A Review of Fuzzy Subring Algebra: Theories and Case Studies

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Abstract:The inherent uncertainty in our understanding of the world presents substantial challenges for standard mathematical approaches. Fuzzy algebraic structures,building on Zadeh's foundational work in 1965, offer a specialized means to manage this uncertainty. This paper provides a comprehensive review of developments in fuzzy subring algebra, examining key elements such as fuzzy subrings, fuzzy ideals, fuzzy zero divisors, and the application of fuzzy polynomials and matrices. Through practical case studies in areas like medical diagnostics and agriculture, this study demonstrates the flexibility and crucial role of fuzzy algebra in addressing uncertainty. This paper serves as a valuable resource for researchers and practitioners seeking to navigate complex uncertainties using fuzzy algebraic frameworks.

Keywords:Fuzzy subring, Fuzzy zero divisor, Fuzzy polynomial, Fuzzy matrix

1. Introduction

Uncertainty is an inherent aspect of life, presenting challenges to human comprehension and traditional mathematical frameworks. Fuzzy sets provide a specialized mathematical tool designed to address this uncertainty, offering a structured way to model and navigate imprecision. Pioneering work by Zadeh [58] in 1965 laid the foundation for the study of fuzzy sets, which quickly gained traction among researchers and found applications across a wide range of scientific disciplines. Since then, the body of literature on fuzzy set theory and its applications has expanded significantly, with numerous studies spanning fields such as economics, psychology, artificial intelligence, network analysis, and decision-making [9, 15].

The process of fuzzification of algebraic structures extends classical algebraic concepts to account for uncertainty and imprecision. It involves integrating degrees of membership or truth values into traditional algebraic definitions and operations, reflecting the inherent fuzziness of real-world data. The fuzzification of algebraic structures began in 1971 with Rosenfeld's work[49], and it introduced the idea of fuzzy subgroups, where closure under group operations is relaxed to allow partial membership. Since then, many researchers have contributed to expanding and refining concepts within fuzzy algebra[3, 4, 17, 45].

Building on this, researchers have extended fuzzification to other algebraic structures, including fuzzy subrings. These generalize the concept of rings by permitting elements to partially satisfy the properties of distributivity and associativity. The introduction of fuzzy subrings and fuzzy ideals by Liu [37] in 1982 was a major milestone, further advanced by researchers such as [1, 2, 11, 23, 48], who significantly broadened and deepened the study within this domain.

As the study of algebraic structures progressed, the concept of zero divisors became central to understanding the properties of rings. Zero divisors identify rings that are not integral domains, making them essential for classifying algebraic systems. In fuzzy subring algebra, the notion of fuzzy zero divisors extends this concept by accounting for uncertainty, allowing for partial adherence to the defining properties. This broader definition offers a framework to study elements with varying degrees of zero divisor characteristics. The concept was first explored by Ray [47], with subsequent developments by Melliani [41], who defined fuzzy zero divisors using fuzzy points, and Ayub [7], who introduced fuzzy zero divisors for non-zero, non-unit elements.The study of fuzzy divisors can be extended to [30, 56].

The exploration of fuzzy polynomials intersects with the study of fuzzy subring algebra, particularly in relation to fuzzy zero divisors. Fuzzy polynomials have found wide applications in engineering and fuzzy mathematics. Eslami's [24] foundational work on fuzzy polynomials has been extended by various authors, with Barhoi's [10] definition based on fuzzy triangular numbers providing a connection between traditional and fuzzy polynomials. Additionally, Melliani [41] introduced fuzzy polynomials defined by fuzzy points, contributing to the variety of approaches in this field. Fuzzy polynomials are widely used in practical applications, such as in neural networks, where Zarandi [59] employed fuzzy polynomial neural networks (FPNN) to predict concrete compressive strength.

Fuzzy matrices play a critical role in many scientific and engineering domains. Introduced by Thomason [55], the theory of fuzzy matrices was later expanded by Hashimoto [27], who investigated properties of nilpotent fuzzy matrices, and Kim [33], who studied their adjoint and determinant properties. Atanassov [6] introduced intuitionistic fuzzy matrices, extending traditional fuzzy matrices to account for both membership and non-membership values. These fuzzy matrices are applied in a variety of fields. For instance, Meenakshi [40] utilized fuzzy matrices for medical diagnosis based on patient data, while Sun [22] applied them to agricultural studies to determine suitable crops for specific land areas.

This paper is organized into eight sections. Section 2 explores various structures of fuzzy subrings, including intuitionistic fuzzy subrings, Q-fuzzy subrings using t-norm T, anti Q-fuzzy subrings using t-conorm C, bipolar fuzzy subrings, and picture fuzzy subrings. Section 3 reviews fuzzy polynomials and fuzzy polynomial subrings. Section 4 discusses fuzzy matrices, including controllable fuzzy matrices and intuitionistic fuzzy matrices, along with their determinant and adjoint properties. The literature reviewed in these sections aims to inspire future research on fuzzy zero divisors in polynomial and matrix rings. Section 5 provides an overview of recent advances in fuzzy zero divisors. Section 6 illustrates practical applications in fields such as medical diagnosis and agriculture, emphasizing the role of fuzzy algebra in managing uncertainty. Section 7 discusses the limitations of the study, while Section 8 presents conclusions and outlines future research directions.
Notation: Henceforth, $X$is the universal set, and R denote commutative ring with unity 1 and additive identity 0.

2. Diverse Structures Fuzzy Subrings

Ray [47]laid the groundwork for the theory of fuzzy subrings and fuzzy ideals in 1982. Since then, many researchers have engaged in extending the concept of fuzzy subring to the broader framework of abstract algebra of rings. This section delves into diverse structures of fuzzy subrings, including intuitionistic fuzzy subrings, Q-fuzzy subrings employing t-norm T, anti Q-fuzzy subrings utilizing t-conorm C, bipolar fuzzy subrings, and picture fuzzy subrings.

2.1 Definition [25]

A mapping from a set $A \subseteq X$ to the interval [0,1] is referred to as a fuzzy subset of A. The collection of all fuzzy subsets of A is denoted by $FS(A)$.

Das [18]introduced the theory of level subsets, which gave new dimension to the study of fuzzy set theory.

2.2 Definition [25]

Letηbe a fuzzy subset of$A \subseteq X$. Then, a crisp set$\eta_\alpha = \{r \in A \mid \eta(r) \geq \alpha\}$ is referred to as the $\alpha -$ cut (or$\alpha -$ level set) ofη.

2.3 Definition [37]

Let ηbe a fuzzy subset of ring$R$. Then, η is called a fuzzy subring of R if and only if, for all $r, s \in R$, the following conditions hold:

i. $\eta(r - s) \geq \min\{\eta(r),\ \eta(s)\}$,
ii. $\eta(rs) \geq \min\{\eta(r),\ \eta(s)\}$.

2.4 Definition [37]

A fuzzy subset $\eta$of ring R is called a fuzzy ideal of R if and only if, for all$r, s \in R$, the following conditions hold:

i. $\eta(r - s) \geq \min\{\eta(r),\ \eta(s)\}$,
ii. $\eta(rs) \geq \max\{\eta(r),\ \eta(s)\}$.

