Effect of Interfacial Tension on Internal Waves in two Superposed Liquids against a Vertical Cliff

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Abstract: A simple and straightforward reduction method is described in this analysis to obtain the linear solution of the obliquely incident incoming waves towards a rigid vertical cliff in two superposed liquids. The Analytical expressions for the velocity potentials in each of the liquids are obtained considering the effect of interfacial tension where the lower liquid is of finite constant depth and the upper liquid is of finite constant height. Various known results are recovered as special cases of the general problem considered here.

Key words: linear theory, irrotational flow, velocity potential, interfacial tension, superposed liquids.
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1. Introduction:

For the vertical cliff problem, waves are supposed to be propagating towards the cliff and at the cliff, the normal component of velocity is assumed to be zero. Physically one would then expect full reflection of waves by the cliff, but that is not correlative with the incoming nature of the waves towards the cliff at infinity. This is accounted for by the assumption of a source/sink type nature in the velocity potential at origin which requires a logarithmic singularity in the potential function at the origin (cf. [1]) in the absence of surface tension effect. However, if the effect of surface tension is considered, this requirement of logarithmic singularity of the potential function at the origin is not necessary, since the wave amplitude remains finite there (cf. [2]).

The objective of the present analysis is to find mathematically, the linear solution for the three dimensional problem of incoming waves, at the interface of two superposed liquids, progressing towards a rigid vertical cliff in the presence of interfacial tension. If no reflection of waves by the cliff is assumed, then it follows that the cliff bound wave carries certain energy with it and is totally reflected back, as there is no mechanism to absorb the incoming energy in an inviscid fluid system. However existing literature on problems involving two superposed liquids is rather complicated because of the coupled boundary conditions at the interface of the two liquids. Since then, some attempts have been taken to study the water wave problems in a single liquid or in a two layered liquid media and few of its generalization by applying different mathematical techniques (cf. [1]-[13]).

The present investigation is concerned with the study of internal waves, incident obliquely, on a rigid vertical cliff, in two superposed liquids using the fact that the liquids are bounded on the left by the cliff where the lower liquid is of uniform finite depth $'a'$ and the upper liquid is of uniform finite height $'b'$. The novelty of the present study is based on the fact that the effect of interfacial tension at the interface of the two liquids is taken into consideration. By applying a simple and straight forward reduction procedure, the analytical expressions for the velocity potentials in each of the two liquids are obtained. In the absence of the upper liquid, various results are obtained and identified with the known results.

2. Formulation of the Problem:

Consider the three-dimensional motion of two inviscid, incompressible, immiscible liquids of densities ρ_1 and $\rho_2(\rho_1 > \rho_2)$, of the lower and upper liquid respectively, under the action of gravity and separated by a common interface. A rectangular cartesian co-ordinate system is chosen in which the y-axis is directed vertically downwards into the lower liquid, the plane $y = 0, x > 0$ is the mean position of the interface, $x = 0$ is the rigid cliff, and the two liquids occupy the regions $x > 0\,, 0 < y \leq a\,, -\infty < z < \infty$ and $x > 0\,, -b \leq y < 0\,, -\infty < z < \infty$ respectively. The origin is taken as the point of intersection of the rigid cliff with the mean interface.

Usual assumptions on linear theory ensure the existence of velocity potentials $\Phi_1(x, y, z, t)$ and $\Phi_2(x, y, z, t)$ for the lower and upper liquid respectively which represent progressive waves moving towards the z-axis, such that the wave crests at a large distance from the shore tend to straight lines which make an arbitrary angle α with the shore line.

Thus for periodic motion, we may write

$$\left.\begin{array}{l} \Phi_1(x, y, z, t) = Re[\ \phi_1(x, y) exp\{-i(\sigma t + \mu z)\}] \\ \Phi_2(x, y, z, t) = Re[\ \phi_2(x, y) exp\{-i(\sigma t + \mu z)\}] \end{array}\right\} \qquad (2.1)$$

where $\mu = \beta_0 \sin \alpha$ and β_0 is the unique positive real root of the transcendental equation $K(coth\, ax + s\, coth\, bx) - x(1 - s) - Mx^3 = 0$and σ is the angular frequency.

