Study and Comparison of Finite Difference Methods for the Numerical Solution of Partial Differential Equations

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Abstract: This paper presents a comprehensive study of Partial Differential Equations (PDEs), beginning with a general definition and classification into three primary types: parabolic, hyperbolic, and elliptic equations. We focus on the development of finite difference schemes for numerically solving specific PDEs and Ordinary Differential Equations (ODEs), demonstrating their effectiveness through selected examples implemented in Matlab. A general study of some finite difference schemes for the numerical resolution of specific PDEs and ODEs has been conducted. Well-selected numerical examples have been proposed, and numerical implementation of different examples has been carried out using MATLAB. Our study concluded that the numerical methods of finite difference schemes for numerical accuracy of definite PDEs and ODEs have had good effectiveness and this has been verified by the numerical examples that have been implemented using MATLAB software.

Keywords: Partial Differential Equations, Finite Difference Method, Numerical Schemes, MATLAB software.

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1. Introduction

The study of Partial Differential Equations (PDEs) has long been a fundamental area of interest within mathematics and various applied sciences. Theoretical advancements, along with the development of numerical methods, have played a significant role in enabling the analysis and solution of these equations across disciplines such as physics, engineering, biology, and economics (Brezis,1983; Ciarlet, P.G. (1978; Ciarlet, 1991; Godlewski & Raviart, 1991; Godunov, 1976; Herbin). PDEs are crucial in modeling a wide range of phenomena, from fluid dynamics and heat conduction to electromagnetism and population dynamics. This broad relevance has motivated extensive research into the existence, uniqueness, and stability of their solutions, as well as the numerical methods required for their practical resolution (Atkinson, 1978; Bartle, 1976; Boehm & Pautzsch, 1993; Burden & Douglas , 2001; Evans, 1995; Gautschi, 1997; Maron, 1982; Quarteroni, Sacco, & Saleri, 2000; Rappaz & Picasso, 1998; Stewart, 1996).

The finite difference method, one of the most widely used numerical techniques, offers an effective framework for solving both ordinary and partial differential equations (Butcher, 1987; Hairer & Wanner, 1996; Henrici, 1962; Lambert, 1991; Stroud, 1974). This method approximates the derivatives in differential equations by employing Taylor series expansions, transforming continuous problems into discrete ones that can be solved using computational techniques. For instance, the derivative of a function $u(\omega)$ with respect to ω is approximated as:

$$u'(\omega) \approx \frac{u(\omega+h)-u(\omega)}{h}, \quad \text{as} \quad h \to 0.$$

Such approximations introduce discretization errors, which must be carefully analyzed to ensure the accuracy and stability of the numerical solution (Causon & Mingham, n.d.). In this work, we focus on a comprehensive analysis of finite difference schemes for solving specific types of PDEs and ordinary differential equations (ODEs). We also evaluate the effectiveness of these schemes through various numerical experiments implemented in Matlab. Key properties such as consistency, stability, and convergence are studied in detail for each method.

This study aims to provide a detailed comparison of these numerical schemes, highlighting their advantages and limitations. By analyzing their error behavior and computational efficiency, we aim to offer insights that will aid in selecting the most appropriate method for solving different types of PDEs in various practical applications.

In this section, we provide a general overview of Partial Differential Equations (PDEs). First, we present the general definition of a PDE. Then, we classify PDEs into three major categories: parabolic, hyperbolic, and elliptic equations.

1.1 Partial Differential Equations

A system of partial differential equations has the following generic form:

$$\frac{\partial \omega}{\partial t} = S\left(t, x, y, \frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y}, \frac{\partial^2 \omega}{\partial x^2}, \frac{\partial^2 \omega}{\partial y^2}, \frac{\partial^2 \omega}{\partial xy}, \cdots\right). \tag{1}$$

The independent variables $t \in D_t$, $x \in D_x$ and $y \in D_y$, where the D_t, D_x, D_y are some domains that can be bounded or unbounded. The variable $\omega = \omega(t, x, y) \in \mathrm{R}^n$ is a solution to the system (1) for all $(t, x, y) \in D_t \times D_x \times D_y$, and the function $S \in \mathrm{R}^n$ is a second member of the system (1). In general, the variable t denotes time, and the variables x, y denote spatial variables. If time exists among the independent variables, we refer to the system as *time-dependent* or *evolutionary*; otherwise, system (1) is referred to as *steady-state* or *equilibrium*.

