A NOVEL APPROACH ON *l*-HILBERT MEAN GRAPHS

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ABSTRACT

Let G be a graph with p vertices and q edges. The q^{th}hilbert number is denoted by$\text{Ꜧ}_q$and is defined by $\text{Ꜧ}_q = 4(q-1)+1$where $q \geq 1$. A *l*-hilbert mean labeling is an injective function $\mu: V(G) \to \{0, 1, 2, \dots, \text{Ꜧ}_{l+q-1}\}$where $l \geq 1$ that induces a bijection $\mu^*: E(G) \to \{\text{Ꜧ}_l, \text{Ꜧ}_{l+1}, \text{Ꜧ}_{l+2}, \dots, \text{Ꜧ}_{l+q-1}\}$defined by

$$\mu^*(uv) = \begin{cases} \dfrac{\mu(u)+\mu(v)+1}{2} & \text{if } \mu(u)+\mu(v) \text{ is odd} \\ \dfrac{\mu(u)+\mu(v)}{2} & \text{if } \mu(u)+\mu(v) \text{ is even} \end{cases}$$

for all $uv \in E(G)$. A graph which admits such labeling is called a*l*-hilbert mean graph.

In this paper, we investigate the *l*-hilbert mean labelingfor union of*l*-hilbert mean graphs with some other graphs.

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KEYWORDS: Hilbert numbers, *l*-Hilbertmean labeling, *l*-Hilbertmean graphs.

1.INTRODUCTION

The graph considered in this paper are finite, undirected and without loops or multiple edges.Let $G = (V, E)$ be a graph with p vertices and q edges. Terms not defined here are used in the sense of Harary [3]. For number theoretic terminology [1] is followed. A graph labeling is an assignment of integers to the vertices or the edges or both subject to certain conditions. If the domain of the mapping is a set of vertices (edges/both) then the labelingis called a vertex (edge /total) labeling.A dynamic survey of graph labeling is regularly updated by Gallian [2] and it is published by Electronic Journal of Combinatorics. The concept of mean labeling was introduced by S.Somasundaram and R.Ponraj [5]. The concept of *l*-hilbert mean labeling was introduced in [4].

2.PRELIMINARIES

Definition 2.1: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two vertex disjoint graphs. Then their union $G = G_1 \cup G_2$ is a graph whose vertex set is $V = V_1 \cup V_2$ and its edge set is $E = E_1 \cup E_2$.

Definition 2.2: The n^{th} hilbert number Ꜧ_n is given by the formula $4(n-1)+1$for $n \geq 1$. The first few hilbert numbers are $1, 5, 9, 13, 17, 21, 25, 29, 33, 37, 41, 45, 49, 53, 57$, etc.

Definition 2.3: Let G be a graph with p vertices and q edges. A hilbert mean labeling is an injective function $\mu : V(G) \rightarrow \{0, 1, 2, \dots, Ꜧ_q\}$ where $Ꜧ_q$ is the q^{th}hilbert number that induces a bijection$\mu^*: E(G) \rightarrow \{Ꜧ_1, Ꜧ_2, \dots, Ꜧ_q\}$defined by

$$\mu^*(uv) = \begin{cases} \dfrac{\mu(u) + \mu(v) + 1}{2} & \text{if } \mu(u) + \mu(v) \text{ } is\ odd \\ \dfrac{\mu(u) + \mu(v)}{2} & \text{if } \mu(u) + \mu(v) \text{ } is\ even \end{cases}$$

for all $uv \in E(G)$. A graph which admits such labeling is called a hilbert mean graph.

3. MAIN RESULTS

Definition 3.1: Let G be a graph with p vertices and q edges. The q^{th} hilbert number is denoted by$Ꜧ_q$ and is defined by $Ꜧ_q = 4(q-1) + 1$ where $q \geq 1$. A l-hilbert mean labeling is an injective function $\mu: V(G) \rightarrow \{0, 1, 2, \dots, Ꜧ_{l+q-1}\}$ where $l \geq 1$, that induces a bijection $\mu^*: E(G) \rightarrow \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q-1}\}$ defined by

$$\mu^*(uv) = \begin{cases} \dfrac{\mu(u) + \mu(v) + 1}{2} & \text{if } \mu(u) + \mu(v) \text{ } is\ odd \\ \dfrac{\mu(u) + \mu(v)}{2} & \text{if } \mu(u) + \mu(v) \text{ } is\ even \end{cases}$$

for all $uv \in E(G)$. A graph which admits such labeling is called al-hilbert mean graph.