Thus the problem under consideration can be investigated by way of determining the potentials $\phi_1(x, y)$ and $\phi_2(x, y)$ satisfying the following boundary value problems given below:

(i) Two-dimensional modified Helmholtz's equations:

$$\left.\begin{array}{l} (\nabla^2 - \mu^2)\phi_1 = 0 \\ (\nabla^2 - \mu^2)\phi_2 = 0 \end{array}\right\} \qquad (2.2)$$

in the respective flow domain, where ∇^2 is the two-dimensional Laplacian.

(ii) Linearized form of the interface conditions:

$$\left.\begin{array}{c} \phi_{1_y} = \phi_{2_y} \\ K\phi_1 + \phi_{1_y} - s\left(K\phi_2 + \phi_{2_y}\right) + M \begin{cases} \phi_{1_{yyy}} \\ \phi_{2_{yyy}} \end{cases} = 0 \end{array}\right\} on\ y = 0, x > 0 \qquad (2.3)$$

where $K = \sigma^2/g$ is the wave number, g is the acceleration due to gravity, $s = \rho_2/\rho_1$and $M = T/(\rho_1 g)$, T being the coefficient of interfacial tension.

(iii) Conditions at the rigid bottom and top:

$$\left.\begin{aligned} \phi_{1_y} &= 0, \quad on \ \ y = a \\ \phi_{2_y} &= 0, \quad on \ \ y = -b \end{aligned}\right\} \ \ x > 0. \tag{2.4}$$

(iv) Conditions on the rigid vertical cliff $x = 0$:

$$\left.\begin{aligned} \phi_{1_x} &= 0, \quad 0 < y \leq a \\ \phi_{2_x} &= \ 0, \quad -b \leq y < 0. \end{aligned}\right\} \tag{2.5}$$

Following Packham [2], we can adopt

$$\phi_1, \phi_2 \sim \ln r \quad as \ \ r = (x^2+y^2)^{1/2} \to 0 \tag{2.6}$$

so long as $T > 0$.

As ϕ_1, ϕ_2 behave as incoming progressive waves at infinity moving towards the cliff, following Gorgui and Kassem [3], we may assume as $x \to \infty$:

$$\left.\begin{aligned} \phi_1 &\sim \ \ \frac{\cosh \beta_0(a-y)}{\sinh a\beta_0} \exp(-i\omega x) \\ \phi_2 &\sim -\frac{\cosh \beta_0(b+y)}{\sinh b\beta_0} \exp(-i\omega x) \end{aligned}\right\} \tag{2.7}$$

where $= \beta_0 \cos \alpha$.

3. Solution of the Problem:

To solve the problem mathematically, we use a procedure which involves the reduction of the original boundary value problem described by (2.2) - (2.6) together with the infinity requirements (2.7), to another boundary value problem. To do this let us introduce the functions $\psi_1(x, y)$, $\psi_2(x, y)$ by:

$$\left.\begin{aligned} \phi_1(x,y) &= \ \ \frac{2\cosh \beta_0(a-y)}{\sinh a\beta_0} \cos \omega x \ + \psi_1(x,y) \\ \phi_2(x,y) &= -\frac{2 \ \cosh \beta_0(b+y)}{\sinh b\beta_0} \cos \omega x + \psi_2(x,y) \end{aligned}\right\} \tag{3.1}$$

where ψ_1, ψ_2 satisfy the partial differential equation (2.2) along with the conditions given by (2.3) – (2.6) and
as $x \to \infty$

$$\left.\begin{aligned} \psi_1 &\to -\frac{cosh\, \beta_0(a-y)}{sinh\, a\beta_0} exp(i\omega x) \\ \psi_2 &\to \ \ \frac{cosh\, \beta_0(b+y)}{sinh\, b\beta_0} exp(i\omega x). \end{aligned}\right\} \tag{3.2}$$

(3.2) implies that ψ_1, ψ_2 represent outgoing wave.

ψ_1, ψ_2 can also be expressed in the form of integrals as follows:

$$\left.\begin{aligned}\psi_1(x,y) &= L\oint_{\beta_0}^{\infty}\frac{k\,cosh\,k(a-y)\,sinh\,bk\cos\gamma x}{\gamma\,\lambda(k)}dk\\ \psi_2(x,y) &= -L\oint_{\beta_0}^{\infty}\frac{k\,cosh\,k(b+y)\,sinh\,ak\cos\gamma x}{\gamma\lambda(k)}dk\end{aligned}\right\} \tag{3.3}$$

where $\gamma = (k^2-\mu^2)^{1/2}$ and

$\lambda(k) = \{k(1-s)+Mk^3\}\sinh ak\sinh bk - K\,(\cosh ak\sinh bk + s\cosh bk\sinh ak)$.

