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New Operators Using Pythagorean Fuzzy Open Sets

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In this chapter, some new operators called pf (resp. $pf\delta$, $pf\delta\mathcal{P}, pf\delta\mathcal{S}, pf\delta\alpha$ and $pf\delta\beta$) frontier respective border and exterior with the help of $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os, pf\delta\mathcal{S}os, pf\delta\alpha os$ and $pf\delta\beta os$)'s in Pythagorean fuzzy topological spaces are introduced. Further, the important properties are discussed with examples. Also, Pythagorean fuzzy set is portrayed by membership and non-membership, more forceful to seize indeterminacy. Here we applied one similarity measures for decision making problem in Industry, and given solution for employee assignment for the project.
Keywords: $pf\delta os$, $pf\delta cs$, $pf\delta int(K)$ and $pf\delta cl(K)$.

1. Introduction

Considering the imprecision in decision-making, Zadeh [30] introduced the idea of fuzzy set which has a membership function, μ that assigns to each element of the universe of discourse, a number from the unit interval [0,1] to indicate the degree of belongingness to the set under consideration. The notion of fuzzy sets generalizes classical sets theory by allowing intermediate situations between the whole and nothing. In a fuzzy set, a membership function is defined to describe the degree of membership of an element to a class. The membership value ranges from 0 to 1, where 0 shows that the element does not belong to a class, 1 means belongs, and other values indicate the degree of membership to a class. For fuzzy sets, the membership function replaced the characteristic function in crisp sets. The concept of fuzzy set theory seems to be inconclusive because of the exclusion of nonmembership function and the disregard for the possibility of hesitation margin.

Atanassov critically studied these shortcomings and proposed a concept called intuitionistic fuzzy sets (IFSs) [1, 2, 4, 5]. The construct (that is, IFS's) incorporates both membership function, μ and nonmembership function, ν with hesitation margin, π (that is, either membership nor non-membership functions), such that $\mu + \nu \leq 1$ and $\mu + \nu + \pi = 1$. Atanassov [3] introduced intuitionistic fuzzy sets of second type (IFSST) with the property that the sum of the square of the membership and non-membership degrees is less than or equal to one. This concept generalizes IFS's in a way. The notion of IFS's provides a flexible framework to elaborate uncertainty and vagueness. The idea of IFS seems to be resourceful in modelling many real-life situations like medical diagnosis [7, 8, 12, 24, 25], career determination [10], selection process [11], and multi-criteria decision-making [15, 16, 17], among others.

There are situations where $\mu + \nu \geq 1$ unlike the cases capture in IFS's. This limitation in IFS naturally led to a construct, called Pythagorean fuzzy sets (pfs's). Pythagorean fuzzy set (pfs) proposed in [27, 28, 29] is a new tool to deal with vagueness considering the membership grade, μ and non-membership grade, ν satisfying the conditions $\mu + \nu \leq 1$ or $\mu + \nu \geq 1$, and also, it follows that $\mu^2 + \nu^2 + \pi^2 = 1$, where π is the Pythagorean fuzzy set index. In fact, the origin of Pythagorean fuzzy sets emanated from IFSST earlier studied in the literature. As a generalized set, PFS has close relationship with IFS. The construct of PFS's can be used to characterize uncertain information more sufficiently and accurately than IFS. Garg [14] presented an improved score function for the ranking order of interval-valued Pythagorean fuzzy sets (IVPFSs). Based on it, a Pythagorean fuzzy technique for order of preference by similarity to ideal solution (TOPSIS) method by taking the preferences of the experts in the form of interval-valued Pythagorean fuzzy decision matrices was discussed. Other explorations of the theory of PFS's can be found in [6, 9, 13, 18, 19, 22, 23].

In this paper, some new operators called neutrosophic δ frontier, neutrosophic δ border and neutrosophic δ exterior with the help of neutrosophic δ-open sets in neutrosophic topological spaces are introduced. Also, the important properties of them and discussed their relations with examples. Finally, we just applied only one similarity measure in the decision making of industry problem.

2 Preliminaries

We recall some basic notions of fuzzy sets, IFS's and pfs's.

Definition 2.1 [30] Let X be a nonempty set. A fuzzy set A in X is characterized by a membership function $\mu_A : X \to [0,1]$. That is:

$$\mu_A(x) = \begin{cases} 1, & \text{if} \quad x \in X \\ 0, & \text{if} \quad x \notin X \\ (0,1) & \text{if x ispartlyin X.} \end{cases}$$

Alternatively, a fuzzy set A in X is an object having the form $A = \{< x, \mu_A(x) > | x \in X\}$ or $A = \left\{\left\langle \frac{\mu_A(x)}{x} \right\rangle | x \in X\right\}$, where the function $\mu_A(x): X \to [0,1]$ defines the degree of membership of the element, $x \in X$.

The closer the membership value $\mu_A(x)$ to 1, the more x belongs to A, where the grades 1 and 0 represent full membership and full nonmembership. Fuzzy set is a collection of objects with graded membership, that is, having degree of membership. Fuzzy set is an extension of the classical notion of set. In classical set theory, the membership of elements in a set is assessed in a binary terms according to a bivalent condition; an element either belongs or does not belong to the set. Classical bivalent sets are in fuzzy set theory called crisp sets. Fuzzy sets are generalized classical sets, since the indicator function of classical sets is special cases of the membership functions of fuzzy sets, if the latter only take values 0 or 1. Fuzzy sets theory permits the gradual assessment of the membership of element in a set; this is described with the aid of a membership function valued in the real unit interval [0,1].

Let us consider two examples:

(i) all employees of XYZ who are over 1.8m in height; (ii) all employees of XYZ who are tall. The first example is a classical set with a universe (all XYZ employees) and a membership rule that divides the universe into members (those over 1.8m) and nonmembers. The second example is a fuzzy set, because some employees are definitely in the set and some are definitely not in the set, but some are borderline.

This distinction between the ins, the outs, and the borderline is made more exact by the membership function, μ. If we return to our second example and let A represent the fuzzy set of all tall employees and x represent a member of the universe X (i.e. all employees), then $\mu_A(x)$ would be $\mu_A(x) = 1$ if x is definitely tall or $\mu_A(x) = 0$ if x is definitely not tall or $0 < \mu_A(x) < 1$ for borderline cases.

Definition 2.2 [1, 2, 4, 5] Let a nonempty set X be fixed. An IFS A in X is an object having the form: $A = \{< x, \mu_A(x), \nu_A(x) > | x \in X\}$ or $A = \left\{\left\langle\frac{\mu_A(x),\nu_A(x)}{x}\right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\nu_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$: $0 \leq \mu_A(x) + \nu_A(x) \leq 1$. For each A in X: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is the intuitionistic fuzzy set index or hesitation margin of x in X. The hesitation margin $\pi_A(x)$ is the degree of nondeterminacy of $x \in X$ to the set A and $\pi_A(x) \in [0,1]$. The hesitation margin is the function that expresses lack of knowledge of whether $x \in X$ or $x \notin X$. Thus: $\mu_A(x) + \nu_A(x) + \pi_A(x) = 1$.

Example 2.1 Let $X = \{x, y, z\}$ be a fixed universe of discourse and $A = \left\{\left\langle\frac{0.6,0.1}{x}\right\rangle, \left\langle\frac{0.8,0.1}{y}\right\rangle, \left\langle\frac{0.5,0.3}{z}\right\rangle\right\}$, be the intuitionistic fuzzy set in X. The hesitation margins of the elements x, y, z to A are as follows: $\pi_A(x) = 0.3, \pi_A(y) = 0.1$ and $\pi_A(z) = 0.2$.

Definition 2.3 [27, 28, 29] Let X be a universal set. Then, a Pythagorean fuzzy set A, which is a set of ordered pairs over X, is defined by the following: $A = \{< x, \mu_A(x), \nu_A(x) | x \in X\}$ or $A = \left\{\left\langle\frac{\mu_A(x),\nu_A(x)}{x}\right\rangle | x \in X\right\}$, where the functions $\mu_A(x): X \to [0,1]$ and $\nu_A(x): X \to [0,1]$ define the degree of membership and the degree of nonmembership, respectively, of the element $x \in X$ to A, which is a subset of X, and for every $x \in X$, $0 \leq (\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$. Supposing $(\mu_A(x))^2 + (\nu_A(x))^2 \leq 1$, then there is a degree of indeterminacy of $x \in X$ to A defined by $\pi_A(x) = \sqrt{1 - [(\mu_A(x))^2 + (\nu_A(x))^2]}$ and $\pi_A(x) \in [0,1]$. In what follows,

$(\mu_A(x))^2 + (\nu_A(x))^2 + (\pi_A(x))^2 = 1$. Otherwise, $\pi_A(x) = 0$ whenever $(\mu_A(x))^2 + (\nu_A(x))^2 = 1$. We denote the set of all PFS's over X by $pfs(X)$.

Definition 2.4 [29] Let A and B be pfs's of the forms $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ and $B = \{< a, \lambda_B(a), \mu_B(a) > | a \in X\}$. Then

1. $A \subseteq B$ if and only if $\lambda_A(a) \leq \lambda_B(a)$ and $\mu_A(a) \geq \mu_B(a)$ for all $a \in X$.
2. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
3. $\overline{A} = \{< a, \mu_A(a), \lambda_A(a) > | a \in X\}$.
4. $A \cap B = \{< a, \lambda_A(a) \wedge \lambda_B(a), \mu_A(a) \vee \mu_B(a) > | a \in X\}$.
5. $A \cup B = \{< a, \lambda_A(a) \vee \lambda_B(a), \mu_A(a) \wedge \mu_B(a) > | a \in X\}$.
6. $\phi = \{< a, \phi, X > | a \in X\}$ and $X = \{< a, X, \phi > | a \in X\}$.
7. $\overline{X} = \phi$ and $\overline{\phi} = X$.

Definition 2.5 [21] An Pythagorean fuzzy topology by subsets of a non-empty set X is a family τ of pfs's satisfying the following axioms.

1. $\phi, X \in \tau$.
2. $G_1 \cap G_2 \in \tau$ for every $G_1, G_2 \in \tau$ and
3. $\bigcup G_i \in \tau$ for any arbitrary family $\{G_i | i \in j\} \subseteq \tau$. The pair (X, τ) is called an Pythagorean fuzzy topological space (pfts in short) and any pfs G in τ is called an Pythagorean fuzzy open set (pfos in short) in X. The complement $\overline{A}$ of an Pythagorean fuzzy open set A in an $pfts(X, \tau)$ is called an Pythagorean fuzzy closed set (pfcs in short).