The solution of the system (1) is called a field. The field is scalar-valued if the system (1) is scalar ($n = 1$), and vector-valued if (1) represents a system of PDEs ($n \geq 2$).

1.2 Classification

A scalar-valued partial differential equation with constant coefficients of order two has the general form:

$$\alpha\frac{\partial^2\omega}{\partial x^2}+\beta\frac{\partial^2\omega}{\partial x\partial y}+\gamma\frac{\partial^2\omega}{\partial y^2}+\varepsilon\frac{\partial\omega}{\partial x}+\sigma\frac{\partial\omega}{\partial y}+\mu\omega=g(x,y). \qquad (2)$$

- If at least one of the scalars α, β, γ is nonzero, then we have:
  - The equation (2) is hyperbolic if: $\beta^2 - 4\alpha\gamma > 0$,
  - The equation (2) is parabolic if: $\beta^2 - 4\alpha\gamma = 0$,
  - The equation (2) is elliptic if: $\beta^2 - 4\alpha\gamma < 0$.
- If $\alpha = \beta = \gamma = 0$, but $\varepsilon \neq 0$ and $\sigma \neq 0$, then (2) is hyperbolic.

1.2.1 Parabolic PDE

The parabolic equations (Bensoussan, Prato, Delfour, & Mitter 1992; Cazenave & Haraux, 1990; Evans, 1990; Neittaanmaki &Tiba, 1994) govern evolutionary or unsteady problems in which diffusion or dissipation mechanisms are involved. These problems are typically defined on a spatial domain $\Omega = (\ell, L)$ with a boundary Γ, where the unknown is subject to boundary conditions of the Dirichlet or Neumann type, along with initial conditions.

- Dirichlet boundary problem:

$$\frac{\partial\omega}{\partial t} - c\frac{\partial^2\omega}{\partial x^2} = f, \qquad (3)$$

$$\omega(x,0) = \omega_0(x), \qquad (4)$$

$$\omega(\ell,t) = q_\ell(t),\ \omega(L,t) = q_L(t), \qquad (5)$$

where $\omega = \omega(x,t)$, $(x,t) \in \Omega \times [0, T]$, f, ω_0, q_ℓ, and q_L are known functions, and c is a positive constant.

- Neumann boundary problem:

$$\frac{\partial\omega}{\partial t} - c\frac{\partial^2\omega}{\partial x^2} = f, \qquad (6)$$

$$\omega(x,0) = \omega_0(x), \qquad (7)$$

$$\omega'(\ell,t) = z_\ell(t),\ \omega'(L,t) = z_L(t), \qquad (8)$$

where z_ℓ and z_L are known functions.

1.2.2. Hyperbolic PDE

The hyperbolic equations model wave propagation without dissipation (Godlewski and P. A. Raviart, 1991; Godlewski & Raviart, 1996; Lions, 1971; Pazy, 1983). For instance, sound propagation in a homogeneous medium. These equations also describe conservation laws, such as mass, momentum, and energy conservation in compressible fluids.

- Homogeneous wave equation:

$$\begin{aligned} \frac{\partial^2\omega}{\partial t^2} - c^2\frac{\partial^2\omega}{\partial x^2} &= 0, \\ \omega(x,0) &= \omega_0(x), \quad \frac{\partial\omega}{\partial t}(x,0) = r_0(x), \\ \omega(\ell,t) &= q_\ell(t), \quad \omega(L,t) = q_L(t), \end{aligned}$$

where $(x,t) \in \Omega \times [0,T]$ and $c \in \mathrm{R}$.

- Advection equation:

$$\begin{aligned} \frac{\partial\omega}{\partial t} + a\frac{\partial\omega}{\partial x} &= 0, \\ \omega(x,0) &= \omega_0(x), \\ \omega(\ell,t) &= q_\ell(t), \end{aligned}$$

where $(x,t) \in \Omega \times [0,T]$ and a is a positive constant.