It is to be noted here that $\mu(k)$ has a simple pole at at $k=\beta_0 > 0$ (say), a simple pole at k = $\beta^{/}$ < 0 (say), and an infinite number of complex poles whose real parts are positive of the form $s\delta_n \pm i\beta_n$ (cf. [8]). Here the contour is indented below the pole at $k=\beta_0$ to account for the outgoing nature of ψ_1, ψ_2 as $x\to\infty$, and L is a constant to be determined such that the conditions given by (3.2), at infinity, are satisfied. By simple manipulation, it can be shown that ψ_1, ψ_2 given by (10) satisfy both the the interface conditions given by (2.3). It may be noted here that the above form of ψ_1, ψ_2 remains finite as $r\to 0$, so long as $\tau > 0$ (cf. [2]).

Evaluating for ψ_1, ψ_2 given by (3.3), we obtain

$$\left.\begin{aligned}\psi_1(x,y) = 2\pi iL\Bigg[&\frac{1}{cos\,\alpha}\frac{f_1(\beta_0)}{f_2(\beta_0)}\,cosh\,\beta_0(a-y)\,sinh\,b\beta_0\,exp(i\omega x)\\ &+\sum\frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)}cosh\,\xi(a-y)\,sinh\,b\xi\,exp(i\omega_1 x)\\ &-\sum\frac{\bar{\xi}f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})}cosh\,\bar{\xi}(a-y)\,sinh\,b\bar{\xi}\,exp(-i\omega_2 x)\Bigg]\\ \psi_2(x,y) = -2\pi iL\Bigg[&\frac{1}{cos\,\alpha}\frac{f_1(\beta_0)}{f_2(\beta_0)}\,cosh\,\beta_0(b+y)\,sinh\,a\beta_0\,exp(i\omega x)\\ &+\sum\frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)}cosh\,\xi(b+y)\,sinh\,a\xi\,exp(i\omega_1 x)\\ &-\sum\frac{\bar{\xi}f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})}cosh\,\bar{\xi}(b+y)\,sinh\,a\bar{\xi}\,exp(-i\omega_2 x)\Bigg]\end{aligned}\right\} \tag{3.4}$$

Where,

$f_1(x) = \cosh ax\sinh bx + s\cosh bx\sinh ax$

$f_2(x) = [2a\{x(1-s)+Mx^3\} + \{(1-s)+3Mx^2\}\sinh 2ax]\sinh^2 bx$

$\quad +s\sinh^2 ax\,[2b\{x(1-s)+Mx^3\} + \{(1-s)+3Mx^2\}\sinh 2bx]$,

$\xi = s\delta_n + i\beta_n,\ \bar{\xi} = s\delta_n - i\beta_n,\ n = 1,2,3,\ldots..$

$\omega_1 = (\xi^2-\mu^2)^{1/2},\ \omega_2 = (\bar{\xi}^2-\mu^2)^{1/2}.$

Conditions at infinity given by (3.2) are satisfied if we choose

$$L = \frac{iN}{2\pi} \tag{3.5}$$

where $N = \left[\frac{f_1(\beta_0)}{f_2(\beta_0)} \sinh a\beta_0 \sinh b\beta_0\right]^{-1} \cos\alpha$.

Exploiting (3.5) into (3.4), the reduced potential functions ψ_1, ψ_2 can be found, which are given by

$$\left.\begin{aligned} \psi_1(x,y) &= -\frac{\cosh\beta_0(a-y)}{\sinh a\beta_0}\exp(i\omega x) \\ &\quad -N\sum \frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)} cosh\,\xi(a-y)\,sinh\,b\xi\,exp(i\omega_1 x) \\ &\quad +N\sum \frac{\bar{\xi} f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})} cosh\,\bar{\xi}(a-y)\,sinh\,b\bar{\xi}\,exp(-i\omega_2 x) \\ \psi_2(x,y) &= \frac{\cosh\beta_0(b+y)}{\sinh b\beta_0}\exp(i\omega x) \\ &\quad +N\sum \frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)} cosh\,\xi(b+y)\,sinh\,a\xi\,exp(i\omega_1 x) \\ &\quad -N\sum \frac{\bar{\xi} f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})} cosh\,\bar{\xi}(b+y)\,sinh\,a\bar{\xi}\,exp(-i\omega_2 x). \end{aligned}\right\} \quad (3.6)$$