2.5 Theorem[20]

A fuzzy subset η is a fuzzy ideal of a ring$R$, if and only if the level subset$\eta_t$is an ideal of the ring$R$, for all$t \in Im\,\eta$.

2.6 Definition[20]

A fuzzy subset η is a fuzzy subring (or ideal) of ring R if and only if its level subsets η_t are subrings (or ideals) of ring R for each$t \in Im\,\eta$.

Atanassov [6]introduced the concept of intuitionistic fuzzy sets in 1986, extending the framework as a generalization of fuzzy sets.Biswas [13] contributed to this study by introducing the intuitionistic fuzzy subgroup concept, offering a more comprehensive illustration of the applications of intuitionistic fuzzy sets. Likewise, Banerjee and Basnet [8] presented intuitionistic fuzzy subrings and ideals. Mohamed [23] explores intuitionistic fuzzy ideals in BE-algebras and establishes several new results related to their structure.

2.7 Definition [6]

An intuitionistic fuzzy set$I$ of set$X$is defined as follows:

$$I = \{\langle x,\ \mu(x),\ \eta(x)\rangle \mid x \in X\},$$

where,$\mu: X \to [0, 1]$ and $\eta: X \to [0, 1]$represent the degree of membership and degree of non-membership functions for every$x \in X$.

2.8 Definition [8]

An intuitionistic fuzzy subset$I = \{\langle r, \mu(r), \eta(r)\rangle \mid r \in R\}$ is said to be an intuitionistic fuzzy subring of$R$, if for every $r, s \in R$, the following conditions hold:

i. $\mu(r-s) \geq \min\{\mu(r), \mu(s)\}$,
ii. $\mu(rs) \geq \min\{\mu(r), \mu(s)\}$,
iii. $\eta(r-s) \leq \max\{\eta(r), \eta(s)\}$,
iv. $\eta(rs) \leq \max\{\eta(r), \eta(s)\}$.

2.9 Definition[8]

An intuitionistic fuzzy subring$I = \{\langle r, \mu(r), \eta(r)\rangle \mid r \in R\}$is said to be an intuitionistic fuzzy ideal of R, if for every$r, s \in R$,

i. $\mu(r-s) \geq \min\{\mu(r), \mu(s)\}$,
ii. $\mu(rs) \geq \min\{\mu(r), \mu(s)\}$,
iii. $\eta(r-s) \leq \min\{\eta(r), \eta(s)\}$,
iv. $\eta(rs) \leq \eta(r)$.

Anthony and Sherwood [5]employed the notion of a triangular norm to redefine fuzzy subgroups. The concept of Q −fuzzy groups was first presented by Solairaju and Nagarajan [54]. Rasuli [46]introduced the concept of Q −fuzzy subrings and anti Q −fuzzy subrings by using a t −norm $T$and a t −conorm C, respectively.

2.10 Definition[46]

A mapping $T: [0,1] \times [0,1] \to [0,1]$ is called t −norm if for every$u, v, w \in [0,1]$,

i. $T(u, 1) = u$,
ii. $T(u, v) \leq T(u, w)$ if $v \leq w$,
iii. $T(u, v) = T(v, u)$,
iv. $T(u, T(v, w)) = T(T(u, v), w)$.

2.11 Definition [46]

Let Q be a non-empty set. A Q − fuzzy subset η of R is said to be a Q −fuzzy subring of R with respect to the t −norm T, if for every $u, v \in [0,1]$ and $q \in Q$,

i. $\eta(u+v, q) \geq T(\eta(u, q), \eta(v, q))$,
ii. $\eta(-u, q) \geq \eta(u, q)$,
iii. $\eta(uv, q) \geq T(\eta(u, q), \eta(v, q))$

2.12 Definition[46]

A mapping $C: [0,1] \times [0,1] \to [0,1]$is called t − conorm if for every $u, v, w \in [0,1]$,

i. $C(u, 0) = u$,
ii. $C(u, v) \leq C(u, w)$ if $v \leq w$,
iii. $C(u, v) = C(v, u)$,
iv. $C(u, C(v, w)) = C(C(u, v), w)$.

2.13 Definition [46]

Let Q be a non-empty set. A Q −fuzzy subset$\mu$of R is said to be an anti$Q$ −fuzzy subring of R with respect to the t −conorm C, if for every $u, v \in [0,1]$and $q \in Q$,

i. $\eta(u+v, q) \leq C(\eta(u, q), \eta(v, q))$,
ii. $\eta(-u, q) \leq \eta(u, q)$,
iii. $\eta(uv, q) \leq C(\eta(u, q), \eta(v, q))$.

Zhang [60] initiated the conceptualization of bipolar fuzzy sets. The study of bipolar fuzzy sets involves considerations of both positive and negative membership values. In a bipolar fuzzy set, one membership value is confined to the interval$[0, 1]$, representing positive membership, while the other belongs to $[-1, 0]$, signifying negative membership. This key distinction stands as a notable departure from intuitionistic fuzzy sets, where membership and non-membership values both fall within the range $[0,1]$. In a fuzzy set with bipolar values, a membership degree of 0 implies that elements are irrelevant to the corresponding property. A membership degree in the range $(0, 1]$ indicates that elements somewhat satisfy the property. On the other hand, a membership degree in the range $[-1, 0)$suggests that elements somewhat satisfy the implicit counter-property.

2.14 Definition [60]
Let$X$be a non-empty set. The bipolar fuzzy set$A$of set$X$is defined as follows:

$$A = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in X\},$$

where,$\mu: X \to [0, 1]$ and $\eta: X \to [0,1]$ represents the degree of positive membership and degree of negative membership function respectively for every $x \in X$.

2.15 Definition [38]
A bipolar fuzzy subset$B = \{\langle r, \mu(r), \eta(r)\rangle \mid r \in X\}$is called a bipolar fuzzy subring of R, if for every $r, s \in R$,

i. $\mu(r - s) \geq \min\{\mu(r), \mu(s)\}$,
ii. $\mu(rs) \geq \min\{\mu(r), \mu(s)\}$,
iii. $\eta(r - s) \leq \max\{\eta(r), \eta(s)\}$,
iv. $\eta(rs) \leq \max\{\eta(r), \eta(s)\}$.

In the realm of intuitionistic fuzzy sets, every element in the set X is characterized by two components: the degree of membership and the degree of non-membership. The picture fuzzy set serves as a broader framework, extending the intuitionistic fuzzy set by incorporating three components for each element in X: the degree of positive membership, the degree of neutral membership, and the degree of negative membership. This enriched model of fuzzy sets was introduced by Cuong [16].

2.16 Definition[16]
Let$X$be a non-empty set. A picture fuzzy set $A$of set$X$is defined as follows:

$$A = \{\langle x, \mu(x), \eta(x), \gamma(x)\rangle \mid x \in X\},$$

where,$\mu: X \to [0, 1]$, $\eta: X \to [0,1]$and $\gamma: X \to [0,1]$ represents the degree of positive membership, the degree of neutral membership,and the degree of negative membership respectively for every$x \in X$.