Definition 2.6 [21] Let (X, τ) be an pfts and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ be an pfs in X. Then the interior and the closure of A are denoted by $pfint(A)$ and $pfcl(A)$ and are defined as follows: $pfcl(A) = \cap \{K | K \text{ isan pfcs and } A \subseteq K\}$ and $pfint(A) = \cup \{G | G \text{ isan pfos and } G \subseteq A\}$. Also, it can be established that $pfcl(A)$ is an pfcs and $pfint(A)$ is an pfos, A is an pfcs if and only if $pfcl(A) = A$ and A is an pfos if and only if $pfint(A) = A$. We say that A is pf-dense if $pfcl(A) = X$.

Lemma 2.1 [26] For any Pythagorean fuzzy set A in (X, τ), we have $X - pfint(A) = pfcl(X - A)$ and $X - pfcl(A) = pfint(X - A)$.

Definition 2.7 [26] Let (X, τ) be an pfts and A be an pfs. Then A is said to be an Pythagorean fuzzy (i) regular open set (pfros in short) if $A = pfint(pfcl(A))$. (ii) regular closed set (pfrcs in short) if $A = pfcl(pfint(A))$. By Lemma 2.1, it follows that A is an pfros iff $\overline{A}$ is an pfrcs.

3 Pythagorean fuzzy frontier

In this section, pf (resp. $pf\delta$, $pf\delta\mathcal{P}, pf\delta\mathcal{S}, pf\delta\alpha$ and $pf\delta\beta$) frontier respective border and exterior with the help of $pf\delta os$ (resp. $pf\delta\mathcal{P}os, pf\delta\mathcal{S}os, pf\delta\alpha os$ and $pf\delta\beta os$) are introduced and discussed their properties in pfts.

Definition 3.1 Let (X, τ) be an pfts and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ be an pfs in X. Then the δ-interior and the δ-closure of A are denoted by pfδint(A) and pfδcl(A) and are defined as follows. $pf\delta cl(A) = \cap \{K | K$ is an pfrcs and $A \subseteq K\}$, ($pf\delta int(A) = \cup \{G | G$ is an pfros and $G \subseteq A\}$.

Definition 3.2 Let (X, τ) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ be an pfs in X. A set A is said to be pf

1. δ-open set (briefly, $pf\delta os$) if $A = pf\delta int(A)$,
2. δ-pre open set (briefly, $pf\delta\mathcal{P}os$) if $A \subseteq pfint(pf\delta cl(A))$.
3. δ-semi open set (briefly, $pf\delta\mathcal{S}os$) if $A \subseteq pfcl(pf\delta int(A))$.
4. δ-α open set or a-open set (briefly, $pf\delta\alpha os$ or $pfaos$) if $A \subseteq pfint(pfcl(pf\delta int(A)))$.
5. δ-β open set or e^*-open set (briefly, $pf\delta\beta os$ or pfe^*os) if $A \subseteq pfcl(pfint(pf\delta cl(A)))$.
6. δ (resp. δ-pre, δ-semi, δ-α and δ-β) dense if $pf\delta cl(A)$ (resp. $pf\delta pcl(A), pf\delta\mathcal{S}cl(A), pf\delta\alpha cl(A)$ and $pf\delta\beta cl(A)) = X$.

The complement of an $pf\delta os$ (resp. $pf\delta\mathcal{P}os, pf\delta\mathcal{S}os, pf\delta\alpha os$ and $pf\delta\beta os$) is called an $pf\delta$ (resp. $pf\delta\mathcal{P}, pf\delta\mathcal{S}, pf\delta\alpha$ and $pf\delta\beta$) closed set (briefly, $pf\delta cs$ (resp. $pf\delta\mathcal{P}cs, pf\delta\mathcal{S}cs, pf\delta\alpha cs$ and $pf\delta\beta cs$ in X.

The family of all $pf\delta os$ (resp. $pf\delta cs, pf\delta\mathcal{P}os, pf\delta\mathcal{P}cs, pf\delta\mathcal{S}os, pf\delta\mathcal{S}cs, pf\delta\alpha os, pf\delta\alpha cs, pf\delta\beta os$ and $pf\delta\beta cs$) of X is denoted by $pf\delta OS(X)$, (resp. $pf\delta CS(X), pf\delta\mathcal{P}OS(X), pf\delta\mathcal{P}CS(X), pf\delta\mathcal{S}OS(X), pf\delta\mathcal{S}CS(X), pf\delta\alpha OS(X), pf\delta\alpha CS(X)$, $pf\delta\beta OS(X)$ and $pf\delta\beta CS(X)$).

Definition 3.3 Let (X, τ) be an $pfts$ and $A = \{< a, \lambda_A(a), \mu_A(a) > | a \in X\}$ be an pfs in X. Then the $pf\delta$-pre (resp. $pf\delta$-semi, $pf\delta\alpha$ and $pf\delta\beta$)-interior and the $pf\delta$-pre (resp. $pf\delta$-semi, $pf\delta\alpha$ and $pf\delta\beta$)-closure of A are denoted by $pf\delta\mathcal{P}int(A)$ (resp. $pf\delta\mathcal{S}int(A)$, $pf\delta\alpha int(A)$ and $pf\delta\beta int(A)$) and the $pf\delta\mathcal{P}cl(A)$ (resp. $pf\delta\mathcal{S}cl(A), pf\delta\alpha cl(A)$ and $pf\delta\beta cl(A)$ and are defined as follows:

$pf\delta\mathcal{P}int(A)$ (resp. $pf\delta\mathcal{S}int(A), pf\delta\alpha int(A)$ and $pf\delta\beta int(A) = \cup \{G | G$ in a $pf\delta\mathcal{P}os$ (resp. $pf\delta\mathcal{S}os, pf\delta\alpha os$ and $pf\delta\beta os$) and $G \subseteq A\}$

and $pf\delta\mathcal{P}cl(A)$ (resp. $pf\delta\mathcal{S}cl(A), pf\delta\alpha cl(A)$ and $pf\delta\beta cl(A) = \cap \{K | K$ is an $pf\delta\mathcal{P}cs$ (resp. $pf\delta\mathcal{S}cs, pf\delta\alpha cs,\ pf\delta\beta cs$) and $A \subseteq K\}$.

Definition 3.4 Let (X, Γ_P) be a $pfts$ and let A be a pfs in $pfts$. Then the pf (resp. $pf\delta$, $pf\delta\mathcal{P}, pf\delta\mathcal{S}, pf\delta\alpha$ and $pf\delta\beta$) frontier of A is denoted by $pfFr(A)$ (resp. $pf\delta Fr(A)$, $pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Fr(A)$ and $pf\delta\beta Fr(A)$) and is defined by $pfFr(A) = pfcl(A) \cap pfcl(A^c)$ (resp. $pf\delta Fr(A) = pf\delta cl(A) \cap pf\delta cl(A^c)$, $pf\delta\mathcal{P}Fr(A) = pf\delta\mathcal{P}cl(A) \cap pf\delta\mathcal{P}cl(A^c)$, $pf\delta\mathcal{S}Fr(A) = pf\delta\mathcal{S}cl(A) \cap pf\delta\mathcal{S}cl(A^c)$, $pf\delta\alpha Fr(A) = pf\delta\alpha cl(A) \cap pf\delta\alpha cl(A^c)$ and $pf\delta\beta Fr(A) = pf\delta\beta cl(A) \cap pf\delta\beta cl(A^c)$).

Example 3.1 Let $X = \{x_1, x_2\}$ and the pfs's A_1, A_2, A_3 and A are defined as

$$\mu_{A_1}(a) = 0.5, \gamma_{A_1}(a) = 0.6, \mu_{A_1}(b) = 0.3, \gamma_{A_1}(b) = 0.4; \mu_{A_2}(a) = 0.6, \gamma_{A_2}(a) = 0.5, \mu_{A_2}(b) = 0.4, \gamma_{A_2}(b) = 0.3; \mu_{A_3}(a) = 0.5, \gamma_{A_3}(a) = 0.7, \mu_{A_3}(b) = 0.2, \gamma_{A_3}(b) = 0.9; \mu_A(a) = 0.6, \gamma_A(a) = 0.5, \mu_A(b) = 0.4, \gamma_A(b) = 0.3.$$

We have $\tau = \{0_X, 1_X, A_1, A_2, A_3\}$ be a $pfts$ on X. Then

1. $pfFr(A) = pf\delta\mathcal{S}Fr(A) = pf\delta\mathcal{P}Fr(A) = pf\delta\alpha Fr(A) = pf\delta\beta Fr(A) = A^c$.
2. $pf\delta Fr(A) = A_2^c$.

Remark 3.1 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), $pfFr(A)$ (resp. $pf\delta Fr(A)$, $pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Fr(A)$ and $pf\delta\beta Fr(A)$) is $pfcs$ (resp. $pf\delta cs$, $pf\delta\mathcal{P}cs$, $pf\delta\mathcal{S}cs$, $pf\delta\alpha cs$ and $pf\delta\beta cs$).

Theorem 3.1 Let (X, Γ_P) be a $pfts$. Let A be a $pfss$ in (X, Γ_P), then the following conditions are true.

1. $pfFr(A) = pfFr(A^c)$.
2. $pf\delta Fr(A) = pf\delta Fr(A^c)$.
3. $pf\delta\mathcal{P}Fr(A) = pf\delta\mathcal{P}Fr(A^c)$.
4. $pf\delta\mathcal{S}Fr(A) = pf\delta\mathcal{S}Fr(A^c)$.
5. $pf\delta\alpha Fr(A) = pf\delta\alpha Fr(A^c)$.
6. $pf\delta\beta Fr(A) = pf\delta\beta Fr(A^c)$.

Proof. (i) Let A be a pfs in (X, Γ_P). Then by Definition 3.4, $pfFr(A) = pfcl(A) \cap pfcl(A^c) = pfcl(A^c) \cap pfcl(A) = pfcl(A^c) \cap (pfcl(A^c)^c)$. Again by Definition 3.4, this is equal to $pfFr(A^c)$. Hence $pfFr(A) = pfFr(A^c)$.