Making use of (3.6) into (3.1), the solutions ϕ_1, ϕ_2 for the original boundary value problem described by (2.2) – (2.7) have been obtained, which are given by

$$\left.\begin{aligned} \phi_1(x,y) &= \frac{\cosh\beta_0(a-y)}{\sinh a\beta_0}\exp(i\omega x) \\ &\quad -N\sum \frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)} cosh\,\xi(a-y)\,sinh\,b\xi\,exp(i\omega_1 x) \\ &\quad +N\sum \frac{\bar{\xi} f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})} cosh\,\bar{\xi}(a-y)\,sinh\,b\bar{\xi}\,exp(-i\omega_2 x) \\ \phi_2(x,y) &= -\frac{\cosh\beta_0(b+y)}{\sinh b\beta_0}\exp(i\omega x) \\ &\quad +N\sum \frac{\xi f_1(\xi)}{\omega_1 f_2(\xi)} cosh\,\xi(b+y)\,sinh\,a\xi\,exp(i\omega_1 x) \\ &\quad -N\sum \frac{\bar{\xi} f_1(\bar{\xi})}{\omega_2 f_2(\bar{\xi})} cosh\,\bar{\xi}(b+y)\,sinh\,a\bar{\xi}\,exp(-i\omega_2 x) \end{aligned}\right\} \quad (3.7)$$

Finally, exploiting (3.7) into (2.1), the velocity potentials $\Phi_1(x,y,z,t)$ and $\Phi_2(x,y,z,t)$ have been found (see Appendix-A). The explicit form of Φ_1, Φ_2 are given by

$$\left.\begin{aligned}\Phi_1(x,y,z,t) &= \frac{\cosh\beta_0(a-y)}{\sinh a\beta_0}\cos(\omega x+\sigma t+\mu z)\\ &\quad -2N\sin(\sigma t+\mu z)\sum \mathrm{h}(\beta_n)\exp(-q_n x)\\ \Phi_2(x,y,z,t) &= -\frac{\cosh\beta_0(H+y)}{\sinh b\beta_0}\cos(\lambda x+\sigma t+\mu z)\\ &\quad +2N\sin(\sigma t+\mu z)\sum \mathrm{H}(\beta_n)\exp(-q_n x)\end{aligned}\right\} \quad (3.8)$$

which are the required velocity potentials for a three-dimensional wave train progressing towards a vertical cliff in two immiscible liquids.

4. Special Case:

To check the results obtained here, analytically, we make the assumption assume $\rho_2 = 0$ which leads to the case of a single liquid. In this case, the expression for Φ_1 given by (3.8) reduces to the velocity potential of a three-dimensional wave train progressing towards a vertical cliff in water of uniform finite depth $'a'$ in the presence of surface tension and the explicit form of the velocity potential becomes (see Appendix-B)

$$\begin{aligned}\Phi_1(x,y,z,t) &= \frac{\cosh\beta_0(a-y)}{\sinh a\beta_0}\cos(\omega x+\sigma t\\ &\quad +\mu z)\\ &+\frac{4\cos\alpha\,[2a\beta_0(1+M{\beta_0}^2)+(1+3M{\beta_0}^2)\sinh 2a\beta_0]}{\sinh 2a\beta_0}\sin(\sigma t+\nu z)\\ &\times\sum\frac{\beta_n\cos\beta_n(a-y)\cos a\beta_n}{2a\beta_n(1-M{\beta_n}^2)+(1-3M{\beta_n}^2)\sin 2a\beta_n}\,\frac{exp\,(-q_n x)}{q_n}\end{aligned} \quad (4.1)$$

where $q_n = ({\beta_n}^2+\mu^2)^{1/2}$.