2.17 Definition[21]
A picture fuzzy subset $P = \{\langle r, \mu(r), \eta(r), \gamma(r)\rangle \mid r \in R\}$ is said to be picture fuzzy subring of R, if for every$r, s \in R$,

i. $\mu(r - s) \geq \min\{\mu(r), \mu(s)\}$, $\eta(r - s) \geq \min\{\eta(r), \eta(s)\}$ and $\gamma(r - s) \leq \max\{\gamma(r), \gamma(s)\}$
ii. $\mu(rs) \geq \min\{\mu(r), \mu(s)\}$, $\eta(rs) \geq \min\{\eta(r), \eta(s)\}$ and $\gamma(r - s) \leq \max\{\gamma(r), \gamma(s)\}$

3 Fuzzy Polynomial
Fuzzy polynomials hold significant importance in both engineering and fuzzy mathematics. This section undertakes a comprehensive investigation into the study of fuzzy polynomials, covering diverse aspects such as their representation using fuzzy numbers, their extension to fuzzy polynomials involving triangularfuzzy numbers, and the intricate relationship existing between crisp polynomials and their fuzzy polynomials. Furthermore, the section explores

advancements in the study of fuzzy polynomial subrings, as proposed by Melliani [41], fuzzy polynomial ideals, as introduced by Kim [33], and the extension to intuitionistic fuzzy polynomial ideals, as investigated by Sharma [53]. This multifaceted exploration sheds light on the wide-ranging applications and theoretical developments surrounding fuzzy polynomials within various mathematical contexts and their relevance to engineering disciplines.

3.1 Definition[19]
A fuzzy subset μ of set X is called fuzzy number if

i. μ is normal, i.e., there exist $x_0 \in X$ such that $\mu(x_0) = 1$,
ii. μ is fuzzy convex, i.e., $\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min\{\mu(x_1), \mu(x_2)\}$ for all $x_1, x_2 \in X$ and $\lambda \in [0, 1]$,
iii. $\mu(x)$ is upper semi continuous,
iv. support of $\mu = \{x \in R \mid \mu(x) > 0\}$ is bounded.

Rouhparvar [50] introduced the concept of fuzzy polynomials, wherein the coefficients are represented as fuzzy numbers.

3.2 Definition [50]
A polynomial of the type,

$$A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n = 0$$

where $x \in X$, the coefficients $A_0, A_1, A_2, \ldots A_n$ are fuzzy numbers, is called fuzzy polynomial equation.

Fuzzy polynomials can be numerically solved through a variety of methods, with the Newton-Raphson method, ranking method, modified Adomian decomposition method, and fuzzy neural network method emerging as the most popular techniques. Each of these methods is characterized by distinctive algorithms, contributing to their effectiveness in accurately determining the real roots of fuzzy polynomials.

Barhoi [10] presented one approach to defining fuzzy polynomial using fuzzy triangular numbers and established a relation between crisp polynomial and fuzzy polynomial.

3.3 Definition [10]
The fuzzy set μ on set A is said to be triangular if the membership function is given by

$$\mu_A(x) = \begin{cases} \dfrac{x-a}{b-a} if a \leq x \leq b \\ \dfrac{c-x}{c-b} if b \leq x \leq c \end{cases}$$

and it is denoted by $\tilde{A} = [a, b, c]$.

3.4 Definition [10]
Let $\tilde{A} = [a_1, b_1, c_1]$ and $\tilde{B} = [a_2, b_2, c_2]$ be two triangular fuzzy numbers.

i. Addition: $\tilde{A} + \tilde{B} = [a_1 + a_2, b_1 + b_2, c_1 + c_2]$
ii. Subtraction: $\tilde{A} - \tilde{B} = [a_1 - a_2, b_1 - b_2, c_1 - c_2]$
iii. Scalar Multiplication: $k\tilde{A} = [ka_1, kb_1, kc_1]$ for any non zero k
iv. Multiplication: $\tilde{A} * \tilde{B} = [\min\{a_i b_i\},\ product\ of\ mid\ point, \max\{a_i b_i\}]$

3.5 Definition [10]
A polynomial of the form

$$\tilde{f}(\tilde{X}) = \sum_i \widetilde{A_i}\widetilde{X^i} = \widetilde{A_0} + \widetilde{A_1}\tilde{X} + \widetilde{A_2}\widetilde{X^2} + \widetilde{A_3}\widetilde{X^3} + \cdots$$

where, $\widetilde{A_0}, \widetilde{A_1}, \widetilde{A_2}, \widetilde{A_3}, \ldots$ and $\tilde{X}$are triangular fuzzy numbers.

3.6 Example

Let $f(x) = 5x + 6$. Then $f(2) = (5 \times 2) + 6 = 16$. Extending this crisp polynomial to a fuzzy polynomial, we have the following expression$f(\tilde{X}) = \widetilde{A_0} + \widetilde{A_1}\tilde{X}$, where$\widetilde{A_0} = [6, 6, 6]$, $\widetilde{A_1} = [5, 5, 5]$ and $\tilde{X} = [1, 2, 3]$. Hence $f(\tilde{2}) = [6, 6, 6] + [5, 5, 5] * [1, 2, 3] = [11, 16, 21]$. The middle value is the same as the crisp value.

3.7 Example

Consider a polynomial with more than one variable $f(x, y) = 3xy^2 + 5x + 6y + 7$. Then $f(2, 3) = 89$. Now extending this crisp polynomial to fuzzy polynomial, we have $f(\tilde{X}, \tilde{Y}) = \tilde{3}\tilde{X}\tilde{Y}^2 + \tilde{5}\,\tilde{X} + \tilde{6}\tilde{Y} + \tilde{7}$ and $f(\tilde{2}, \tilde{3}) = [3, 3, 3] * [1, 2, 3] * [2, 3, 4]^2 + [5, 5, 5] * [1, 2, 3] + [6, 6, 6] * [2, 3, 4] + [7, 7, 7] = [36, 89, 190]$. Thus, a fuzzy value in the middle is the same as the crisp value.

The study of fuzzy polynomial subrings involves investigating the algebraic properties of the polynomials. Eslami [24] pioneered the study of the fuzzy polynomial ring. Expanding on this work, Melliani [41]defined fuzzy polynomials using fuzzy points as coefficients and established that$F_\eta(R)[X]$ constitutes the fuzzy polynomial ring. Here, ηdenotes a fuzzy subring of R, and $F_\eta(R)$represents the set encompassing all fuzzy points of ηwithin the ring$R$.