(ii) Let A be a pfs in (X, Γ_P). Then by Definition 3.4, $pf\delta Fr(A) = pf\delta cl(A) \cap pf\delta cl(A^c) = pf\delta cl(A^c) \cap pf\delta cl(A) = pf\delta cl(A^c) \cap (pf\delta cl(A^c)^c)$. Again by Definition 3.4, this is equal to $pf\delta Fr(A^c)$. Hence $pf\delta Fr(A) = pf\delta Fr(A^c)$. The proof of other cases are similar.

Theorem 3.2 Let (X, τ) be an $pfts$ and let A and B be Pythagorean fuzzy sets. Then the following hold. [(i)]

1. $pf\delta int(\phi) = \phi$ and $pf\delta int(X) = X$.
2. A is a $pf\delta os$ iff $A = pf\delta int(A)$.
3. $pf\delta int(A)$ is the greatest $pf\delta os$ containing A.
4. $pf\delta int(pf\delta int(A)) = pf\delta int(A)$.
5. $A \subseteq B$ implies that $pf\delta int(A) \subseteq pf\delta int(B)$.
6. $pf\delta int(A \cap B) = pf\delta int(A) \cap pf\delta int(B)$.

7. $pf\delta int(A \cup B) \supseteq pf\delta int(A) \cup pf\delta int(B)$.
8. $pf\delta int(A) \subseteq A$.
9. $(X - pf\delta int(A)) = pf\delta cl(X - A)$.
10. $pf\delta int(A)$ is the greatest $pf\delta os$ contained in A.

Theorem 3.3 Let (X, τ) be an $pfts$ and let A and B be Pythagorean fuzzy sets. Then the following hold. [(i)]

1. $pf\delta cl(\phi) = \phi$ and $pf\delta cl(X) = X$.
2. A is an $pf\delta cs$ iff $A = pf\delta cl(A)$.
3. $pf\delta cl(A)$ is the smallest $pf\delta cs$ contained in A.
4. $pf\delta cl(pf\delta cl(A)) = pf\delta cl(A)$.
5. $A \subseteq B$ implies that $pf\delta cl(A) \subseteq pf\delta cl(B)$.
6. $pf\delta cl(A \cap B) \subseteq pf\delta cl(A) \cap pf\delta cl(B)$.
7. $pf\delta cl(A \cup B) = pf\delta cl(A) \cup pf\delta cl(B)$.
8. $A \subseteq pf\delta cl(A)$.
9. $(X - pf\delta cl(A)) = pf\delta int(X - A)$.
10. $x \in pf\delta cl(A)$ iff $A \cap B \neq \phi$ for every $pf\delta os$ B containing x.
11. $pf\delta cl(A) = A$ iff A is a $pf\delta cs$.

Theorem 3.4 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P). Then the following statements are true.

1. $pfFr(A) = pfcl(A) - pfint(A)$.
2. $pf\delta Fr(A) = pf\delta cl(A) - pf\delta int(A)$.
3. $pf\delta\mathcal{P}Fr(A) = pf\delta\mathcal{P}cl(A) - pf\delta\mathcal{P}int(A)$.
4. $pf\delta\mathcal{S}Fr(A) = pf\delta\mathcal{S}cl(A) - pf\delta\mathcal{S}int(A)$.
5. $pf\delta\alpha Fr(A) = pf\delta\alpha cl(A) - pf\delta\alpha int(A)$.
6. $pf\delta\beta Fr(A) = pf\delta\beta cl(A) - pf\delta\beta int(A)$.

Proof. (i) Let A be a pfs in (X, Γ_P). By Theorem 3.3 (ix), $(pfcl(A^c))^c = pfint(A)$ and by Definition 3.4, $pfFr(A) = pfcl(A) \cap (pfcl(A^c)) = pfcl(A) \cap (pfint(A^c))^c$. By using $A - B = A \cap B^c$, $pfFr(A) = pfcl(A) - pfint(A)$. Hence $pfFr(A) = pfcl(A) - pfint(A)$.

(ii) Let A be a pfs in (X, Γ_P). By Theorem 3.3 (ix), $(pf\delta cl(A^c))^c = pf\delta int(A)$ and by Definition 3.4, $pf\delta Fr(A) = pf\delta cl(A) \cap (pf\delta cl(A^c)) = pf\delta cl(A) \cap (pf\delta\ int(A^c))^c$. By using $A - B = A \cap B^c$, $pf\delta Fr(A) = pf\delta cl(A) - pf\delta int(A)$. Hence $pf\delta Fr(A) = pf\delta cl(A) - pf\delta int(A)$. The proof of others are similar.

Theorem 3.5 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P) is $pfcs$ (resp. $pf\delta cs$, $pf\delta\mathcal{P}cs$, $pf\delta\mathcal{S}cs$, $pf\delta\alpha cs$ and $pf\delta\beta cs$) iff $pfFr(A)\subseteq A$ (resp. $pf\delta Fr(A)\subseteq A$, $pf\delta\mathcal{P}Fr(A)\subseteq A$, $pf\delta\mathcal{S}Fr(A)\subseteq A$, $pf\delta\alpha Fr(A)\subseteq A$ and $pf\delta\beta Fr(A)\subseteq A$).

Proof. Let A be a $pf\delta cs$ in (X,Γ_P). Then by Definition 3.4, $pf\delta Fr(A)=pf\delta cl(A)\cap pf\delta cl(A^c)\subseteq pf\delta cl(A)$. By using Theorem 3.3 (ix), $pf\delta cl(A)=A$. Hence $pf\delta Fr(A)\subseteq A$, if A is $pf\delta cs$ in X.

Conversely, Assume that, $pf\delta Fr(A)\subseteq A$. Then $pf\delta cl(A)-pf\delta int(A)\subseteq A$. Since $pf\delta int(A)\subseteq A$, we conclude that $pf\delta cl(A)=A$ and hence A is $pf\delta cs$.

The proof of the others are similar.

Theorem 3.6 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P). If A is a $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pf\delta\alpha os$ and $pf\delta\beta os$) in X, then $pfFr(A)\subseteq A^c$ (resp. $pf\delta Fr(A)\subseteq A^c$, $pf\delta\mathcal{P}Fr(A)\subseteq A^c$, $pf\delta\mathcal{S}Fr(A)\subseteq A^c$, $pf\delta\alpha Fr(A)\subseteq A^c$ and $pf\delta\beta Fr(A)\subseteq A^c$).

Proof. Let A be a $pf\delta os$ in (X,Γ_P). By Definition 3.2, A^c is $pf\delta cs$ in X. By Theorem 3.5, $pf\delta Fr(A^c)\subseteq A^c$ and by Theorem 3.5, we get $pf\delta Fr(A)\subseteq A^c$.

The proof of the others are similar.

Theorem 3.7 Let (X,Γ_P) be a $pfts$. Let A and B be a pfs's in (X,Γ_P). Let $A\subseteq B$ and B be a $pfcs$ (resp. $pf\delta cs$, $pf\delta\mathcal{P}cs$, $pf\delta\mathcal{S}cs$, $pf\delta\alpha cs$ and $pf\delta\beta cs$) in X. Then $pfFr(A)\subseteq B$ (resp. $pf\delta Fr(A)\subseteq B$, $pf\delta\mathcal{P}Fr(A)\subseteq B$, $pf\delta\mathcal{S}Fr(A)\subseteq B$, $pf\delta\alpha Fr(A)\subseteq B$ and $pf\delta\beta Fr(A)\subseteq B$).

Proof. By Theorem 3.3 (v), $A\subseteq B$, $pf\delta cl(A)\subseteq pf\delta cl(B)$. By Definition 3.4, $pf\delta Fr(A)=pf\delta cl(A)\cap pf\delta cl(A^c)\subseteq pf\delta cl(B)\cap pf\delta cl(A^c)\subseteq pf\delta cl(B)$. Then by Theorem 3.3 (ix), this is equal to B. Hence $pf\delta Fr(A)\subseteq B$.

The proof of the others are similar.

Theorem 3.8 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P). Then $(pfFr(\ A))^c=pfint(A)\cup pfint(A^c)$ (resp. $(pf\delta Fr(A))^c=pf\delta int(A)\cup pf\delta int(A^c)$, $(pf\delta\mathcal{P}Fr(A))^c=pf\delta\mathcal{P}int(A)\cup pf\delta\mathcal{P}int(A^c)$, $(pf\delta\mathcal{S}Fr(A))^c=pf\delta\mathcal{S}int(A)\cup pf\delta\mathcal{S}int(A^c)$, $(pf\delta\alpha Fr(A))^c=pf\delta\alpha int(A)\cup pf\delta\alpha int(A^c)$ and $(pf\delta\beta Fr(A))^c=pf\delta\beta int(A)\cup pf\delta\beta int(A^c)$).

Proof. Let A be a pfs in (X,Γ_P). Then by Definition 3.4, $(pf\delta Fr(A))^c=(pf\delta cl(A)\cap pf\delta cl(A^c))^c=((pf\delta cl(A))^c\cup(pf\delta cl(A^c))^c$. By Theorem 3.3 (ix), which is equal to $pf\delta int(A^c)\cup pf\delta int(A)$. Hence $(pf\delta Fr(A))^c=pf\delta int(A)\cup pf\delta int(A^c)$.

The proof of the others are similar.

Proposition 3.1 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P), then [(i)]

1. $pf\delta int(A)\subseteq pfint(A)$
2. $pf\delta int(A)\subseteq pf\delta\mathcal{S}int(A)\subseteq pf\delta\beta int(A)$
3. $pf\delta int(A)\subseteq pf\delta\mathcal{P}int(A)\subseteq pf\delta\beta int(A)$
4. $pf\delta\alpha int(A)\subseteq pf\delta\mathcal{S}int(A)\subseteq pf\delta\beta int(A)$

5. $pf\delta\alpha int(A) \subseteq pf\delta\mathcal{P}int(A) \subseteq pf\delta\beta int(A)$
6. $pf\delta cl(A) \supseteq pfcl(A)$
7. $pf\delta cl(A) \supseteq pf\delta\mathcal{S}cl(A) \supseteq pf\delta\beta cl(A)$
8. $pf\delta cl(A) \supseteq pf\delta\mathcal{P}cl(A) \supseteq pf\delta\beta cl(A)$
9. $pf\delta\alpha cl(A) \supseteq pf\delta\mathcal{S}cl(A) \supseteq pf\delta\beta cl(A)$
10. $pf\delta\alpha cl(A) \supseteq pf\delta\mathcal{P}cl(A) \supseteq pf\delta\beta cl(A)$.