If in addition, we assume $\alpha = 0$, then the above expression for Φ_1, given by (4.1), reduces to

$$\begin{aligned}\Phi_1(x,y,t) &= \frac{\cosh\beta_0(a-y)}{\sinh a\beta_0}\cos(\beta_0 x\\ &\quad +\sigma t)\\ &+\frac{4[2a\beta_0(1+M{\beta_0}^2)+(1+3M{\beta_0}^2)\sinh 2a\beta_0]}{\sinh 2a\beta_0}\sin\sigma t\\ &\times\sum\frac{\cos\beta_n(a-y)\cos a\beta_n}{2a\beta_n(1-M{\beta_n}^2)+(1-3M{\beta_n}^2)\sin 2a\beta_n}\,exp\,(-\beta_n x)\end{aligned} \quad (4.2)$$

which is the velocity potential for a wave train, incident, normally on a vertical cliff, in a liquid of uniform finite depth $'a'$, in the presence of interfacial tension effect at the interface. The above results are somewhat similar to that obtained by Mandal and Kundu [5].

5. Conclusions:

A formal analytical derivation representing the velocity potentials for an incoming wave train obliquely incident on a rigid vertical cliff at the interface of two superposed liquids, have been found here. Assuming no reflection of waves by the cliff, the solutions have been obtained here by using a straightforward analysis based on a simple reduction procedure in presence of interfacial tension effect at the interface of the two liquids where the lower and upper liquid to be of finite depth and finite height respectively. Mixing near the cliff is likely to produce an intrusive flow that blurs the interface. However, as the problem is formulated within the framework of linearized theory, and as the liquids are assumed to be immiscible, these effects are not considered here. Various results can be found as special case of the general problem considered here. In particular, if we assume $\rho_2 = 0$, the expression $\Phi_1(x, y, z, t)$ for an incoming progressive wave train obliquely incident on a rigid cliff in water of uniform finite depth $'a'$ can be found. The corresponding two-dimensional problem can also be obtained by putting $= 0$, which are in complete agreement with those found earlier by Mandal and Kundu [5]. This problem is a simplified mathematical model of the well known sloping beach problem arising in oceanography. The problem considered here may further be developed analytically.

Appendix-A:

$P(x) = \cosh s\delta_n x \cos\beta_n x,$ $\quad Q(x) = \sinh s\delta_n x \sin\beta_n x,$

$R(x) = \sinh s\delta_n x \cos\beta_n x,$ $\quad S(x) = \cosh s\delta_n x \sin\beta_n x,$

$F_1 = P(a)R(b) - Q(a)S(b),$ $\quad F_2 = P(a)S(b) + Q(a)R(b),$

$F_3 = s[P(b)R(a) - Q(b)S(a)],$ $\quad F_4 = s[P(b)S(a) + Q(b)R(a)],$

$T(x) = \cosh 2s\delta_n x \cos 2\beta_n x,$ $\quad U(x) = \sinh 2s\delta_n x \sin 2\beta_n x,$

$V(x) = \sinh 2s\delta_n x \cos 2\beta_n x,$ $\quad W(x) = \cosh 2s\delta_n x \sin 2\beta_n x,$

$I_1 = 1 - s + 3M(s^2\delta_n^2 - \beta_n^2),$ $\quad I_2 = 6Ms\delta_n\beta_n,$

$I_3 = s(1 - s)\delta_n + M(s^3\delta_n^2 - 3s\delta_n\beta_n^2),$ $\quad I_4 = \beta_n(1 - s) + M(3s^2\delta_n^2\beta_n - \beta_n^3),$

$$E_1 = \frac{1}{2}[I_1\{V(a)(T(b) - 1) - W(a)U(b)\} - I_2\{W(a)(T(b) - 1) + V(a)U(b)\}] ,$$

$$E_2 = \frac{1}{2}[I_2\{V(a)(T(b) - 1) - W(a)U(b)\} + I_1\{W(a)(T(b) - 1) + V(a)U(b)\}] ,$$

$E_3 = I_3 a[T(b) - 1] - I_4 a\, U(b) ,$ $\quad E_4 = I_3 a\, U(b) + I_4 a[T(b) - 1] ,$

$$E_5 = \frac{s}{2}[I_1\{V(b)(T(a) - 1) - W(b)U(a)\} - I_2\{W(b)(T(a) - 1) + V(b)U(a)\}],$$

$$E_6 = \frac{s}{2}[I_2\{V(b)(T(a) - 1) - W(b)U(a)\} + I_1\{W(b)(T(a) - 1) + V(b)U(a)\}],$$

$E_7 = s\{I_3 b[T(a) - 1] - I_4 b\, U(a)\}$, $\quad E_8 = s\{I_3 b\, U(a) + I_4 b[T(a) - 1]\}$,