Theorem 3.2: Let Gbe al-hilbert mean graph. Then $G \cup B_{(n,n)}$ is al-hilbert mean graph for all $n \geq 2$.

Proof: Let G be al-hilbert mean graphwith p vertices and q edges. Then$G$ has$l$-hilbert mean labeling$\mu: V(G) \rightarrow \{0, 1, 2, \dots, Ꜧ_{l+q-1}\}$ and its induced edge labelingis $\mu^*: E(G) \rightarrow \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q-1}\}$. Let $G' = G \cup B_{(n,n)}$with $V(B_{(n,n)}) = \{u, v, u_{i_1}, v_{i_1} : 1 \leq i_1 \leq n\}andE(B_{(n,n)}) = \{uv, uu_{i_1}, vv_{i_1} : 1 \leq i_1 \leq n\}$. Then G'has$p + 2n + 2$ vertices and $q + 2n + 1$ edges. Define an injective function $\lambda : V(G') \rightarrow \{0, 1, 2, \dots, Ꜧ_{l+q+2n}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

$$\lambda(u) = 4[l + q - 1] + 1, \lambda(v) = 4[l + q + 2n - 1],$$
$$\lambda(u_{i_1}) = 4[l + q + 2i_1 - 3], \ 1 \leq i_1 \leq n,$$
$$\lambda(v_{i_1}) = 4[l + q + 2i_1 - 1] + 1, \ 1 \leq i_1 \leq n$$

Then the induced edge labeling$\lambda^*: E(G') \rightarrow \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+2n}\}$ is defined as follows. $\lambda^*(e) = \mu^*(e), \lambda^*(uv) = 4[l + q + n - 1] + 1$,

$$\lambda^*(uu_{i_1}) = 4[l + q + i_1 - 2] + 1, 1 \leq i_1 \leq n,$$
$$\lambda^*(vv_{i_1}) = 4[l + q + i_1 + n - 1] + 1, 1 \leq i_1 \leq n.$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+2n}\}$.Hence$G \cup B_{(n,n)}$ is al-hilbert mean graph for all $n \geq 2$.

Theorem 3.3: Let G be al-hilbert mean graph. Then $G \cup nK_2$ is al-hilbert mean graph for all $n \geq 1$.

Proof: Let G be al-hilbert mean graphwith p vertices and q edges. Then Ghas$l$-hilbert mean labeling$\mu: V(G) \rightarrow \{0, 1, 2, \dots, Ꜧ_{l+q-1}\}$ and its induced edge labeling is $\mu^*: E(G) \rightarrow \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q-1}\}$. Let $G' = G \cup nK_2$ with $V(nK_2) = \{u_{i_1 i_2}: 1 \leq i_1 \leq n, 1 \leq i_2 \leq 2\}$ and $E(nK_2) = \{u_{i_1 i_2} u_{i_1 i_{2+1}} : 1 \leq i_1 \leq n, 1 \leq i_2 \leq 2\}$.Then G' has $p + 2n$ vertices and $q + n$ edges. Define an injective function $\lambda : V(G') \rightarrow \{0, 1, 2, \dots, Ꜧ_{l+q+n-1}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

For $1 \leq i_1 \leq n, 1 \leq i_2 \leq 2$,

$$\lambda(u_{i_1 i_2}) = \begin{cases} 4[l + q + i_1 + i_2 - 3] & \text{if } i_2 \text{ } is\ odd \\ 4[l + q + i_1 + i_2 - 4] + 1 & \text{if } i_2 \text{ } is\ even \end{cases}$$

Then the induced edge labeling $\lambda^*: E(G') \to \{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q+n-1}\}$ is defined as follows.

$$\lambda^*(e) = \mu^*(e), \lambda^*(u_{i_1\ i_2}u_{i_1\ i_{2+1}}) = 4[l + q + i_1 + i_2 - 3] + 1, 1 \le i_1 \le n, 1 \le i_2 \le 2$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q+n-1}\}$. Hence $G \cup nK_2$ is a l-hilbert mean graph for all $n \ge 1$.