3.8 Definition [24]

Let S be a commutative ring with unity. Let $R = S[x_1, x_2, \ldots, x_n]$ be the polynomial ring in the indeterminates $x_1, x_2, \ldots, x_n$over S. Let ηbe fuzzy subring over R. For all

$$P = \sum_{i_n=0}^{m_n} \sum_{i_{n-1}=0}^{m_{n-1}} \ldots \sum_{i_1=0}^{m_1} c_{i_1 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n} \in R$$

where $c_{i_1 \ldots i_n} \in S$. Let

$$\eta(P) = \min\{\min\{\eta(c_{i_1 \ldots i_n}) \mid i_j = 0 \; to \; m_j, j = 1 \; to \; n\}, \min\{t_j \mid j = 1 \; to \; n\}\}$$

then $\eta$is called fuzzy polynomial ring over$R$, where

$$t_j = \begin{cases} \eta(x_j) if \eta(x_j) is nontrivial, \\ 1 \qquad otherwise. \end{cases}$$

3.9 Definition [41]

A fuzzy polynomial ring in one determinate on$F_\eta(R)$is a set of sequences$(a_{t_1}, a_{t_2}, a_{t_3}, \ldots) = (a_{t_k})_{k \in \mathbb{N}}$with$(a_{t_k}) \in F_\eta(R)$such that there exist$n \in \mathbb{N}$such that$\forall\, p \geq n,\ a_{t_p} = 0_s$with$t_i, s \in (0, 1]$. It is denoted by$F_\eta(R)[X]$.

Let$P = (a_{t_k})_{k \in \mathbb{N}}$ with$a_{t_p} = 0_s$ for all$p \geq n$and$Q = (b_{s_k})_{k \in \mathbb{N}}$ with$b_{t_p} = 0_s$for all $p \geq m$. The ring operations on the fuzzy polynomial ring are as follows:

Addition:$P + Q = (a + b)_{t_k \wedge s_k}$ with $(a + b)_{t_p \wedge s_p} = 0_s$ for all $p \geq \max\{n, m\}$

Multiplication:$P \times Q = (d_{\beta_k})_{k \in \mathbb{N}}$ with $d_{\beta_k} = \sum_{i+j=p} a_{t_i} b_{s_j}$, $\beta_k = min_{0 \geq i.j \geq k}\{t_i, s_j\}$and $d_{\beta_p} = 0_s$ for all $p > m + n$.

3.10 Theorem [41]

$(F_\eta(R)[X], +, \times)$ is a commutative ring.

3.11 Definition [41]
$P \in F_\eta(R)[X]$ is said to be fuzzy polynomial on $F_\eta(R)$if there exist$a_{t_i} \in F_\eta(R)$such that $P = \sum_{i=0}^{n} a_{t_i} X^i$.

3.12 Definition [41]
A polynomial $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + \cdots + a_{t_n}X^n$ is said to be non zero polynomial if there exist non zero coefficients $a_{t_0}, a_{t_1}, a_{t_3}, \dots, a_{t_n}$.

3.13 Definition [41]
Let $P = a_{t_0} + a_{t_1}X + a_{t_2}X^2 + \cdots + a_{t_n}X^n \in F_\eta(R)[X]$. The fuzzy degree of P is denoted by$\deg(P)$or$d^0$and is defined as the maximal number n such that$a_{t_n} \neq 0_{t_n}$. In this$a_{t_n}$is called as leading coefficient of P.

3.14 Definition [41]
The$\alpha_s \in F_\eta(R)$is said to be zero of the polynomial$P \in F_\eta(R)[X]$if and only if
$P(\alpha_s) = \sum_{i=0}^{n} a_{t_i} \alpha_s^i = 0_\beta, \beta \leq s$.

Let$I(b_t) = \{P \in F_\eta(R)[X] \mid P(b_t) = 0_s, s \leq t\}$. It can be easily shown that$I(b_t)$is ideal of$F_\eta(R)[X]$.

3.15 Definition [41]
The fuzzy point$b_t \in F_\eta(R)$is said be algebraic fuzzy point if$I(b_t) \neq \{0\}$. Otherwise, b_t is called a transcendent fuzzy point.

3.16 Theorem [41]
Let R be a ring. Then R is called integral domain if and only if$F_\eta(R)[X]$ is integral ring.

Kim [35] introduced the concept of a fuzzy polynomial ideal, denoted as η_x, within the polynomial ring$R[x]$. This fuzzy polynomial ideal is induced by a fuzzy idealη existing in the ring$R$, thereby establishing a relationship between fuzzy ideals in the base ring and the corresponding induced fuzzy polynomial ideals.

3.17 Theorem [35]
Let $\eta: R \to [0,1]$ be a fuzzy ideal of R, and consider $\eta_x: R[x] \to [0,1]$, a fuzzy subset of $R[x]$ defined by $\eta_x(f(x)) = \min_i\{\eta(a_i)\}$ for any polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$in $R[x]$. Then η_x is fuzzy ideal of $R[x]$.

3.18 Theorem [35]
Let $\eta: R \to [0,1]$ be a fuzzy ideal of R. Then the set $D = \{f(x) \in R[x] \mid \eta_x(f(x)) = \eta_x(0)\}$ is a subring of $R[x]$.

The intuitionistic fuzzy polynomial idealA_x of a polynomial ring$R[x]$induced by intuitionistic fuzzy ideal$A$of a ring$R$was introduced by Sharma [53].

3.19 Theorem [53]
Let$I = \{\langle x, \mu(x), \eta(x)\rangle \mid x \in R\}$be intuitionistic fuzzy ideal of$R$and let$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ be any polynomial in$R[x]$. Define an intuitionistic fuzzy ideal

$$I_x = \{\langle f(x), \mu_x(f(x)), \eta_x(f(x))\rangle \mid f(x) \in R[x]\}$$

of $R[x]$by$\mu_x(f(x)) = \min_i\{\mu(a_i)\}$ and $\eta_x(f(x)) = \max_i\{\eta(a_i)\}$. Then$I_x$is an intuitionistic fuzzy ideal of$R[x]$.

3.20 Theorem [53]
Let$I$be an intuitionistic fuzzy ideal of$R$. Then the set

$$S = \{f(x) \in R\,[x] \mid \mu_x(f(x)) = \mu_x(0), \eta_x(f(x)) = \eta_x(0)\}$$

is subring of$R[x]$.

4 Fuzzy Matrix

The concept of fuzzy matrices was initially introduced by Thomason [55], who also studied the convergence of powers of the fuzzy matrix. In this section, we explore the fuzzy matrix concept as studied by various authors, encompassing controllable fuzzy matrices, intuitionistic fuzzy matrices, and their associated determinant and adjoint properties. This comprehensive study holds significant importance in various areas of science and engineering.

4.1 Definition [55]
A fuzzy matrix is a matrix with its elements from the unit interval$[0, 1]$.

The following are the operations on fuzzy matrices,
Let$A = [a_{ij}]_{m \times k}$,$B = [b_{ij}]_{k \times n}$and $C = [c_{ij}]_{k \times n}$be any two fuzzy matrices with$a_{ij}, b_{ij}, c_{ij} \in [0, 1]$then,

i. $B + C = [d_{ij}]$ where $d_{ij} = \max\{b_{ij}, c_{ij}\}$,
ii. $AB = \sum_{p=1}^{k} a_{ip}\, b_{pj}$ where $a_{ip}\, b_{pj} = \min\{a_{ip}, b_{pj}\}$,
iii. $B \leq C$ if $b_{ij} \leq c_{ij}$ for all i and j.