$A_1 = E_1 + E_3 + E_5 + E_7,$ $\quad A_2 = E_2 + E_4 + E_6 + E_8$, $\quad B_1 = F_1 + F_3,$ $\quad B_2 = F_2 + F_4$,

$p_n + iq_n = (s^2{\delta_n}^2 - {\beta_n}^2 - \mu^2 + 2is\delta_n\beta_n)^{1/2},$

$G_1 = P(a - y)R(b) - Q(a - y)S(b),$ $\quad G_2 = P(a - y)S(b) + Q(a - y)R(b),$

$$G_3 = \frac{[G_1(p_n s\delta_n + q_n\beta_n) - G_2(p_n\beta_n - \eta q_n s\delta_n)}{{p_n}^2 + {q_n}^2}, \quad G_4 = \frac{[G_1(p_n\beta_n - q_n s\delta_n) + G_2(p_n s\delta_n + q_n\beta_n)}{{p_n}^2 + {q_n}^2},$$

$C_1 = (G_3 \cos p_n x - G_4 \sin p_n x),$ $\quad C_2 = (G_3 \sin p_n x + G_4 \cos p_n x),$

$$J_1 = P(b+y)R(a) - Q(b+y)S(a), \qquad J_2 = P(b+y)S(a) + Q(b+y)R(a),$$

$$J_3 = \frac{[J_1(p_n s\delta_n + q_n\beta_n) - J_2(p_n\beta_n - q_n s\delta_n)}{{p_n}^2 + {q_n}^2}, \qquad J_4 = \frac{[J_1(p_n\beta_n - q_n s\delta_n) + J_2(p_n s\delta_n + q_n\beta_n)}{{p_n}^2 + {q_n}^2},$$

$$D_1 = (J_3 \cos p_n x - J_4 \sin p_n x), \qquad D_2 = (J_3 \sin p_n x + J_4 \cos p_n x),$$

$$h(\beta_n) = \frac{[C_1A_1B_2 + C_2A_1B_1 - C_1A_2B_1 + C_2A_2B_2]}{{A_1}^2 + {A_2}^2}, \quad H(\beta_n) = \frac{[D_1A_1B_2 + D_2A_1B_1 - D_1A_2B_1 + D_2A_2B_2]}{{A_1}^2 + {A_2}^2}.$$

Appendix-B:

Upon substitution $\rho_2 = 0$ we find

$f_1(x) = \cosh ax \sinh bx$,
$f_2(x) = \{(1 + 3Mx^2)\sinh 2ax + 2ax(1 + Mx^2)\}\sinh^2 bx$

$$N = \frac{2\cos\alpha\,[2a\beta_0(1 + M{\beta_0}^2) + (1 + 3M{\beta_0}^2)\sinh 2a\beta_0]}{\sinh 2a\beta_0}.$$

$P(x) = \cos\beta_n x$, $Q(x) = R(x) = 0$, $S(x) = \sin\beta_n x$, $p_n = 0$, $q_n = ({\beta_n}^2 + \mu^2)^{1/2}$,

$T(x) = \cos 2\beta_n x$, $U(x) = V(x) = 0$, $W(x) = \sin 2\beta_n x$, $I_1 = 1 - 3M\beta_n^2$, $I_2 = I_3 = 0$,

$I_4 = \beta_n(1 - M\beta_n^2)$, $E_1 = 0$, $E_2 = -(1 - 3M\beta_n^2)\, sin\, 2a\beta_n\, \sin^2 b\beta_n$, $E_3 = 0$, $F_1 = 0$,

$F_2 = \cos a\beta_n \sin b\beta_n$, $F_3 = F_4 = 0, B_1 = 0$, $B_2 = F_2$, $G_1 = 0$, $G_2 = \cos\beta_n(a - y)\sin b\beta_n$,

$G_3 = 0$, $G_4 = \frac{\beta_n G_2}{q_n}$. $C_1 = 0$, $C_2 = \frac{\beta_n}{q_n}\cos\beta_n(a - y)\sin b\beta_n$,

$$u(\beta_n) = -\frac{\beta_n\, cos\,\beta_n(a - y)\, cos\, a\beta_n}{q_n[2a\beta_n(1 - M{\beta_n}^2) + (1 - 3M{\beta_n}^2)\, sin\, 2a\beta_n]}.$$

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