1.2.3. Elliptic PDE

The elliptic equations govern stationary problems (Ciarlet, 1978; Ciarlet, 1991; Lions, 1971, Pazy, 1983) defined on a spatial domain $\Omega \subset \mathrm{R}^n$ with boundary Γ, where the unknown is subjected to Dirichlet or Neumann boundary conditions.

Dirichlet problem:

$$\begin{aligned} -\Delta\omega &= f, \quad \text{on} \quad \Omega, \\ \omega &= \omega_0, \quad \text{on} \quad \Gamma. \end{aligned}$$

Neumann problem:

$$\begin{aligned} -\Delta\omega &= f, \quad \text{on} \quad \Omega, \\ \frac{\partial\omega}{\partial \mathbf{n}} &= g, \quad \text{on}\, \Gamma, \end{aligned}$$

where $\Delta\omega = \sum_{i=1}^{n} \frac{\partial^2\omega}{\partial x_i{}^2}$ and $\mathbf{n}$ is the normal vector to the boundary Γ.

In this section, we provided a general overview of the definition of partial differential equations and their classifications. We outlined the main categories: parabolic, hyperbolic, and elliptic equations, each representing different types of physical phenomena. These equations have been extensively studied across various scientific fields, including mathematics, engineering, physics, and beyond.

2. Finite difference approximations

It's well known that the finite differential method is an excellent tool for solving several different ODEs and PDEs (Butcher, 1987; Hairer & Wanner, 1996; Henrici, 1962; Lambert, 1991; Stroud, 1974). In fact, this method is based on the expansion of differential operators in Taylor series (Causon, & Mingham n.d.). For example: for $x \in$ R and using the definition of $\frac{du}{}(x) = u'(x)$, we get:

$$\frac{du}{dx}(x) = u'(x) = \lim_{h \longrightarrow 0} \frac{u(x+h) - u(x)}{h}$$

Then, we can consider the following approximation of the first derivative:

$$u'(x) \approx \frac{u(x+h) - u(x)}{h}, \quad \text{as} \quad h \longrightarrow 0.$$

The domain, (on which the problem is defined), is partitioned in space and time, and approximations of the answer are computed at the space or time points. The error resulting between the exact and numerical solutions is determined by transforming a PD operator to a finite difference operator. In this case, the error that occurs is referred to as the discretization error or truncated error, indicating that only a finite portion of the Taylor series (Causon, & Mingham n.d.) is employed in the approximation.

2.1 Taylor series

2.2.1 Approximation of the first derivative

Let $I_h =]\omega - h, \omega + h[$ be neighborhood of ω. Consider a function u $C^2(I_h)$. Then, for all $h > 0$, we have the following development:

$$u(\omega + h) = u(\omega) + hu'(\omega) + \frac{h^2}{2} u''(\omega + h_1) \tag{9}$$

where $h_1 \in$ $(0,h)$. To solve problems containing the first derivative u', it is appropriate to keep the first two terms of equation (9)

$$u(\omega + h) = u(\omega) + hu'(\omega) + O(h^2) \quad .$$

Note that $O(h^2)$ is the error of the approximation. From the relation (9), we can deduce without difficulty that: a ∃ constant $C > 0$:

$$\left| \frac{u(\omega + h) - u(\omega)}{h} - u'(\omega) \right| \leq Ch, \quad C = 0.5 \sup_{y \in [\omega, \omega + h_0]} |u''(y)|$$

for $h_0 \geq h$ (where h_0 is a given strictly positive real number). The error caused after compensation the derivative with the differential product is of order h. We say that the approximation of the first derivative u' at the spatial point ω is consistent to the first order. This type of approximation is called *forward difference* approximation of the first derivative u'. In generally, we have the following definition for the order.