Theorem 3.4: Let G be a l-hilbert mean graph. Then $G \cup H(P_n)$ is a l-hilbert mean graph for all $n \ge 3$.

Proof: Let G be a l-hilbert mean graph with p vertices and q edges. Then G has l-hilbert mean labeling $\mu: V(G) \to \{0, 1, 2, \dots, Ꝕ_{l+q-1}\}$ and its induced edge labeling is $\mu^*: E(G) \to \{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q-1}\}$. Let $G' = G \cup H(P_n)$ with $V(H(P_n)) = \{u_{i_1}, v_{i_1} : 1 \le i_1 \le n\}$ and $E(H(P_n)) = \left\{u_{i_1}u_{i_{1+1}}, v_{i_1}v_{i_{1+1}}: 1 \le i_1 \le n-1, \left(u_{\frac{n+1}{2}}, v_{\frac{n+1}{2}}\right)\right\}$ or $E(H(P_n)) = \left\{u_{i_1}u_{i_{1+1}}, v_{i_1}v_{i_{1+1}}: 1 \le i_1 \le n-1, \left(u_{\frac{n+2}{2}}, v_{\frac{n}{2}}\right)\right\}$. Then G' has $p + 2n$ vertices and $q + 2n - 1$ edges. Define an injective function $\lambda: V(G') \to \{0, 1, 2, \dots, Ꝕ_{l+q+2n-2}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

Case 1: n is odd,

For $1 \le i_1 \le n$,

$$\lambda(u_{i_1}) = \begin{cases} 4[l + q + i_1 - 2] & \text{if } i_1 \text{ is odd} \\ 4[l + q + i_1 - 3] + 1 & \text{if } i_1 \text{ is even} \end{cases}$$

$$\lambda(v_{i_1}) = \begin{cases} 4[l + q + i_1 + n - 3] + 1 & \text{if } i_1 \text{ is odd} \\ 4[l + q + i_1 + n - 2] & \text{if } i_1 \text{ is even} \end{cases}$$

Case 2: n is even,

For $1 \le i_1 \le n$,

$$\lambda(u_{i_1}) = \begin{cases} 4[l + q + i_1 - 2] & \text{if } i_1 \text{ is odd} \\ 4[l + q + i_1 - 3] + 1 & \text{if } i_1 \text{ is even} \end{cases}$$

$$\lambda(v_{i_1}) = \begin{cases} 4[l + q + i_1 + n - 2] & \text{if } i_1 \text{ is odd} \\ 4[l + q + i_1 + n - 3] + 1 & \text{if } i_1 \text{ is even} \end{cases}$$

Then the induced edge labeling $\lambda^*: E(G') \to \{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q+2n-2}\}$ is defined as follows.

$\lambda^*(e) = \mu^*(e), \lambda^*(u_{i_1}\ u_{i_{1+1}}) = Ꝕ_{l+q+i_1-1}, 1 \le i_1 \le n-1,$

$\lambda^*(v_{i_1}\ v_{i_{1+1}}) = Ꝕ_{l+q+n+i_1-1}, \quad 1 \le i_1 \le n-1$

$\lambda^*\left(u_{\frac{n+1}{2}}, v_{\frac{n+1}{2}}\right) =, Ꝕ_{l+q+n-1}$ if n *is odd*

$$\lambda^*\left(u_{\frac{n+2}{2}}, v_{\frac{n}{2}}\right) = Ꝕ_{l+q+n-1} \text{ if } n \text{ is even}$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q+2n-2}\}$. Hence $G \cup H(P_n)$ is a l-hilbert mean graph for all $n \ge 3$.

Theorem 3.5: Let G be a l-hilbert mean graph. Then $G \cup T(n)$ is a l-hilbert mean graph for all $n \ge 4$ and n is even.