4.2 Definition [28]
If A is a fuzzy square matrix of order $m \times m$, then

i. $A$is symmetric if and only if $A = A^t$,
ii. A is reflexive if and only if $A \geq I_n$,
iii. $A$is transitive if and only if $A^2 \leq A$.

Hence A is called reflexive, symmetric, and transitive (or idempotent) if and only if $A^2 = A$.

Kim [33] presented several properties related to the determinant and adjoint of a fuzzy square matrix

4.3 Definition [33]
The determinant of a fuzzy matrix$A$ is denoted by $|A|$and defined as,

$$|A| = \sum_{\sigma \in S_k} a_{1\sigma(1)} a_{2\sigma(2)} a_{3\sigma(3)} \ldots a_{k\sigma(k)}$$

where $S_k$is the permutation group of all permutations.

4.4 Example
Let $A = \begin{bmatrix} 0.3 & 0.2 & 0.5 \\ 0.5 & 0.6 & 0.1 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}$be fuzzy matrix then$|A| = 0.3(0.3 + 0.1) + 0.2(0.3 + 0.1) + 0.5(0.4 + 0.2) = 0.3(0.3) + 0.2(0.3) + 0.5(0.4) = 0.3 + 0.2 + 0.4 = 0.4$

4.5 Definition[33]

The adjoint of a fuzzy square matrix is denoted by $Adj\ A$ and defined as, $Adj\ A = [|A_{ij}|]_{m \times m}$, where $|A_{ij}|$ is a determinant obtained by deleting i^{th} row and j^{th} column of matrix A.

Kim and Roush [32] conducted a study on the canonical form of an idempotent matrix in the years 1980. In a related context, Hashimoto [27] investigated the canonical form of a transitive matrix in the year 1983.

4.6 Theorem [27]
If R is idempotent fuzzy matrix (transitive, strongly transitive), then there exist a permutation matrix P such that the matrix $T = [t_{ij}] = P \times R \times P'$ satisfies $t_{ij} \geq t_{ji}$, for $i > j$.

In the above matrix, T is referred to as the canonical form of an idempotent fuzzy matrix (transitive, strongly transitive).

Xin [57] studied the controllable fuzzy matrix as below:

4.7 Definition [57]
A fuzzy matrix R is said to be controllable from below (above), if there exist a permutation matrix P such that $T = [t_{ij}] = P \times R \times P'$ satisfies $t_{ij} \geq t_{ji}$ $(t_{ij} \leq t_{ji})$ for $i > j$.

A fuzzy matrix $R = [r_{ij}]$ is said to be controlled from below (above), if $r_{ij} \geq r_{ji}$ $(r_{ij} \leq r_{ji})$ for $i > j$.

Intuitionistic fuzzy matrix deals with both membership and non-membership value. Atanassov [6] has first introduced the concept of intuitionistic fuzzy matrix which is the extension of fuzzy matrix.

4.8 Definition [43]
Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be two fuzzy matrices such that $a_{ij} + b_{ij} \leq 1$ for every $i \leq m$ and $j \leq n$. The pair $\langle A, B\rangle$ is called intuitionistic fuzzy matrix.

5 Fuzzy Zero Divisor
In this section, various methodologies for defining fuzzy zero divisors are examined. The introduction of the concept of fuzzy zero divisors for rings was initially put forth by Ray [47] in 2004. Ray also introduced the notions of fuzzy characteristics and fuzzy unit within the ring, extending the idea of fuzzy order for group elements introduced by Kim [34].

5.1 Definition [34]
Let μ be a fuzzy subgroup of G. For a given $x \in [0, 1]$, if there exists a smallest positive integer n such that $\mu(x^n) = \mu(e)$, then x has a fuzzy order of n with respect to μ. If no such n exists, x is of infinite order with respect to μ.

For any $x, y \in G$, let the fuzzy order of elements x and y be denoted by $FO_\mu(x)$ and $FO_\mu(y)$, respectively. If $O(x)$ and $O(y)$ are the orders of elements x and y in the group G, respectively, then $O(x) = O(y)$ does not imply that $FO_\mu(x) = FO_\mu(y)$, as shown in the following example.

5.2 Definition [47]

Let η be a fuzzy subring of a ring R. If $\eta(mx) = \eta(0)$ for all x in R and the smallest positive integer m, then m is said to be the fuzzy characteristic of the ring R with respect to η. If $m = 0$ is the only integer such that $\eta(mx) = \eta(0)$ for all x in R, then the fuzzy characteristic of ring R is 0 (or infinity).

In general, the fuzzy characteristic is not necessarily equal to the ring characteristic. However, if the set $\{x \in R \mid \eta(x) = \eta(0)\} = \{0\}$, then $FC_\eta(R)$ is equal to the characteristic of the ring R.

5.3 Theorem[47]
If η is a fuzzy ideal and e is the identity of ring R, then $FC_\eta(R) = FO_\eta(e)$.

5.4 Theorem[47]
Let η be a fuzzy subring of R, and let $a, b \in R$. If $\eta(ab) \neq \eta(0)$ and $\eta(a) \neq \eta(0)$, $\eta(b) \neq \eta(0)$, then a is referred to as the left fuzzy divisor of b, and b is referred to as the right fuzzy divisor of a with respect to η.

If $\{x \in R \mid \eta(x) = \eta(0)\} = \{0\}$ then the fuzzy zero divisors with respect to η are equal to zero divisors of ring.

5.5 Theorem[47]
Let η be a fuzzy ideal of a ring R, and let $a, b \in R$. If η has no fuzzy zero divisors with respect to η and $\eta(a) \neq \eta(0), \eta(b) \neq \eta(0)$, then $FO_\eta(a) = FO_\eta(b)$ in the additive group of R.

5.6 Theorem[47]
Let ηbe a fuzzy ideal of a ring R with no fuzzy zero divisors with respect to η and $FC_\eta(R) \neq 0$. If $\eta(a) \neq \eta(0)$ for some $a \in R$, then $FC_\eta(R)$ is prime number.

5.7 Corollary[47]
Let η be a fuzzy ideal of the ring R with no fuzzy zero divisors with respect to η then $FC_\eta(R) = 0$ if and only if $FO_\eta(a)$ is infinite for every$a \in R$with $\eta(a) \neq \eta(0)$. Otherwise $FC\ (R)$ and $FO_\eta(a)$ is prime p for every $a \in R$ with $\eta(a) \neq \eta(0)$.

5.8 Definition [47]
Suppose η be a fuzzy subring of R with identity 1 such that $0 \neq \eta(1) \neq \eta(0)$. An element $a \in R$ with $\eta(a) \neq \eta(0)$ is said to be fuzzy unit of R denoted by $FU_\eta(R)$ if there exist $b \in R$ with $\eta(b) \neq \eta(0)$ such that $\eta(ab) = \eta(ba) = \eta(1)$.

