, S_1; 0.40, 0.60 >$	$<, S_2; 0.41, 0.62 >$	$<, S_3; 0.45, 0.75 >$	$<, S_4; 0.38, 0.63 >$
Mansoon (M)	$<, S_1; 0.39, 0.70 >$	$<, S_2; 0.39, 0.67 >$	$<, S_3; 0.41, 0.55 >$	$<, S_4; 0.41, 0.65 >$

Clearly, all values in the Table 1 are pfs's. Now we calculate the ε_{pfs} of each value.

Table 2. Entropy measure of each item through each salesman.

	Salesman1 (S_1)	Salesman 2 (S_2)	Salesman 3 (S_3)	Salesman 4 (S_4)
Summer (S)	0.99	0.92	0.81	0.97
Winter (W)	0.96	0.96	0.91	0.94
Mansoon (M)	0.90	0.92	0.98	0.94

From Table 2, it is clear that $\varepsilon_{pfs}(M, S_1) < \varepsilon_{pfs}(W, S_1) < \varepsilon_{pfs}(S, S_1)$.

Similarly $\varepsilon_{pfs}(W, S_2) < \varepsilon_{pfs}(S, S_2) < \varepsilon_{pfs}(W, S_2)$;

$< \varepsilon_{pfs}(S, S_3)\ \varepsilon_{pfs}(W, S_3)\ \varepsilon_{pfs}(M, S_3)$

and $\varepsilon_{pfs}(M, S_4) < \varepsilon_{pfs}(W, S_4) < \varepsilon_{pfs}(S, S_4)$.

Hence Salesman 1 and Salesman 4 work efficiency during summer season compared to the other seasons. Hence their incentive can be calculated and provided at the end of summer season.

Salesman 2 works efficiency during winter season compared to the other seasons. Hence their incentive can be calculated and provided at the end of winter season.

Salesman 3 works efficiency during Mansoon season compared to the other seasons. Hence their incentive can be calculated and provided at the end of Mansoon season.

6 Conclusion

In this paper, using pfMos we have defined pfMCts map and analyzed its properties. After that we have compared Pythagorean fuzzy continuity maps to pfM -continuity maps. Furthermore, we have extended these maps to pfM -irresolute maps. Also we applied entropy measure for decision making problem of calculation of incentive based on the performance of salesman during the seasons. It makes an employer to felt satisfied for getting immediate appreciation and reward for their hardware. In future, we decide to apply entropy measure for decision making in various fields.

References

[1] K. T. Atanassov (1983), Intuitionistic fuzzy sets, VII ITKRâ€™s Session, Sofia.

[2] K. T. Atanassov (1986), Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20, 87-96.

[3] K. T. Atanassov (1989), Geometrical interpretation of the elements of the intuitionistic fuzzy objects, Preprint IM-MFAIS-1-89, Sofia.

[4] K. T. Atanassov (1999), Intuitionistic fuzzy sets: theory and applications, Physica, Heidelberg.

[5] K. T. Atanassov (2012), On intuitionistic fuzzy sets theory, Springer, Berlin.

[6] G. Beliakov and S. James (2014), Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs, In: Proceedings of the IEEE international conference on fuzzy systems (FUZZ-IEEE), 298-305.

[7] B. Davvaz and E. H. Sadrabadi (2016), An application of intuitionistic fuzzy sets in medicine, Int. J. Biomath 9, (3) 1650037.

[8] S. K. De, R. Biswas and A. R. Roy (2001), An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets Syst. 117 (2), 209-213.

[9] S. Dick, R. R. Yager and O. Yazdanbakhsh (2016), On Pythagorean and complex fuzzy set operations. IEEE Trans Fuzzy Syst. 24 (5), 1009-1021.

[10] P. A. Ejegwa, A. J. Akubo and O. M. Joshua (2014), Intuitionistic fuzzzy sets in career determination, J Info Comput. Sci. 9 (4), 285-288.

[11] P. A. Ejegwa (2015), Intuitionistic fuzzy sets approach in appointment of positions in an organization via maxâ€“minâ€“max rule, Glob. J Sci. Front Res F Math. Decis. Sci. 15 (6), 1-6.

[12] P. A. Ejegwa and E. S. Modom (2015), Diagnosis of viral hepatitis using new distance measure of intuitionistic fuzzy sets, Int. J. Fuzzy Math. Arch. 8 (1), 1-7.

[13] P. A. Ejegwa (2018), Distance and similarity measures of Pythagorean fuzzy sets, Granul Comput. https://doi.org/10.1007/s41066-018-00149-z.

[14] H. Garg (2017), A new improved score function of an interval-valued Pythagorean fuzzy set based TOPSIS method, Int. J Uncertain Quantif 7 (5), 463-474.

[15] H. Garg and S. Singh (2018), A novel triangular interval type-2 intuitionistic fuzzy set and their aggregation operators, Iran J Fuzzy Syst. 15 (5), 69-93.

[16] H. Garg and K. Kumar (2018), Distance measures for connection number sets based on set pair analysis and its applications to decision making process, Appl. Intell 48 (10), 3346-3359.

[17] H. Garg and K. Kumar (2018), An advance study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making, Soft Comput. 22 (15), 4959-4970.
[18] X. J. Gou, Z. S. Xu and P. J. Ren (2016), The properties of continuous Pyhagorean fuzzy information, Int. J Intell. Syst. 31 (5), 401-424.
[19] X. He, Y. Du and W. Liu (2016), Pythagorean fuzzy power average operators, Fuzzy Syst. Math. 30 (6), 116-124.
[20] Murat Olgun, Mehmet Unver and Seyhmus Yardimci (2019), Pythagorean fuzzy topological spaces, Complex & Intelligent Systems. https://doi.org/10.1007/s40747-019-0095-2.
[21] X. Peng and Y. Yang (2015), Some results for Pythagorean fuzzy sets, Int. J Intell Syst. 30, 1133-1160.
[22] X. Peng and G. Selvachandran (2017), Pythagorean fuzzy set state of the art and future directions, Artif Intell Rev. https://doi.org/10.1007/s10462-017-9596-9.
[23] E. Szmidt and J. Kacprzyk (2001), Intuitionistic fuzzy sets in some medical applications, Note IFS 7 (4), 58-64.
[24] E. Szmidt and J. Kacprzyk (2004), Medical diagnostic reasoning using a similarity measure for intuitionistic fuzzy sets, Note IFS 10 (4), 61-69.
[25] M. Udhaya Shalini and A. Stanis Arul Mary (2022), Generalized pre-closed sets in Pythagorean fuzzy topological spaces, International Journal of Creative Research Thoughts (IJCRT), 10 (30), e142-e147.
[26] R. R. Yager (2013), Pythagorean membership grades in multicriteria decision making, In: Technical report MII-3301. Machine Intelligence Institute, Iona College, New Rochelle.
[27] R. R. Yager (2013), Pythagorean fuzzy subsets, In: Proceedings of the joint IFSA world congress NAFIPS annual meeting, 57-61.
[28] R. R. Yager (2014), Pythagorean membership grades in multicriteria decision making, IEEE Trans Fuzzy Syst. 22 (4), 958-965.
[29] L. A. Zadeh (1965), Fuzzy sets, Inf. Control, 8, 338-353.