Proof: Let G be a l-hilbert mean graph with p vertices and q edges. Then G has l-hilbert mean labeling $\mu: V(G) \to \{0, 1, 2, \dots, Ꝕ_{l+q-1}\}$ and its induced edge labeling is $\mu^*: E(G) \to \{Ꝕ_l, Ꝕ_{l+1}, Ꝕ_{l+2}, \dots, Ꝕ_{l+q-1}\}$. Let $G' = G \cup T(n)$ with $V(T(n)) = \{v_{i_1} : 1 \le i_1 \le n, u_{i_1}, w_{i_1}: 1 \le i_1 \le n-2\}$ and $E(T(n)) = \{v_{i_1}v_{i_{1+1}}: 1 \le i_1 \le n-1, u_{i_1}v_{i_{1+1}}, w_{i_1}v_{i_{1+1}} : 1 \le i_1 \le n-2\}$. Then G' has $p + 3n - 4$ vertices and $q + 3n - 5$ edges. Define an injective function $\lambda: V(G') \to \{0, 1, 2, \dots, Ꝕ_{l+q+3n-6}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

For $1 \le i_1 \le n$,

$$\lambda(v_{i_1}) = \begin{cases} 4[l+q+3i_1-4] & \text{if } i_1 \text{ is odd} \\ 4[l+q+3i_1-7]+1 & \text{if } i_1 \text{ is even} \end{cases}$$

For $1 \leq i_1 \leq n-2$,

$$\lambda(u_{i_1}) = \begin{cases} 4[l+q+3i_1-2] & \text{if } i_1 \text{ is odd} \\ 4[l+q+3i_1-5]+1 & \text{if } i_1 \text{ is even} \end{cases}$$

$$\lambda(w_{i_1}) = \begin{cases} 4[l+q+3i_1] & \text{if } i_1 \text{ is odd} \\ 4[l+q+3i_1-3]+1 & \text{if } i_1 \text{ is even} \end{cases}$$

Then the induced edge labeling $\lambda^*: E(G') \to \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+3n-6}\}$ is defined as follows.

$\lambda^*(e) = \mu^*(e), \lambda^*(v_{i_1}v_{i_{1+1}}) = Ꜧ_{l+q+3i_1-3}$, $1 \leq i_1 \leq n-1$,

$$\lambda^*(w_{i_1}v_{i_{1+1}}) = Ꜧ_{l+q+3i_1-1} 1 \leq i_1 \leq n-2$$

$$\lambda^*(u_{i_1}v_{i_{1+1}}) =, Ꜧ_{l+q+3i_1-2} \quad 1 \leq i_1 \leq n-2$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+3n-6}\}$. Hence$G \cup T(n)$is a$l$-hilbert mean graph for all $n \geq 4$ and n is even.

Theorem 3.6: Let G be al-hilbert mean graph. Then $G \cup BG$ is al-hilbert mean graph.

Proof: Let G be al-hilbert mean graphwith p vertices and q edges. Then Ghas$l$-hilbert mean labeling$\mu: V(G) \to \{0, 1, 2, \dots, Ꜧ_{l+q-1}\}$ and its induced edge labeling is $\mu^*: E(G) \to \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q-1}\}$. Let$G' = G \cup BG$with $V(BG) = \{v_{i_1} : 1 \leq i_1 \leq n\}$and $E(BG) = \{v_{i_1}v_{i_{1+1}}: 1 \leq i_1 \leq n-1, v_2 v_4\}$. Then G'has$p+5$ vertices and $q+5$ edges. Define an injective function $\lambda : V(G') \to \{0, 1, 2, \dots, Ꜧ_{l+q+4}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

For $i_1 = 1, 2$

$$\lambda(v_{i_1}) = \begin{cases} 4[l+q+i_1-2] & \text{if } i_1 \text{ is odd} \\ 4[l+q+i_1-3]+1 & \text{if } i_1 \text{ is even} \end{cases}$$

For $3 \leq i_1 \leq 5$,

$$\lambda(v_{i_1}) = \begin{cases} 4[l+q+i_1-2]+1 & \text{if } i_1 \text{ is odd} \\ 4[l+q+i_1-1] & \text{if } i_1 \text{ is even} \end{cases}$$

Then the induced edge labeling$\lambda^*: E(G') \to \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+4}\}$ is defined as follows.

$$\lambda^*(e) = \mu^*(e), \lambda^*(v_{i_1}v_{i_{1+1}}) = Ꜧ_{l+q+i_1-1} , i_1 = 1, 2$$

$\lambda^*(v_{i_1}v_{i_{1+1}}) = Ꜧ_{l+q+i_1}, 3 \leq i_1 \leq 4$

$$\lambda^*(v_2 v_4) = Ꜧ_{l+q+2}$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q+4}\}$. Hence$G \cup BG$ is al-hilbert mean graph.