5.9 Definition [42]
Letη_G and$\eta_H$be fuzzy subsets of the sets$G$and$H$respectively, then the product of$\eta_G$and η_H, denoted by$\eta_G \times \eta_H$ and defined by

$$(\eta_G \times \eta_H)(x, y) = \min\{\eta_G(x), \eta_H(x)\},$$

for all $x \in G$ and $y \in H$.

5.10 Theorem[47]
If η_R and η_S are fuzzy subrings of the rings$R$and$S$, respectively, then

$$FC_{\eta_R \times \eta_S}(R \times S) = \begin{cases} 0 & if\ FC_{\eta_R}(R) = 0\ and\ FC_{\eta_S}(S) = 0, \\ p & if\ FC_{\eta_R}(R) \neq 0\ and\ FC_{\eta_S}(S) \neq 0, \end{cases}$$

where, $p = lcm\left\{FC_{\eta_R}, FC_{\eta_S}\right\}$.

Pu and Liu[44] introduced the concept of fuzzy points. Building upon this idea, Melliani[41]proposed the fuzzy ring $\left(F_\eta(R), +, \times\right)$ using the notion of fuzzy points, where ηrepresents a fuzzy subring of R. Subsequently, Melliani presented the notion of zero divisors within this fuzzy ring $\left(F_\eta(R), +, \times\right)$, termed fuzzy zero divisors. Additionally, he introduced the concept of an integral ring within this framework.

5.11 Definition[44]
Let Abe a non-empty set. For $x \in A$ and $\alpha \in (0,1]$, define $x_\alpha : A \to [0,1]$ such that

$$x_\alpha(y) = \begin{cases} \alpha if x = y, \\ 0 \quad if x \neq y. \end{cases}$$

Then, x_α is called a fuzzy point(singleton).

5.12 Definition[44]
The fuzzy point$x_\alpha$is said to be contained in a fuzzy set ηor belongs to η, denoted by $x_\alpha \in \eta$ if and only if$\eta(x) \geq \alpha$. Generally, every fuzzy set ηcan be expressed as the union of all fuzzy points belonging to η.

By the principal extension of Zadeh, we have

$$x_t + y_s = (x + y)_{t \wedge s},$$
$$x_t y_s = (xy)_{t \wedge s}.$$

The set of all fuzzy points of η of ring R is denoted by $F_\eta(R)$ and defined as

$$F_\eta(R) = \{x_\alpha \mid \eta(x) \geq \alpha, x \in R, \alpha \in (0,1]\}.$$

5.13 Theorem[41]
Let η be a fuzzy subset of$R$, then η is a fuzzy subring of R if and only if η_t is a subring of R for each $t \in [0, \eta(0)]$.

5.14 Theorem[41]
Let η be a fuzzy subset of R. Then η is a fuzzy subring of$R$if and only if, for each point $x_t y_s \in \eta$, we have $x_t - y_s \in \eta$ and $x_t y_s \in \eta$.

5.15 Theorem[41]
Let Rring with unity. If η is subring of$R$then $(F_\eta(R, +, \times)$ is ring.

5.16 Theorem[41]
Let$R$be a commutative ring with unity. Let η and ν be two fuzzy subrings of R such that $\eta \subset \nu$. Then $F_\eta(R)$ is a subring of $F_\nu(R)$.

5.17 Theorem[41]
Let $a_t(\neq 0_t) \in F_\eta(R)$. Then a_t is called fuzzy zero divisor if there exist$b_s(\neq 0_s) \in F_\eta(R)$ such that $a_t b_s = 0_{t \wedge s}$.

5.18 Definition[41]
Ring $F_\eta(R)$ is said to be an integral ring if it has no zero divisors.

5.19 Theorem[41]

$F_\eta(R)$ is integral ring if and only if R is an integral domain.

Ayub[7]presented an additional approach to fuzzy zero divisors, and subsequently, the concept of a fuzzy integral domain is explored.

5.20 Definition [7]
Let η be a fuzzy subring of R. A non zero and non unit element a is called fuzzy zero divisor over η if there exist non zero and non unit element b such that $\eta(ab) = \eta(0)$.

5.21 Definition [7]
Let R be an integral domain, and η is a fuzzy subring of R. Then η is called the fuzzy integral domain of R if there is no fuzzy zero divisor over η. The set of all fuzzy integral domains of R is denoted by $FID(R)$.

From now, R is assumed to be an integral domain. R^* denotes the set of all units of R. Also $\eta_* = \{x \in R \mid \eta(x) = \eta(0)\}$.

5.22 Theorem[7]
Let η be a fuzzy subring of R. Then the following conditions are equivalent

i. $\eta \in FID(R)$,
ii. $\eta(x) \neq \eta(0)$ for all non zero and non unit $x \in R$,
iii. $\eta_* \subseteq \{0\} \cup R^*$.

6 Applications
A fuzzy set proves to be a valuable tool in decision-making processes. Sun[22] introduced a multi-level comprehensive fuzzy evaluation function that analyzes environmental issues, management skills, and economic benefits along with biophysical suitability. This function aids in making informed recommendations for land usage. The practical applications of fuzzy set theory extend into the medical field. Sanchez[51] developed a diagnostic model using a fuzzy matrix, which encapsulates medical knowledge regarding symptoms and diseases. Meenakshi[40] further expanded on Sanchez's approach by presenting a theory of interval-valued fuzzy matrices, providing an extension to medical diagnosis methods. Meenakshi also introduced the arithmetic mean matrix of an interval-valued fuzzy matrix and directly applied Sanchez's medical diagnosis method to it. Kavitha[29] introduced the concept of a circulant fuzzy matrix and demonstrated its application in animal disease diagnosis.

The concept of a fuzzy zero divisor can be used to define a fuzzy zero divisor graph. Kuppan[36] initiated the study of fuzzy zero divisor graphs, which have various applications in graph theory. One significant application is in pharmaceutical chemistry, where the challenge lies in providing a mathematical graphical representation for a set of chemical compounds such that distinct representations correspond to distinct compounds. Additionally, in computer networks, the structure of fuzzy zero divisor graphs can be applied, representing servers, hubs, and nodes as vertices and connections as edges. Moreover, fuzzy zero divisor graphs have potential applications in navigation, robotics, and coding theory.

This section presents a comprehensive examination of two case studies, centered on medical diagnosis and agricultural analysis. The initial study, conducted by Beaula[12], introduces an algorithm designed for medical diagnosis. Subsequently, the second case study elaborates on the practical implementation of $(\in, \in \vee Q)$ fuzzy ideals. This approach aids in the selection of effective symptoms and streamlines the diagnosis of diseases[31].

6.1 Case Study 1

Fuzzy algebra finds significant applications in the field of medicine, particularly in diagnostic models. Sanchez[52] pioneered the use of fuzzy matrices to represent medical knowledge relating symptoms to diseases. Building on this work, Meenakshi and Kaliraja[40]further developed the approach by introducing interval-valued fuzzy matrices. In 2010, Cagman et al.[14] introduced fuzzy soft matrix theory and its application in decision making.