Theorem 3.9 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfFr(A) \subseteq pf\delta Fr(A)$ (resp. $pf\delta\mathcal{P}Fr(A) \subseteq pf\delta Fr(A)$, $pf\delta\mathcal{S}Fr(A) \subseteq pf\delta Fr(A)$, $pf\delta\alpha Fr(A) \subseteq pf\delta Fr(A)$, $pf\delta\beta Fr(A) \subseteq pfFr(A)$, $pf\delta\mathcal{S}Fr(A) \subseteq pf\delta\beta Fr(A)$, $pf\delta\alpha Fr(A) \subseteq pf\delta\mathcal{S}Fr(A)$ and $pf\delta\alpha Fr(A) \subseteq pf\delta\mathcal{P}Fr(A)$).

Proof. Let A be a pfs in (X, Γ_P). Then by Proposition 3.1, $pfcl(A) \subseteq pf\delta cl(A)$ and $pfcl(A^c) \supseteq pf\delta cl(A^c)$. By Definition 3.4, $pf\delta Fr(A) = pf\delta cl(A) \cap pf\delta cl(A^c) \supseteq pfcl(A) \cap pfcl(A^c)$, this is equal to $pfFr(A)$. Hence $pfFr(A) \subseteq pf\delta Fr(A)$. The proof of other cases are similar.

Theorem 3.10 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfcl(pfFr(A)) \subseteq pfFr(A)$ (resp. $pf\delta cl(pf\delta Fr(A)) \subseteq pf\delta Fr(A)$, $pf\delta\mathcal{P}cl(pf\delta\mathcal{P}Fr(A)) \subseteq pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}cl(pf\delta\mathcal{S}Fr(A)) \subseteq pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha cl(pf\delta\alpha Fr(A)) \subseteq pf\delta\alpha Fr(A)$ and $pf\delta\beta cl(pf\delta\beta Fr(A)) \subseteq pf\delta\beta Fr(A)$).

Proof. Let A be a pfs in (X, Γ_P). Then by Definition 3.4, $pf\delta cl(pf\delta Fr(A)) = pf\delta cl(pf\delta cl(A) \cap (pf\delta cl(A^c))) \subseteq (pf\delta cl(pf\delta cl(A))) \cap (pf\delta cl(pf\delta cl(A^c)))$. By Theorem 3.3 (iv), $pf\delta cl(pf\delta Fr(A)) = pf\delta cl(A) \cap (pf\delta cl(A^c))$. By Definition 3.4, this is equal to $pf\delta Fr(A)$.

The proof of the others are similar.

Theorem 3.11 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfFr(pfint(A)) \subseteq pfFr(A)$ (resp. $pf\delta Fr(pf\delta int(A)) \subseteq pf\delta Fr(A)$, $pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}int(A)) \subseteq pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}int(A)) \subseteq pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Fr(pf\delta\alpha int(A)) \subseteq pf\delta\alpha Fr(A)$ and $pf\delta\beta Fr(pf\delta\beta int(A)) \subseteq pf\delta\beta Fr(A)$).

Proof. Let A be a pfs in (X, Γ_P). Then

$$pf\delta Fr(pf\delta int(A)) = pf\delta cl(pfint(A)) \cap (pf\delta cl(pf\delta int(A))^c)[byDefinition 3.4]$$

$$= pf\delta cl(pf\delta int(A)) \cap (pf\delta cl(pf\delta cl(A^c)))[byTheorem 3.43.2(ix)]$$

$$= pf\delta cl(pf\delta int(A)) \cap (pf\delta cl(A^c))[byTheorem 3.43.23.3(iv)]$$

$$\subseteq pf\delta cl(A) \cap pf\delta cl(A^c)[byTheorem 3.43.23.33.2(viii)]$$

$$= pf\delta Fr(A)[byDefinition 3.43.23.33.23.4].$$

Hence $pf\delta Fr(pf\delta int(A)) \subseteq (pf\delta Fr(A))$.

The proof of the others are similar.

Theorem 3.12 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfFr(pfcl(A)) \subseteq pfFr(A)$ (resp. $pf\delta Fr(pf\delta cl(A)) \subseteq pf\delta Fr(A)$, $pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}cl(A)) \subseteq pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}cl(A)) \subseteq pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Fr(pf\delta\alpha cl(A)) \subseteq pf\delta\alpha Fr(A)$ and $pf\delta\beta Fr(pf\delta\beta cl(A)) \subseteq pf\delta\beta Fr(A)$).

Proof. Let A be a pfs in (X, Γ_P). Then

$pf\delta Fr(pf\delta cl(A)) = pf\delta cl(pf\delta cl(A)) \cap (pf\delta cl(pf\delta cl(A))^c)[by Definition 3.4]$

$= pf\delta cl(A) \cap (pf\delta cl(pf\delta int(A^c)))[by Theorems 3.43.3(iv)]$

$[and 3.43.33.2(v) \& (ix)]$

$\subseteq pf\delta cl(A) \cap pf\delta cl(A^c)[by Theorem 3.43.33.23.3(viii)]$

$= pf\delta Fr(A)[by Definition 3.43.33.23.33.4]$

Hence $pf\delta Fr(pf\delta cl(A)) \subseteq pf\delta Fr(A)$.

The proof of the others are similar.

Theorem 3.13 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P). Then $pfint(A) \subseteq A - pfFr(A)$ (resp. $pf\delta int(A) \subseteq A - pf\delta Fr(A)$, $pf\delta\mathcal{P}int(A) \subseteq A - pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}int(A) \subseteq A - pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha int(A) \subseteq A - pf\delta\alpha Fr(A)$ and $pf\delta\beta int(A) \subseteq A - pf\delta\beta Fr(A)$).

Proof. Let A be a pfs in (X, Γ_P). Now by Definition 3.4,

$A - pf\delta Fr(A) = A \cap (pf\delta Fr(A))^c$

$= A \cap [pf\delta cl(A) \cap pf\delta cl(A^c)]^c$

$= A \cap [pf\delta int(A^c) \cup pf\delta int(A)]$

$= [A \cap pf\delta int(A^c)] \cup [A \cap pf\delta int(A)]$

$= [A \cap pf\delta int(A^c)] \cup pf\delta int(A) \supseteq pf\delta int(A)$

Hence $pf\delta int(A) \subseteq A - pf\delta Fr(A)$.

The proof of the others are similar.

Remark 3.2 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P). Then the following conditions are hold:

1. $pfFr(A) \cap pfint(A) = 0_P$ (resp. $pf\delta Fr(A) \cap pf\delta int(A) = 0_P$, $pf\delta\mathcal{P}Fr(A) \cap pf\delta\mathcal{P}int(A) = 0_P$, $pf\delta\mathcal{S}Fr(A) \cap pf\delta\mathcal{S}int(A) = 0_P$, $pf\delta\alpha Fr(A) \cap pf\delta\alpha int(A) = 0_P$ and $pf\delta\beta Fr(A) \cap pf\delta\beta int(A) = 0_P$),

2. $pfint(A) \cup pfFr(A) = pfcl(A)$ (resp. $pf\delta int(A) \cup pf\delta Fr(A) = pf\delta cl(A)$, $pf\delta\mathcal{P}int(A) \cup pf\delta\mathcal{P}Fr(A) = pf\delta\mathcal{P}cl(A)$, $pf\delta\mathcal{P}int(A) \cup pf\delta\mathcal{P}Fr(A) = pf\delta\mathcal{P}cl(A)$, $pf\delta\mathcal{S}int(A) \cup pf\delta\mathcal{S}Fr(A) = pf\delta\mathcal{S}cl(A)$, $pf\delta\alpha int(A) \cup pf\delta\alpha Fr(A) = pf\delta\alpha cl(A)$ and $pf\delta\beta int(A) \cup pf\delta\beta Fr(A) = pf\delta\beta cl(A)$),

3. $pfint(A) \cup pfint(A^c) \cup pfFr(A) = 1_P$ (resp. $pf\delta int(A) \cup pf\delta int(A^c) \cup pf\delta Fr(A) = 1_P$, $pf\delta\mathcal{P}int(A) \cup pf\delta\mathcal{P}int(A^c) \cup pf\delta\mathcal{P}Fr(A) = 1_P$, $pf\delta\mathcal{S}int(A) \cup pf\delta\mathcal{S}int(A^c) \cup pf\delta\mathcal{S}Fr(A) = 1_P$, $pf\delta\alpha int(A) \cup pf\delta\alpha int(A^c) \cup pf\delta\alpha Fr(A) = 1_P$ and $pf\delta\beta int(A) \cup pf\delta\beta int(A^c) \cup pf\delta\beta Fr(A) = 1_P$).

Theorem 3.14 Let (X, Γ_P) be a $pfts$. Let A and B be a pfs's in (X, Γ_P). Then $pfFr(A \cup B) \subseteq pfFr(A) \cup pfFr(B)$ (resp. $pf\delta Fr(A \cup B) \subseteq pf\delta Fr(A) \cup pf\delta Fr(B)$, $pf\delta\mathcal{P}Fr(A \cup B) \subseteq pf\delta\mathcal{P}Fr(A) \cup pf\delta\mathcal{P}Fr(B)$, $pf\delta\mathcal{S}Fr(A \cup B) \subseteq pf\delta\mathcal{S}Fr(A) \cup pf\delta\mathcal{S}Fr(B)$, $pf\delta\alpha Fr(A \cup B) \subseteq pf\delta\alpha Fr(A) \cup pf\delta\alpha Fr(B)$ and $pf\delta\beta Fr(A \cup B) \subseteq pf\delta\beta Fr(A) \cup pf\delta\beta Fr(B)$).