Theorem 3.7: Let G be al-hilbert mean graph. Then $G \cup P_m \odot P_n$ is al-hilbert mean graph where$m \geq 3, n \geq 4$and n is even.`

Proof: Let G be al-hilbert mean graphwith p vertices and q edges. Then Ghas$l$-hilbert mean labeling$\mu: V(G) \to \{0, 1, 2, \dots, Ꜧ_{l+q-1}\}$ and its induced edge labeling is $\mu^*: E(G) \to \{Ꜧ_l, Ꜧ_{l+1}, Ꜧ_{l+2}, \dots, Ꜧ_{l+q-1}\}$. Let$G' = G \cup P_m \odot P_n$with$V(P_m \odot P_n) = \{u_{i_1 \, i_2} : 1 \leq i_1 \leq m, 1 \leq i_2 \leq n\}$and $E(P_m \odot P_n) = \{u_{i_1}u_{i_{1+1}}: 1 \leq i_1 \leq m, 1 \leq i_2 \leq n-1, u_{i_1 \, n}u_{i_{1+1} \, n} : 1 \leq i_1 \leq m-1\}$. Then G'has$p+mn$ vertices and $q+mn-1$ edges. Define an injective function $\lambda : V(G') \to \{0, 1, 2, \dots, Ꜧ_{l+q+mn-2}\}$ such that $\lambda(w) = \mu(w)$ for each $w \in V(G)$.

For i_1 is odd and $1 \leq i_1 \leq m$, $1 \leq i_2 \leq n$,

$$\lambda(u_{i_1 \, i_2}) = \begin{cases} 4[l+q+n(i_1-1)+i_2-2] & \text{if } i_2 \text{ is odd} \\ 4[l+q+n(i_1-1)+i_2-3]+1 & \text{if } i_2 \text{ is even} \end{cases}$$

For i_1 is even and $1 \leq i_1 \leq m$, $1 \leq i_2 \leq n$,

$$\lambda(u_{i_1\ i_2}) = \begin{cases} 4[l+q+n(i_1-2)+2n-i_2-2]+1 & \text{if } i_2 \text{ is odd} \\ 4[l+q+n(i_1-2)+2n-i_2-1] & \text{if } i_2 \text{ is even} \end{cases}$$

Then the induced edge labeling $\lambda^*: E(G') \to \{Ӈ_l, Ӈ_{l+1}, Ӈ_{l+2}, \dots, Ӈ_{l+q+mn-2}\}$ is defined as follows.

$$\lambda^*(e) = \mu^*(e),$$

For $1 \le i_1 \le m, 1 \le i_2 \le n$,

$\lambda^*(u_{i_1\ i_2} u_{i_1\ i_2+1}) = Ӈ_{l+q+n(i_1-1)+i_2-1}, i_1$ is odd

$\lambda^*(u_{i_1\ i_2} u_{i_1\ i_2+1}) = Ӈ_{l+q+n(i_1-2)+2n+i_2-1}, i_1$ is even

$$\lambda^*(u_{i_1\ n} u_{i_1+1\ n}) = Ӈ_{l+q+n(i_1-1)+n-1}, 1 \le i_1 \le m-1$$

Clearly λ^* is bijective. Thus we get the induced edge labels as $\{Ӈ_l, Ӈ_{l+1}, Ӈ_{l+2}, \dots, Ӈ_{l+q+mn-2}\}$. Hence $G \cup (P_m \odot P_n)$ is a l-hilbert mean graph where $m \ge 3, n \ge 4$ and n is even.

4. CONCLUSION

In this paper, we have studied the *l*-hilbert meanlabelingfor union of *l*-hilbert mean graphswith some other graphs. This work contributes several new results to the theory of graph labeling.

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