This section presents a case study by Beaula[12] that illustrates the practical application of fuzzy matrices in medical diagnosis.

6.1.1 Definition[12]

Let $[a_{ij}]_{n\times n}$ and $[b_{ij}]_{n\times n}$be the fuzzy matrices of order $n\times\ n$. Then the $supi -$ composition is defined as,$A^i \circ B = \sup i\ (A(x,y)\ B(x,y))$

6.1.2 Definition [12]

A relativity function between variables u and v \in X, denoted by $g\left(\frac{u}{v}\right)$, is defined as:

$$g\left(\frac{u}{v}\right) = \frac{\eta_v(u) - \eta_u(v)}{\max\{\eta_v(u), \eta_u(v)\}}$$

where $\eta_v(u)$ and $\eta_u(v)$ are membership functions mapping from u to v and v to u, respectively.

6.1.3 Procedure of medical diagnosis under a fuzzy environment

Let S denote the set of symptoms associated with certain diseases,D represent the set of diseases, and P the set of patients.

1) Begin by considering the patient-symptom fuzzy matrix A.
2) Next, consider the symptom-disease fuzzy matrix$B$.
3) Compute$C = A^i \circ B$, where∘represents the sup $i -$ composition operation.
4) Find the complement fuzzy matrices A^c and B^c for A and B, respectively.
5) Calculate $D = A^{c^i} \circ B^c$and $M\ =\ C\ -\ D$, using$-$ for the min operator and $\circ$ for the $sup\ i -$ composition operation.
6) Determine the relative values and construct the comparison matrix $R = [r_{ij}]_{n\times n} = g\left(\frac{x_i}{y_i}\right)_{i=1,2,\ldots,n}$.
7) Identify the maximum value in each row of the R matrix for ranking purposes, providing the solution to the problem at hand.

6.1.4 Illustrative Example

Let $P = \{P_1,\ P_2,\ P_3\}$ be the set of patients with symptoms$S = \{High\,Temp.(s_1), Headache(s_2), Cough(s_3)\}$, and possible diseases related to these symptoms$D = \{Dengue(d_1),\ Viral\ Fever(d_2),\ Malaria(d_3)\}$.

1) Consider the patient-symptom fuzzy matrix:

$$A = \begin{bmatrix} 0.5 & 0.6 & 0.4 \\ 0.2 & 0.7 & 0.3 \\ 0.6 & 0.4 & 0.7 \end{bmatrix}$$

2) Consider the symptom-disease fuzzy matrix:

$$A = \begin{bmatrix} 0.4 & 0.2 & 0.7 \\ 0.3 & 0.6 & 0.9 \\ 0.7 & 0.5 & 0.8 \end{bmatrix}$$

3) Compute $C = A^i \circ\ B$, where∘is $sup\ i -$ composition:

$$C = \begin{bmatrix} 0.3 & 0.4 & 0.5 \\ 0.2 & 0.4 & 0.6 \\ 0.5 & 0.4 & 0.6 \end{bmatrix}$$

4) The complement of fuzzy matrices A and B are:
$$A^c = \begin{bmatrix} 0.5 & 0.4 & 0.6 \\ 0.8 & 0.3 & 0.7 \\ 0.4 & 0.6 & 0.3 \end{bmatrix}, B^c = \begin{bmatrix} 0.6 & 0.8 & 0.3 \\ 0.7 & 0.4 & 0.1 \\ 0.3 & 0.5 & 0.2 \end{bmatrix}$$
5) Compute $D = A^{c^i} \circ B^c$:
$$D = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.5 & 0.6 & 0.2 \\ 0.4 & 0.3 & 0.1 \end{bmatrix} \text{ and } M = C - D = \begin{bmatrix} 0.3 & 0.4 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.4 & 0.3 & 0.1 \end{bmatrix}$$
6) Calculate relative values$g\left(\frac{p_i}{d_i}\right)$ for all i,
$$g\left(\frac{p_1}{d_1}\right) = \frac{\eta_{d_1}(p_1) - \eta_{p_1}(d_1)}{\max\{\eta_{d_1}(p_1),\ \eta_{p_1}(d_1)\}} = \frac{0.3 - 0.3}{\max\{0.3, 0.3\}} = 0$$
Similarly, we can calculate, $g\left(\frac{p_1}{d_2}\right) = 0.5$, $g\left(\frac{p_1}{d_3}\right) = -0.5$, $g\left(\frac{p_2}{d_1}\right) = -0.5$, $g\left(\frac{p_2}{d_2}\right) = 0$, $g\left(\frac{p_2}{d_3}\right) = -0.6$, $g\left(\frac{p_3}{d_1}\right) = 0.5$, $g\left(\frac{p_3}{d_2}\right) = 0.3$ and $g\left(\frac{p_3}{d_3}\right) = 0$.
Hence, $R = \begin{bmatrix} 0 & 0.5 & -0.5 \\ -0.5 & 0 & -0.6 \\ 0.5 & 0.3 & 0 \end{bmatrix}$ and the maximum i^{th}row is $\begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$
7) Therefore, it can be concluded that patientP_1 is highly susceptible to**Viral Fever**, patient P_2 is affected by**Viral Fever**, and patient$P_3$is likely affected by**Dengue**.

6.2 Case Study 2

The theory of $(\in, \in\vee\, Q)$ fuzzy ideals is employed in the domain of medical diagnosis systems. This section details the practical application of $(\in, \in\vee\, Q)$fuzzy ideals for the purpose of selecting effective symptoms and facilitating the diagnosis of diseases [31]. The conclusions drawn from the analysis of 15 cases are summarized to underscore the robustness and effectiveness of the application in medical diagnostics.

6.2.1 Definition [31]

In the defined operations, where $a \vee b$ denotes the supremum (least upper bound) and $a \wedge b$ denotes the infimum (greatest lower bound), a partially ordered set (poset) $(L; \leq)$is termed a lattice if, for any elements a and b in L, both $a \vee b$ and $a \wedge b$ exist within the set L.

6.2.2 Definition [31]

A fuzzy subset μ is said to be an $(\in, \in \vee\, q)$ fuzzy ideal of X if and only if the following conditions are hold for all $x, y \in X$.

i. $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$,
ii. $\mu(x \vee y) \geq \mu(x) \wedge \mu(y) \wedge 0.5$,
iii. $x \wedge a = x$ implies $\mu(x) \geq \mu(a) \wedge 0.5$ for all $a \in X$.

A sixty-year-old male patient presenting with a complaint of burning micturation, the doctor considered potential underlying conditions, including Urinary Tract Infection (UTI) (P1), Renal Stone (P2), and Diabetes (P3). To confirm the diagnosis, the doctor recommended a comprehensive evaluation through blood tests, urine analysis, and an abdominal ultrasound (USG). Subsequently, the obtained reports were subjected to a process of fuzzification for further analysis and interpretation.