Proof. Let A and B be a pfs's in (X, Γ_P). Then

$pf\delta Fr(A \cup B) = pf\delta cl(A \cup B) \cap pf\delta cl(A \cup B)^c [by Definition 3.4]$

$= pf\delta cl(A \cup B) \cap pf\delta cl(A^c \cap B^c)$

$\subseteq (pf\delta cl(A) \cup pf\delta cl(B) \cap ((pf\delta cl(A^c))) \cap (pf\delta cl(B^c))[by Theorem 3.43.3(vii)\&(ix)]$

$= [(pf\delta cl(A) \cup (pf\delta cl(B)) \cap (pf\delta cl(A^c)))] \cap [(pf\delta cl(A) \cup (pf\delta cl(B)) \cap (pf\delta cl(B^c)))]$

$= [(pf\delta cl(A) \cap pf\delta cl(A^c)) \cup ((pf\delta cl(B) \cap (pf\delta cl(A^c))))] \cap [(pf\delta cl(A) \cap (pf\delta cl(B^c)))$

$\cup ((pf\delta cl(B) \cap (pf\delta cl(B^c))))]$

$= [pf\delta Fr(A) \cup (pf\delta cl(B)) \cap (pf\delta cl(A^c))] \cap [(pf\delta cl(A) \cap (pf\delta cl(B^c))) \cup (pf\delta Fr(B))]$

$[by Definition 3.43.33.4]$

$= (pf\delta Fr(A) \cup pf\delta Fr(B)) \cap [(pf\delta cl(B) \cap (pf\delta cl(A^c))) \cup ((pf\delta cl(A) \cap pf\delta cl(B^c)))]$

$\subseteq pf\delta Fr(A) \cup pf\delta Fr(B)$.

Hence, $pf\delta Fr(A \cup B) \subseteq pf\delta Fr(A) \cup pf\delta Fr(B)$.

The proof of the others are similar.

Theorem 3.15 Let (X, Γ_P) be a $pfts$. Let A and B be a pfs's in (X, Γ_P), then $pfFr(A \cap B) \subseteq (pfFr(A) \cap (pfcl(B))) \cup (pfFr(B) \cap pfcl(A))$ (resp. $pf\delta Fr(A \cap B) \subseteq (pf\delta Fr(A) \cap (pf\delta cl(B))) \cup (pf\delta Fr(B) \cap N\delta cl(A))$, $pf\delta\mathcal{P}Fr(A \cap B) \subseteq (pf\delta\mathcal{P}Fr(A) \cap (pf\delta\mathcal{P}cl(B))) \cup (pf\delta\mathcal{P}Fr(B) \cap N\delta\mathcal{P}cl(A))$, $pf\delta\mathcal{S}Fr(A \cap B) \subseteq (pf\delta\mathcal{S}Fr(A) \cap (pf\delta\mathcal{S}cl(B))) \cup (pf\delta\mathcal{S}Fr(B) \cap N\delta\mathcal{S}cl(A))$, $pf\delta\alpha Fr(A \cap B) \subseteq (pf\delta\alpha Fr(A) \cap (pf\delta\alpha cl(B))) \cup (pf\delta\alpha Fr(B) \cap N\delta\alpha cl(A))$ and $pf\delta\beta Fr(A \cap B) \subseteq (pf\delta\beta Fr(A) \cap (pf\delta\beta cl(B))) \cup (pf\delta\beta Fr(B) \cap N\delta\beta cl(A))$).

Proof. Let A and B be a pfs's in (X, Γ_P). Then

$pf\delta Fr(A \cap B) = pf\delta cl(A \cap B) \cap (pf\delta cl(A \cap B)^c)[by Definition 3.4]$

$= pf\delta cl(A \cap B) \cap (pf\delta cl(A^c \cup B^c))$

$\subseteq (pf\delta cl(A) \cap pf\delta cl(B)) \cap (pf\delta cl(A^c) \cup pf\delta cl(B^c))[by Theorem 3.43.3(vii)\&(ix)]$

$= [(pf\delta cl(A) \cap pf\delta cl(B)) \cap pf\delta cl(A^c)] \cup [(pf\delta cl(A) \cap pf\delta cl(B)) \cap pf\delta cl(B^c)]$

$= (pf\delta Fr(A) \cap pf\delta cl(B)) \cup (pf\delta Fr(B) \cap pf\delta cl(A))[by Definition 3.43.33.4]$.

Hence $pf\delta Fr(A \cap B) \subseteq ((pf\delta Fr(A) \cap (pf\delta cl(B))) \cup (pf\delta Fr(B) \cap (pf\delta cl(A))))$.

The proof of the others are similar.

Corollary 3.1 Let (X, Γ_P) be a $pfts$. Let A and B be a pfs's in (X, Γ_P), then $pfFr(A \cap B) \subseteq pfFr(A) \cup pfFr(B)$ (resp. $pf\delta Fr(A \cap B) \subseteq pf\delta Fr(A) \cup pf\delta Fr(B)$, $pf\delta\mathcal{P}Fr(A \cap B) \subseteq pf\delta\mathcal{P}Fr(A) \cup pf\delta\mathcal{P}Fr(B)$, $pf\delta\mathcal{S}Fr(A \cap B) \subseteq pf\delta\mathcal{S}Fr(A) \cup pf\delta\mathcal{S}Fr(B)$, $pf\delta\alpha Fr(A \cap B) \subseteq pf\delta\alpha Fr(A) \cup pf\delta\alpha Fr(B)$ and $pf\delta\beta Fr(A \cap B) \subseteq pf\delta\beta Fr(A) \cup pf\delta\beta Fr(B)$).

Proof. Let A and B be a pfs's in (X, Γ_P). Then

$pf\delta Fr(A \cap B) = pf\delta cl(A \cap B) \cap (pf\delta cl(A \cap B)^c)[by Definition 3.4]$

$= pf\delta cl(A \cap B) \cap (pf\delta cl(A^c \cup B^c)$

$\subseteq (pf\delta cl(A) \cap pf\delta cl(B)) \cap (pf\delta cl(A^c) \cup pf\delta cl(B^c))[by Theorem 3.43.3(vii)\&(ix)]$

$= (pf\delta cl(A) \cap pf\delta cl(B)) \cap (pf\delta cl(A^c) \cup (pf\delta cl(A) \cap pf\delta cl(B)) \cap (pf\delta cl(B^c)))$

$= (pf\delta Fr(A) \cap pf\delta cl(B)) \cup (pf\delta cl(A) \cap pf\delta Fr(B))[by Definition 3.43.33.4]$

$\subseteq pf\delta Fr(A) \cup (pf\delta Fr(B)$.

Hence $pf\delta Fr(A \cap B) \subseteq pf\delta Fr(A) \cup pf\delta Fr(B)$.

The proof of the others are similar.

Theorem 3.16 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P),

1.
   (a) $pfFr(pfFr(A)) \subseteq pfFr(A)$,

   (b) $pfFr(pfFr(pfFr(A))) \subseteq pfFr(pfFr(A))$.

2.
   (a) $pf\delta Fr(pf\delta Fr(A)) \subseteq pf\delta Fr(A)$,

   (b) $pf\delta Fr(pf\delta Fr(pf\delta Fr(A))) \subseteq pf\delta Fr(pf\delta Fr(A))$.

3.

(a) $pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}Fr(A)) \subseteq pf\delta\mathcal{S}Fr(A)$,

(b) $pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}Fr(A))) \subseteq pf\delta\mathcal{S}Fr(pf\delta\mathcal{S}Fr(A))$.

4.

(a) $pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}Fr(A)) \subseteq pf\delta\mathcal{P}Fr(A)$,

(b) $pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}Fr(A))) \subseteq pf\delta\mathcal{P}Fr(pf\delta\mathcal{P}Fr(A))$.

5.

(a) $pf\delta\alpha Fr(pf\delta\alpha Fr(A)) \subseteq pf\delta\alpha Fr(A)$,

(b) $pf\delta\alpha Fr(pf\delta\alpha Fr(pf\delta\alpha Fr(A))) \subseteq pf\delta\alpha Fr(pf\delta\alpha Fr(A))$.

6.

(a) $pf\delta\beta Fr(pf\delta\beta Fr(A)) \subseteq pf\delta\beta Fr(A)$,

(b) $pf\delta\beta Fr(pf\delta\beta Fr(pf\delta\beta Fr(A))) \subseteq pf\delta\beta Fr(pf\delta\beta Fr(A))$.

Proof. (ii) (a) Let A be a pfs in (X, Γ_P). Then

$pf\delta Fr(N\delta Fr(A)) = N\delta cl(pf\delta Fr(A)) \cap N\delta cl(pf\delta Fr(A)^c)[by Definition 3.4]$

$= N\delta cl(pf\delta cl(A) \cap (pf\delta cl(A^c)) \cap (pf\delta cl(pf\delta cl(A)) \cap (pf\delta cl(A^c))^c))$

$[by Definition 3.43.4]$

$\subseteq (pf\delta cl(pf\delta cl(A)) \cap (pf\delta cl(pf\delta cl(A^c))) \cap (pf\delta cl(pf\delta int(A^c))) \cup (pf\delta int(A)))$

$[by Theorem 3.43.43.3(iv)\&(ix)]$

$= (pf\delta cl(A) \cap (pf\delta cl(A^c)) \cap (pf\delta cl(pf\delta int(A) \cup pf\delta int(A))))$

$[by Theorem 3.43.43.33.3(iv)]$

$\subseteq pf\delta cl(A) \cap pf\delta cl(A^c)$

$= pf\delta Fr(A)[by Definition 3.43.43.33.33.4]$.

Therefore $pf\delta Fr(pf\delta Fr(A)) \subseteq pf\delta Fr(A)$.

(b) Again, $pf\delta Fr(pf\delta Fr(pf\delta Fr(A))) \subseteq pf\delta Fr(pf\delta Fr(A))$.

The proof of the others are similar.

4 Pythagorean fuzzy border and exterior

In this section, the Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) border, Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) exterior using Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) open sets are introduced and discussed their properties in $pfts$.

Definition 4.1 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P). The Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) border of A (briefly, $pfBr(A)$ (resp. $pf\delta Br(A)$, $pf\delta\mathcal{P}Br(A)$, $pf\delta\mathcal{S}Br(A)$, $pf\delta\alpha Br(A)$ and $pf\delta\beta Br(A)$)) is defined by $pfBr(A) = A \cap pfint(A)$ (resp. $pf\delta Br(A) = A \cap pf\delta int(A)$, $pf\delta\mathcal{P}Br(A) = A \cap pf\delta\mathcal{P}int(A)$, $pf\delta\mathcal{S}Br(A) = A \cap pf\delta\mathcal{S}int(A)$, $pf\delta\alpha Br(A) = A \cap pf\delta\alpha int(A)$ and $pf\delta\beta Br(A) = A \cap pf\delta\beta int(A)$).