SN.	Test	Values	Normal Values	A_i	τ_1 (P_1)	τ_2 (P_2)	τ_2 (P_3)	$\mu(A_i)$	λ_1 $(\tau_1(P_1))$	λ_2 $(\tau_2(P_2))$	λ_3 $(\tau_3(P_3))$
1	Haemoglobin (gm %)	6.50	13-15	0.81	0.2	0	0.1	0.81	0.5	0.5	0.5
2	ESR mm	34	0-14	0.77	0.2	0.1	0.2	0.77	0.5	0.5	0.5
3	Urea (mg/dl)	27.50	20-40	0	0	0	0	0.5	0.5	0.5	0.5
4	RBSL (mb/dl)	92.50	90-110	0	0	0	0	0.5	0.5	0.5	0.5
5	Calcium (mg %)	7.69	9-10.4	0.33	0.1	0	0.1	0.5	0.5	0.5	0.5
6	Urine(R) protiens	Trace	Trace/Absent	0	0	0	0	0.5	0.5	0.5	0.5
7	Urine culture	Abnormal*	Normal/Abnormal	1	1	0	0	1	1	0.5	0.5
8	USG abdomen	Abnormal**	Normal/Abnormal	1	0.8	0	0	1	0.8	0.5	0.5

* Abnormal due to organism
** Abnormal due to internal echoes
The fuzzy sets defined for various test values are given below,

$$A_1(x) = \begin{cases} 1 & if x < 5 \\ \frac{13-x}{8} & if\ 5 \le x \le 13 \\ 0 & if x > 13 \end{cases}, \qquad A_2(x) = \begin{cases} 0 & if x < 14 \\ \frac{x-14}{26} & if\ 14 \le x \le 40 \\ 1 & if x > 40 \end{cases}$$

$$A_3(x) = \begin{cases} 0 & if x < 40 \\ \frac{x-40}{60} & if\ 40 \le x \le 100, \\ 1 & if x > 100 \end{cases} \qquad A_4(x) = \begin{cases} 0 & if x < 110 \\ \frac{x-110}{90} & if\ 110 \le x \le 200 \\ 1 & if x > 200 \end{cases}$$

$$A_5(x) = \begin{cases} 1 & if x < 5 \\ \frac{9-x}{4} & if\ 5 \le x \le 9, \\ 0 & if x > 9 \end{cases} \qquad A_6(x) = \begin{cases} 1 & if absent \\ 0 & if traced \end{cases}'$$

$$A_7(x) = \begin{cases} 1 & if abnormal \\ 0 & if normal \end{cases}, \qquad A_8(x) = \begin{cases} 1 & if abnormal \\ 0 & if normal \end{cases}$$

The values $\left(\tau_i(P_i)\right)_j$ for $i = 1\ to\ 3$ and $j = 1\ \ to\ 8$ represent the grade membership values provided by experts. These values indicate the degree to which the test is deemed useful in identifying diseases, as assessed by experts relying on their knowledge and experience.

Define fuzzy sets$\mu\left(A_j(x)\right) = A_j(x) \vee 0.5$and $\lambda\left(\tau_i(P_i)_j\right) = \tau_i(P_i)_j \vee 0.5$ for all$x$and $i = 1\ to\ 3$and$j = 1\ to\ 8$. Clearly$\mu$and$\lambda$are$(\in, \in\vee\, q)$ −fuzzy ideals of X. Hence$\mu \cap \lambda$is$(\in, \in\vee\, q)$ −fuzzy ideals of$X$.

Define$\gamma_i(P_i) = \vee_{j=1}^{8} \left(\mu(A_j) \wedge \lambda_i(P_i)_j\right)$for all$i = 1\ to\ 3andj = 1\ to\ 8$. Thus$\gamma_1(P_1) = 1$, $\gamma_2(P_2) = 0.1$and $\gamma_3(P_3) = 0.2$.

Hence, the analysis indicates that urinary tract infection (UTI) has the highest grade of membership, leading to the conclusion that the patient is indeed experiencing a urinary tract infection. This finding aligns with the medical opinion provided by the doctor.

7 Limitation of study

This study faces limitations due to the absence of a standardized process for the fuzzification of algebraic structures. These limitations encompass several challenges associated with the process of fuzzification:

1) **Complexity of transition:** Converting crisp concepts into fuzzy ones while preserving their essential properties poses a significant challenge. This complexity stems from the necessity to maintain the core characteristics of the original concept while adapting it to a fuzzy framework.
2) **Preservation of meaning:** Fuzzifying a concept from a classical set to a fuzzy set without altering its meaning in the classical set is a difficult task.

3) **Computational complexity:** The fuzzification process often involves intricate computations, potentially impacting the scalability and efficiency of fuzzy systems. Managing this computational complexity is crucial to ensure the practicality and usability of fuzzy algebraic techniques.
4) **Loss of precision:** Fuzzification entails approximating crisp concepts with fuzzy ones, leading to a loss of precision. This loss of precision can affect the accuracy of computations, and the reliability of results obtained using fuzzy algebraic structures.
5) **Limited theoretical framework:**The theoretical framework for fuzzy algebraic structures is still under development. This limited theoretical foundation can make it challenging to rigorously analyze and prove properties of fuzzy algebraic structures.

Addressing these limitations involves ongoing research and development efforts aimed at refining fuzzy algebraic techniques, enhancing computational methods, and establishing standardized approaches to ensure consistency and reliability in fuzzy systems.

8 Conclusion and Future Scope

A comprehensive study was conducted by analyzing research papers published in reputable journals from 1965 to 2024, focusing on the structures and applications of fuzzy subrings, fuzzy zero divisors, fuzzy polynomials, and fuzzy matrices. The examination of these papers provided valuable insights into the evolution of these concepts and their current state in research, contributing to a deeper understanding of fuzzy sets' role in ring theory.

The analysis revealed various approaches for defining and studying fuzzy subrings, fuzzy zero divisors, fuzzy polynomials, and fuzzy matrices, highlighting the complexity and challenges involved in fuzzifying algebraic structures. The study emphasized the need for standardized processes in fuzzification to ensure consistency and reliability in fuzzy systems.

Furthermore, the research highlighted the practical applications of fuzzy subrings/ideals and fuzzy matrices in agriculture and the medical field. These applications demonstrate the potential impact of fuzzy algebraic techniques in real-world scenarios, opening up new avenues for research and development in these fields.

In conclusion, this survey offers valuable insights into recent advances in the study of fuzzy algebraic structures and their practical applications. The findings have the potential to enrich the exploration of fuzzification in the algebraic properties of rings, particularly in areas related to zero divisors. These areas include the analysis of zero divisor graphs, the study of annihilator properties of rings, the investigation of Baer rings, principal projective rings, and projective socle rings, as well as the development of error-correcting codes in coding theory. By refining fuzzy algebraic techniques, improving computational methods, and establishing standardized approaches for fuzzification, future research holds promise for further advancing the field and expanding its practical applications.

This study lays the groundwork for future research in fuzzy algebra and its applications, providing a foundation for exploring new theoretical frameworks and practical implementations in diverse fields.

Ethics declarations

Conflicts of interest: The authors declare that they have no conflict of interest.

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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