Example 4.1 In Example 3.1, the pfs A is defined as

$$\mu_A(a) = 0.6, \gamma_A(a) = 0.5, \mu_A(b) = 0.4, \gamma_A(b) = 0.3.$$

Then $pfBr(A) = pf\delta Br(A) = pf\delta\mathcal{S}Br(A) = pf\delta\mathcal{P}Br(A) = pf\delta\alpha Br(A) = pf\delta\beta Br(A) = A^c$.

Theorem 4.1 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P). If A is $pfcs$ (resp. $pf\delta cs$, $pf\delta\mathcal{P}cs$, $pf\delta\mathcal{S}cs$, $pf\delta\alpha cs$ and $pf\delta\beta cs$), then $pfBr(A) = pfFr(A)$ (resp. $pf\delta Br(A) = pf\delta Fr(A)$, $pf\delta\mathcal{P}Br(A) = pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Br(A) = pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Br(A) = pf\delta\alpha Fr(A)$ and $pf\delta\beta Br(A) = pf\delta\beta Fr(A)$).

Proof. Let A be a $pf\delta cs$ of X. Then by Theorem 3.3 (ix), $pf\delta cl(A) = A$. Now, $pf\delta Fr(A) = pf\delta cl(A) - pf\delta int(A) = A - pf\delta int(A) = pf\delta Br(A)$.

The proof of the others are similar.

Theorem 4.2 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P), then $A = pfint(A) \cup pfBr(A)$ (resp. $A = pf\delta int(A) \cup pf\delta Br(A)$, $A = pf\delta\mathcal{P}int(A) \cup pf\delta\mathcal{P}Br(A)$, $A = pf\delta\mathcal{S}int(A) \cup pf\delta\mathcal{S}Br(A)$, $A = pf\delta\alpha int(A) \cup pf\delta\alpha Br(A)$, $A = pf\delta\beta int(A) \cup pf\delta\beta Br(A)$).

Proof. Let $x_{(\alpha,\beta,\gamma)} \in A$. If $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A)$, then the result is obvious. If $x_{(\alpha,\beta,\gamma)} \not\in pf\delta int(A)$, then by the definition of $pf\delta Br(A), x_{(\alpha,\beta,\gamma)} \in pf\delta Br(A)$. Hence $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A) \cup pf\delta Br(A)$ and so $A \subseteq pf\delta int(A) \cup pf\delta Br(A)$. On the other hand, since $pf\delta int(A) \subseteq A$ and $pf\delta Br(A) \subseteq A$, we have $pf\delta int(A) \cup pf\delta Br(A) \subseteq A$.

The proof of the others are similar

Theorem 4.3 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P), then $pfint(A) \cap pfBr(A) = 0_P$ (resp. $pf\delta int(A) \cap pf\delta Br(A) = 0_P$, $pf\delta\mathcal{P}int(A) \cap pf\delta\mathcal{P}Br(A) = 0_P$, $pf\delta\mathcal{S}int(A) \cap pf\delta\mathcal{S}Br(A) = 0_P$, $pf\delta\alpha int(A) \cap pf\delta\alpha Br(A) = 0_P$ and $pf\delta\beta int(A) \cap pf\delta\beta Br(A) = 0_P$).

Proof. Suppose $pf\delta int(A) \cap pf\delta Br(A) \neq 0_P$. Let $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A) \cap pf\delta Br(A)$. Then $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A)$ and $x_{(\alpha,\beta,\gamma)} \in pf\delta Br(A)$. Since $pf\delta Br(A) = A - pf\delta int(A)$, then $x_{(\alpha,\beta,\gamma)} \in A$. But $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A), x_{(\alpha,\beta,\gamma)} \in A$. There is a contradiction. Hence $pf\delta int(A) \cap pf\delta Br(A) = 0_P$.

The proof of the others are similar

Theorem 4.4 Let (X,Γ_P) be a $pfts$. Let A be a pfs in (X,Γ_P), then A is a $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pf\delta\alpha os$ and $pf\delta\beta os$) iff $pfBr(A) = 0_P$ (resp. $pf\delta Br(A) = 0_P$, $pf\delta\mathcal{P}Br(A) = 0_P$, $pf\delta\mathcal{S}Br(A) = 0_P$, $pf\delta\alpha Br(A) = 0_P$ and $pf\delta\beta Br(A) = 0_P$).

Proof. Necessity: Suppose A is $pf\delta os$. Then by Theorem 3.2 (x), $pf\delta int(A) = A$. Now, $pf\delta Br(A) = A - pf\delta int(A) = A - A = 0_P$.

Sufficiency: Suppose $pf\delta Br(A) = 0_P$. This implies, $A - pf\delta int(A) = 0_P$. Therefore $A = pf\delta int(A)$ and hence A is $pf\delta os$.

The proof of the others are similar.

Corollary 4.1 Let (X, Γ_P) be a $pfts$, $pfBr(0_P) = 0_P$ and $pfBr(1_P) = 0_P$ (resp. $pf\delta Br(0_P) = 0_P$ and $pf\delta Br(1_P) = 0_P$).

Proof. Since 0_P and 1_P are $pf\delta os$, by Theorem 4.4, $pf\delta Br(0_P) = 0_P$ and $pf\delta Br(\ 1_P) = 0_P$.

The proof of the others are similar.

Theorem 4.5 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfBr(pf\ int(A)) = 0_P$ (resp. $pf\delta Br(pf\delta int(A)) = 0_P$, $pf\delta\mathcal{P}Br(pf\delta\mathcal{P}int(A)) = 0_P$, $pf\delta\mathcal{S}Br(pf\delta\mathcal{S}int(A)) = 0_P$, $pf\delta\alpha Br(pf\delta\alpha int(A)) = 0_P$ and $pf\delta\beta Br(pf\delta\beta int(A)) = 0_P$).

Proof. By the Definition 4.1, $pf\delta Br(pf\delta int(A)) = pf\delta int(A) - pf\delta int(pf\delta int(A))$. By Theorem 3.2 (iv), $pf\delta int(pf\delta int(A)) = pf\delta int(A)$ and hence $pf\delta Br(pf\delta int(A)) = 0_P$.

The proof of the others are similar

Theorem 4.6 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfint(pfBr(A)) = 0_P$ (resp. $pf\delta int(pf\delta Br(A)) = 0_P$, $pf\delta\mathcal{P}int(pf\delta\mathcal{P}Br(A)) = 0_P$, $pf\delta\mathcal{S}int(pf\delta\mathcal{S}Br(A)) = 0_P$, $pf\delta\alpha int(pf\delta\alpha Br(A)) = 0_P$ and $pf\delta\beta int(pf\delta\beta Br(A)) = 0_P$).

Proof. Let $x_{(\alpha,\beta,\gamma)} \in pf\delta int(pf\delta Br(A))$. Since $pf\delta Br(A) \subseteq A$, by Theorem 3.2 (i), $pf\delta int(pf\delta Br(\ A)) \subseteq pf\delta int(A)$. Hence $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A)$. Since $pf\delta int(pf\delta Br(A)) \subseteq pf\delta Br(A), x_{(\alpha,\beta,\gamma)} \in pf\delta Br(A)$. Therefore $x_{(\alpha,\beta,\gamma)} \in pf\delta int(A) \cap pf\delta Br(A), x_{(\alpha,\beta,\gamma)} = 0_P$.

The proof of the others are similar.

Theorem 4.7 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfBr(pfBr(A)) = pfBr(A)$ (resp. $pf\delta Br\ (pf\delta Br(A)) = pf\delta Br(A)$, $pf\delta\mathcal{P}Br(pf\delta\mathcal{P}Br(A)) = pf\delta\mathcal{P}Br(A)$, $pf\delta\mathcal{S}Br(pf\delta\mathcal{S}Br(A)) = pf\delta\mathcal{S}Br(A)$, $pf\delta\alpha Br(pf\delta\alpha\ Br(A)) = pf\delta\alpha Br(A)$ and $pf\delta\beta Br(pf\delta\beta Br(A)) = pf\delta\beta Br(A)$).

Proof. By the Definition 4.1, $pf\delta Br(pf\delta Br(A)) = pf\delta Br(A) - pf\delta int(pf\delta Br(A))$. By Theorem 4.6 $pf\delta int(pf\delta Br(A)) = 0_P$ and hence $pf\delta Br(pf\delta Br(A)) = pf\delta Br(A)$.

The proof of the others are similar.

Theorem 4.8 Let (X, Γ_P) be a $pfts$ and A be a pfs in (X, Γ_P). Then the following statements are equivalent.

1. A is $pfos$ (resp. $pf\delta os$, $pf\delta\mathcal{P}os$, $pf\delta\mathcal{S}os$, $pf\delta\alpha os$ and $pf\delta\beta os$);
2. $A = pfint(A)$ (resp. $A = pf\delta int(A)$, $A = pf\delta\mathcal{P}int(A)$, $A = pf\delta\mathcal{S}int(A)$, $A = pf\delta\alpha int(A)$ and $A = pf\delta\beta int(A)$);

3. $pfBr(A) = 0_P$ (resp. $pf\delta Br(A) = 0_P$, $pf\delta\mathcal{P}Br(A) = 0_P$, $pf\delta\mathcal{S}Br(A) = 0_P$, $pf\delta\alpha Br(A) = 0_P$ and $pf\delta\beta Br(A) = 0_P$).

Proof. (i) → (ii): Obvious from Theorem ??.

(ii) → (iii): Suppose that $A = pf\delta int(A)$. Then by Definition, $pf\delta Br(A) = pf\delta int(A) - pf\delta int(A) = 0_P$.

(iii) → (i): Let $pf\delta Br(A) = 0_P$. Then by Definition 4.1, $A - pf\delta int(A) = 0_P$ and hence $A = pf\delta int(A)$.

The proof of the others are similar

Theorem 4.9 Let (X, Γ_P) be a $pfts$ and A be a pfs in (X, Γ_P). Then, $pfBr(A) = A \cap pfcl(A^c)$ (resp. $pf\delta Br(A) = A \cap pf\delta cl(A^c)$, $pf\delta\mathcal{P}Br(A) = A \cap pf\delta\mathcal{P}cl(A^c)$, $pf\delta\mathcal{S}Br(A) = A \cap pf\delta\mathcal{S}cl(A^c)$, $pf\delta\alpha Br(A) = A \cap pf\delta\alpha cl(A^c)$ and $pf\delta\beta Br(A) = A \cap pf\delta\beta cl(A^c)$).

Proof. Since $pf\delta Br(A) = A - pf\delta int(A)$ and by Theorem ??, $pf\delta Br(A) = A - (pf\delta cl(A^c))^c = A \cap (pf\delta cl(A^c)^c) = A \cap pf\delta cl(A^c)$.

The proof of the others are similar

Theorem 4.10 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfBr(A) \subseteq pfFr(A)$ (resp. $pf\delta Br(A) \subseteq pf\delta Fr(A)$, $pf\delta\mathcal{P}Br(A) \subseteq pf\delta\mathcal{P}Fr(A)$, $pf\delta\mathcal{S}Br(A) \subseteq pf\delta\mathcal{S}Fr(A)$, $pf\delta\alpha Br(A) \subseteq pf\delta\alpha Fr(A)$ and $pf\delta\beta Br(A) \subseteq pf\delta\beta Fr(A)$).

Proof. Since $A \subseteq pf\delta cl(A)$, $A - pf\delta int(A) \subseteq pf\delta cl(A) - pf\delta int(A)$. That implies, $pf\delta Br(A) \subseteq pf\delta Fr(A)$.

The proof of the others are similar.

Definition 4.2 Let (X, Γ_P) be a $pfts$ and A be a pfs in (X, Γ_P). The Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) interior of A^c is called the Pythagorean fuzzy (resp. δ, $\delta\mathcal{P}$, $\delta\mathcal{S}$, $\delta\alpha$ and $\delta\beta$) exterior of A and it is denoted by $pfExt(A)$ (resp. $pf\delta Ext(A)$, $pf\delta\mathcal{P}Ext(A)$, $pf\delta\mathcal{S}Ext(A)$, $pf\delta\alpha Ext(A)$ and $pf\delta\beta Ext(A)$). That is, $pfExt(A) = pfint(A^c)$ (resp. $pf\delta Ext(A) = pf\delta int(A^c)$, $pf\delta\mathcal{P}Ext(A) = pf\delta\mathcal{P}int(A^c)$, $pf\delta\mathcal{S}Ext(A) = pf\delta\mathcal{S}int(A^c)$, $pf\delta\alpha Ext(A) = pf\delta\alpha int(A^c)$ and $pf\delta\beta Ext(A) = pf\delta\beta int(A^c)$).

Example 4.2 In Example 3.1, the pfs A is defined as

$$\mu_A(a) = 0.6, \gamma_A(a) = 0.5, \mu_A(b) = 0.4, \gamma_A(b) = 0.3.$$

Then $pfExt(A) = pf\delta Ext(A) = pf\delta\mathcal{S}Ext(A) = pf\delta\mathcal{P}Ext(A) = pf\delta\alpha Ext(A) = pf\delta\beta Ext(A) = A_1$.

Theorem 4.11 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfExt(A) = (pfcl(A))^c$ (resp. $pf\delta Ext(A) = (pf\delta cl(A))^c$, $pf\delta\mathcal{P}Ext(A) = (pf\delta\mathcal{P}cl(A))^c$, $pf\delta\mathcal{S}Ext(A) = (pf\delta\mathcal{S}cl(A))^c$, $pf\delta\alpha Ext(A) = (pf\delta\alpha cl(A))^c$ and $pf\delta\beta Ext(A) = (pf\delta\beta cl(A))^c$).

Proof. We know that, $1_P - pf\delta cl(A) = pf\delta int(A^c)$, then $pf\delta Ext(A) = pf\delta int(A^c) = (pf\delta cl(A))^c$.

The proof of the others are similar.

Theorem 4.12 Let (X, Γ_P) be a $pfts$. Let A be a pfs in (X, Γ_P), then $pfExt(pf\ Ext(A)) = pfint(pfcl(A)) \supseteq pfint(A)$ (resp. $pf\delta Ext(pf\delta Ext(A)) = pf\delta int(pf\delta cl(A)) \supseteq pf\delta int(A)$, $pf\delta\mathcal{P}Ext(pf\delta\mathcal{P}Ext(A)) = pf\delta\mathcal{P}int(pf\delta\mathcal{P}cl(A)) \supseteq pf\delta\mathcal{P}\ int(A)$, $pf\delta\mathcal{S}Ext(pf\delta\mathcal{S}Ext(A)) = pf\delta\mathcal{S}int(pf\delta\mathcal{S}cl(A)) \supseteq pf\delta\mathcal{S}int(A)$, $pf\delta\alpha Ext(pf\delta\alpha Ext(A)) = pf\delta\alpha int(pf\delta\alpha cl(A)) \supseteq pf\delta\alpha int(A)$ and $pf\delta\beta Ext(pf\delta\beta Ext(A)) = pf\delta\beta int(pf\delta\beta cl(A)) \supseteq pf\delta\beta int(A)$).

Proof. Now, $\boldsymbol{pf\delta Ext(pf\delta Ext(A)) = pf\delta Ext(pf\delta int(A^c)) = pf\delta int((pf\delta int(A^c))^c) = pf\delta int(pf\ \delta cl(A)) \supseteq pf\delta int(A)}$.

The proof of the others are similar.

Theorem 4.13 Let $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$ be a $\boldsymbol{pfts}$. Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be a $\boldsymbol{pfs}$'s in $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$. If $\boldsymbol{A \subseteq B}$, then $\boldsymbol{pfExt(B) \subseteq pfExt(A)}$ (resp. $\boldsymbol{pf\delta Ext(B) \subseteq pf\delta Ext(A)}$, $\boldsymbol{pf\delta\mathcal{P}Ext(B) \subseteq pf\delta\mathcal{P}Ext(A)}$, $\boldsymbol{pf\delta\mathcal{S}Ext(B) \subseteq pf\delta\mathcal{S}Ext(A)}$, $\boldsymbol{pf\delta\alpha Ext(B) \subseteq pf\delta\alpha Ext(A)}$ and $\boldsymbol{pf\delta\beta Ext(B) \subseteq pf\delta\beta Ext(A)}$).

Proof. Suppose $\boldsymbol{A \subseteq B}$. Now, $\boldsymbol{pf\delta Ext(B) = pf\delta int(B^c) \subseteq pf\delta int(A^c) = pf\delta Ext(A)}$.

The proof of the others are similar

Theorem 4.14 Let $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$ be a $\boldsymbol{pfts}$. Let $\boldsymbol{A}$ be a $\boldsymbol{pfs}$ in $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$, then $\boldsymbol{pfExt(\ 1_P) = 0_P}$ and $\boldsymbol{pfExt(0_P) = 1_P}$ (resp. $\boldsymbol{pf\delta Ext(1_P) = 0_P}$ and $\boldsymbol{pf\delta Ext(0_P) = 1_P}$, $\boldsymbol{pf\delta\mathcal{P}Ext(1_P) = 0_P}$ and $\boldsymbol{pf\delta\mathcal{P}Ext(0_P) = 1_P}$, $\boldsymbol{pf\delta\mathcal{S}Ext(1_P) = 0_P}$ and $\boldsymbol{pf\delta\mathcal{S}Ext(0_P) = 1_P}$, $\boldsymbol{pf\delta\alpha Ext(1_P) = 0_P}$ and $\boldsymbol{pf\delta\alpha Ext(0_P) = 1_P}$ and $\boldsymbol{pf\delta\beta Ext(1_P) = 0_P}$ and $\boldsymbol{pf\delta\beta Ext(0_P) = 1_P}$).

Proof. Now, $\boldsymbol{pf\delta Ext(1_P) = pf\delta int((1_P)^c) = pf\delta int(0_P)}$ and $\boldsymbol{pf\delta Ext(0_P) = pf\delta int((\ 0_P)^c) = pf\delta int(1_P)}$. Since $\boldsymbol{0_P}$ and $\boldsymbol{1_P}$ are $\boldsymbol{pf\delta os}$s, then $\boldsymbol{pf\delta int(0_P) = 0_P}$ and $\boldsymbol{pf\delta int(1_P) = 1_P}$. Hence $\boldsymbol{pf\delta Ext(0_P) = 1_P}$ and $\boldsymbol{pf\delta Ext(1_P) = 0_P}$.

The proof of the others are similar

Theorem 4.15 Let $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$ be a $\boldsymbol{pfts}$. Let $\boldsymbol{A}$ be a $\boldsymbol{pfs}$ in $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$, then $\boldsymbol{pfExt(\ A) = pfExt((pfExt(A))^c)}$ (resp. $\boldsymbol{pf\delta Ext(A) = pf\delta Ext((pf\delta Ext(A))^c)}$, $\boldsymbol{pf\delta\mathcal{P}Ext(A) = pf\delta\mathcal{P}Ext((pf\delta\mathcal{P}Ext(A))^c)}$, $\boldsymbol{pf\delta\mathcal{S}Ext(A) = pf\delta\mathcal{S}Ext((pf\delta\mathcal{S}Ext\ (A))^c)}$, $\boldsymbol{pf\delta\alpha Ext(A) = pf\delta\alpha Ext((pf\delta\alpha Ext(A))^c)}$ and $\boldsymbol{pf\delta\beta Ext(A) = pf\delta\beta Ext((pf\delta\beta Ext(A))^c)}$).

Proof. Now, $\boldsymbol{pf\delta Ext((pf\delta Ext(A))^c) = pf\delta Ext((pf\delta int(A^c))^c) = pf\delta int((((pf\delta int(\ A^c))^c))^c) = pf\delta int(pf\delta int(A^c)) = pf\delta int(A^c) = pf\delta Ext(A)}$.

The proof of the others are similar.

Theorem 4.16 Let $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$ be a $\boldsymbol{pfts}$. Let $\boldsymbol{A}$ be a $\boldsymbol{pfs}$ in $(\boldsymbol{X}, \boldsymbol{\Gamma_P})$. Then the followings statements are true.

1. $\boldsymbol{pfExt(A \cup B) \subseteq pfExt(A) \cap pfExt(B)}$ (resp. $\boldsymbol{pf\delta Ext(A \cup B) \subseteq pf\delta Ext(A) \cap pf\delta Ext(B)}$, $\boldsymbol{pf\delta\mathcal{P}Ext(A \cup B) \subseteq pf\delta\mathcal{P}Ext(A)} \cap \mathbf{p}f\delta\mathcal{P}Ext(B)$, $pf\delta\mathcal{S}Ext(A \cup B) \subseteq pf\delta\mathcal{S}Ext(A) \cap pf\delta\mathcal{S}Ext(B)$, $pf\delta\alpha Ext(A \cup B) \subseteq pf\delta\alpha Ext(A) \cap pf\delta\alpha Ext(B)$ and $pf\delta\beta Ext(A \cup$

B) $\subseteq$ pfδβExt(A) ∩ pfδβExt(B)).

2. pfExt(A ∩ B) ⊇ pfExt(A) ∪ pfExt(B) (resp. pfδExt(A ∩ B) ⊇ pfδExt(A) ∪ pfδExt(B), $pf\delta\mathcal{P}Ext(A \cap B) \supseteq pf\delta\mathcal{P}Ext(A) \cup pf\delta\mathcal{P}Ext(B)$, $pf\delta\mathcal{S}Ext(A \cap B) \supseteq pf\delta\mathcal{S}Ext(A) \cup pf\delta\mathcal{S}Ext(B)$, pfδαExt(A ∩ B) ⊇ pfδαExt(A) ∪ pfδαExt(B) and pfδβExt(A ∩ B) ⊇ pfδβExt(A) ∪ pfδβExt(B)).

Proof. (i) $pf\delta Ext(A \cup B) = pf\delta int((A \cup B)^c) = pf\delta int((A^c) \cap (B^c)) \subseteq pf\delta cl(A^c) \cap pf\delta cl(B^c) = pf\delta Ext(A) \cap pf\delta Ext(B)$.

(ii) $pf\delta Ext(A \cap B) = pf\delta int((A \cap B)^c) = pf\delta int((A^c) \cup (B^c)) \supseteq pf\delta cl(A^c) \cup pf\delta cl(B^c) = pf\delta Ext(A) \cup pf\delta Ext(B)$.

The proof of the others are similar.

5 Application

In a company, we have to make a decision for assigning a project to two employees among here we have four employees in choice and each of their knowledge, skill, aptitude, soft skill and management skills, data's were collected and represented as a Pythagorean fuzzy sets. Now our aim is to select a best pair to complete the project with best co-ordination and get the best output. For that decision making we us similarity measure for Pythagorean fuzzy set of employees. There are 'n' number of similarity measures available for Pythagorean fuzzy sets and we choose Li et al. [20] similarity measure for our problem.

Table represents the Pythagorean fuzzy set of the four employees.

	Knowledge	Skill	Aptitude	Soft skill	Managing skill
	$< \lambda(x), \mu(x) >$	$< \lambda(x), \mu(x) >$	$< \lambda(x), \mu(x) >$	$< \lambda(x), \mu(x) >$	$< \lambda(x), \mu(x) >$
employee 1	$< 0.4, 0.6 >$	$< 0.8, 0.6 >$	$< 0.3, 0.6 >$	$< 0.4, 0.9 >$	$< 0.5, 0.7 >$
employee 2	$< 0.8, 0.5 >$	$< 0.7, 0.5 >$	$< 0.9, 0.2 >$	$< 0.4, 0.8 >$	$< 0.6, 0.3 >$
employee 3	$< 0.7, 0.6 >$	$< 0.8, 0.5 >$	$< 0.75, 0.4 >$	$< 0.7, 0.2 >$	$< 0.9, 0.2 >$
employee 4	$< 0.9, 0.3 >$	$< 0.7, 0.4 >$	$< 0.8, 0.3 >$	$< 0.8, 0.01 >$	$< 0.5, 0.6 >$

The formula for Li et al. [20] similarity measure is,

$$S_L(P_1, P_2) = 1 - \sqrt{\frac{\sum_{i=1}^{n} \{[\lambda_{P_1}(x_i) - \lambda_{P_2}(x_i)]^2 + [\mu_{P_1}(x_i) - \mu_{P_2}(x_i)]^2\}}{2n}}$$

where, $\lambda_{P_1}(x_i)$ is the degree of membership of x_i in the pfs P_1, $\forall\, x_i \in U$; $\mu_{P_1}(x_i)$ is the degree of non-membership of x_i in the pfs P_1, $x_i \in U$; $\lambda_{P_2}(x_i)$ is the degree of membership of x_i in the pfs P_2, $\forall\, x_i \in U$; $\mu_{P_2}(x_i)$ is the degree of non-membership of x_i in the pfs P_2, $\forall\, x_i \in U$, and $S_L(P_1, P_2)$ is the Li et al [20] similarity measure between two Pythagorean fuzzy set P_1 and P_2 on U, the universe of discourse.

Applying this similarity measure to the above problem we get the following results.

S_L (employee 1, employee 2) = 0.90

S_L (employee 1, employee 3) = 0.93

S_L (employee 1, employee 4) = 0.78

S_L (employee 2, employee 3) = 0.75

S_L (employee 2, employee 4) = 0.80

S_L (employee 3, employee 4) = 0.79.

From the above results we could make a decision that the employee 1 and employee 3 will be best pair for assigning the project.

6 Conclusion

In this article, Pythagorean fuzzy δ frontier, Pythagorean fuzzy δ border and Pythagorean fuzzy δ exterior with the help of Pythagorean fuzzy δ-open sets in Pythagorean fuzzy topological space are introduced and also specialized some of their basic properties with examples. Finally, we just applied only one similarity measure in the decision making of industry problem. In future we will employee some similarity measures for comparing or decision making in the field of medical diagnosis and teaching learning process. We also take into diverse fuzzy environment.

References

[1] K. T. Atanassov (1983), Intuitionistic fuzzy sets, VII ITKRâ€™s Session, Sofia.
[2] K. T. Atanassov (1986), Intuitionistic fuzzy sets, Fuzzy Sets Syst. 20, 87-96.
[3] K. T. Atanassov (1989), Geometrical interpretation of the elements of the intuitionistic fuzzy objects, Preprint IM-MFAIS-1-89, Sofia.
[4] K. T. Atanassov (1999), Intuitionistic fuzzy sets: theory and applications, Physica, Heidelberg.
[5] K. T. Atanassov (2012), On intuitionistic fuzzy sets theory, Springer, Berlin.
[6] G. Beliakov and S. James (2014), Averaging aggregation functions for preferences expressed as Pythagorean membership grades and fuzzy orthopairs, In: Proceedings of the IEEE international conference on fuzzy systems (FUZZ-IEEE), 298-305.
[7] B. Davvaz and E. H. Sadrabadi (2016), An application of intuitionistic fuzzy sets in medicine, Int. J. Biomath 9, (3) 1650037.
[8] S. K. De, R. Biswas and A. R. Roy (2001), An application of intuitionistic fuzzy sets in medical diagnosis, Fuzzy Sets Syst. 117 (2), 209-213.
[9] S. Dick, R. R. Yager and O. Yazdanbakhsh (2016), On Pythagorean and complex fuzzy set operations. IEEE Trans Fuzzy Syst. 24 (5), 1009-1021.
[10] P. A. Ejegwa, A. J. Akubo and O. M. Joshua (2014), Intuitionistic fuzzzy sets in career determination, J Info Comput. Sci. 9 (4), 285-288.
[11] P. A. Ejegwa (2015), Intuitionistic fuzzy sets approach in appointment of positions in an organization via maxâ€“minâ€“max rule, Glob. J Sci. Front Res F Math. Decis. Sci. 15 (6), 1-6.
[12] P. A. Ejegwa and E. S. Modom (2015), Diagnosis of viral hepatitis using new distance measure of intuitionistic fuzzy sets, Int. J. Fuzzy Math. Arch. 8 (1), 1-7.
[13] P. A. Ejegwa (2018), Distance and similarity measures of Pythagorean fuzzy sets, Granul

Comput. https://doi.org/10.1007/s41066-018-00149-z.
[14] H. Garg (2017), A new improved score function of an interval-valued Pythagorean fuzzy set based TOPSIS method, Int. J Uncertain Quantif 7 (5), 463-474.
[15] H. Garg and S. Singh (2018), A novel triangular interval type-2 intuitionistic fuzzy set and their aggregation operators, Iran J Fuzzy Syst. 15 (5), 69-93.
[16] H. Garg and K. Kumar (2018), Distance measures for connection number sets based on set pair analysis and its applications to decision making process, Appl. Intell 48 (10), 3346-3359.
[17] H. Garg and K. Kumar (2018), An advance study on the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory and their application in decision making, Soft Comput. 22 (15), 4959-4970.
[18] X. J. Gou, Z. S. Xu and P. J. Ren (2016), The properties of continuous Pyhagorean fuzzy information, Int. J Intell. Syst. 31 (5), 401-424.
[19] X. He, Y. Du and W. Liu (2016), Pythagorean fuzzy power average operators, Fuzzy Syst. Math. 30 (6), 116-124.
[20] Li Y. Olson DL, Qin Z (2007), Similarity measures between intuitionstic fuzzy (vague) sets: a comparative analysis, Pattern Recognit Lett 28, 278-285.
[21] Murat Olgun, Mehmet Unver and Seyhmus Yardimci (2019), Pythagorean fuzzy topological spaces, Complex & Intelligent Systems.
[22] X. Peng and Y. Yang (2015), Some results for Pythagorean fuzzy sets, Int. J Intell Syst. 30, 1133-1160.
[23] X. Peng and G. Selvachandran (2017), Pythagorean fuzzy set state of the art and future directions, Artif Intell Rev. https://doi.org/10.1007/s10462-017-9596-9.
[24] E. Szmidt and J. Kacprzyk (2001), Intuitionistic fuzzy sets in some medical applications, Note IFS 7 (4), 58-64.
[25] E. Szmidt and J. Kacprzyk (2004), Medical diagnostic reasoning using a similarity measure for intuitionistic fuzzy sets, Note IFS 10 (4), 61-69.
[26] M. Udhaya Shalini and A. Stanis Arul Mary (2022), Generalized pre-closed sets in Pythagorean fuzzy topological spaces, International Journal of Creative Research Thoughts (IJCRT), 10 (30), e142-e147.
[27] R. R. Yager (2013), Pythagorean membership grades in multicriteria decision making, In: Technical report MII-3301. Machine Intelligence Institute, Iona College, New Rochelle.
[28] R. R. Yager (2013), Pythagorean fuzzy subsets, In: Proceedings of the joint IFSA world congress NAFIPS annual meeting, 57-61.
[29] R. R. Yager (2014), Pythagorean membership grades in multicriteria decision making, IEEE Trans Fuzzy Syst. 22 (4), 958-965.
[30] L. A. Zadeh (1965), Fuzzy sets, Inf. Control, 8, 338-353.