Definition 1. (Abramowitz, 1970; Greenberg, 1998; Roy, 2021) *We say that the approximation of the first derivative $u'(\omega)$ is of order $r(r > 0)$, if $\exists$ a strictly positive constant $K > 0$, independent of h, such that the error between the derivative and its approximation is bounded by the term Kh^r (i.e. is exactly $O(h^r)$).*

In a similar way, we define a *backward difference* approximation of $u'(\omega)$ as follows:

$$u(\omega - h) = u(\omega) - hu'(\omega) + O(h^2)$$

Moreover, we can define a *central difference* approximation. Assume that the function u is three times differentiable in I_h. We have:

$$u(\omega+h) = u(\omega) + hu'(\omega) + \frac{h^2}{2}u''(\omega) + \frac{h^3}{6}u^{(3)}(\xi^+)$$
$$u(\omega-h) = u(\omega) - hu'(\omega) + \frac{h^2}{2}u''(\omega) - \frac{h^3}{6}u^{(3)}(\xi^-)$$

where $\xi^+ \in]\omega, \omega+h[$ and $\xi^- \in]\omega-h, \omega[$. From these two equations, we get:

$$\frac{u(\omega+h) - u(\omega-h)}{2h} = u'(\omega) + h^2 u^{(3)}(\xi)$$

where $\omega - h < \xi < \omega + h$. Then, for every $0 < h < h_0$, we have:

$$\left|\frac{u(\omega+h) - u(\omega-h)}{2h} - u'(\omega)\right| \le Ch^2, \quad C = \frac{1}{6} \sup_{y \in [\omega-h_0, \omega+h_0]} \left|u^{(3)}(y)\right|$$

Thus, the approximation is consistent of second order.

Remark 1. 1. *Let w and v be two functions. Then, we say that:*

$$w(\zeta) = O(v(\zeta)), \qquad \zeta \longrightarrow 0,$$

if there exists a constant C:

$$\left|\frac{w(\zeta)}{v(\zeta)}\right| < C, \quad \forall\zeta \qquad \textit{sufficiently small.}$$

2. *Let w and v be two functions. Then, we say that:*

$$w(\zeta) = o(v(\zeta)), \qquad \zeta \longrightarrow 0,$$

if we have:

$$\left|\frac{w(\zeta)}{v(\zeta)}\right| \longrightarrow 0, \quad \zeta \longrightarrow 0$$

3. *If $w(\zeta) = o(v(\zeta))$, then $w(\zeta) = O(v(\zeta))$. The converse may not be true.*

2.1.2 Approximation of the second derivative

Let $I_{h0} = [\omega - h_0, \omega + h_0]$, $h_0 > 0$. Then, we have:

Lemma 1. *(*Abramowitz, 1970; Greenberg, 1998; Roy, 2021*) Suppose* $u \in C^4(I_{h_0})$*. Then,* $\exists$ *a constant* $M > 0$*:* $\forall h \in]0, h_0[$*, we have:*

$$\left| \Big(u(\omega+h) - 2u(\omega) + u(\omega-h) \Big) h^{-2} - u''(\omega) \right| \leq Mh^2 \qquad (10)$$

Then, the quotient $(u(\omega+h) - 2u(\omega) + u(\omega-h))h^{-2}$ *is a consistent second order approximation of* $u''(\omega)$*.*

Proof. By Taylor expansions, we get:

$$u(\omega+h) = u(\omega) + hu'(\omega) + \frac{h^2}{2}u''(\omega) + \frac{h^3}{6}u^{(3)}(\omega) + \frac{h^4}{24}u^{(4)}(\xi^+)$$
$$u(\omega-h) = u(\omega) - hu'(\omega) + \frac{h^2}{2}u''(\omega) - \frac{h^3}{6}u^{(3)}(\omega) + \frac{h^4}{24}u^{(4)}(\xi^-)$$

where $\xi^+ \in]x, x+h[$ and $\xi^- \in]x-h, x[$. Using the mean value theorem:

$$\Big(u(\omega+h) - 2u(\omega) + u(\omega-h) \Big) h^{-2} = u''(\omega) + \frac{h^2}{12}u^{(4)}(\xi)$$

where $\xi \in]\omega-h, \omega+h[$. Hence, we deduce the relation (10) with the constant

$$M = \frac{1}{12} \sup_{y \in [\omega-h_0, \omega+h_0]} \left| u^{(4)}(y) \right|.$$

2.2 Finite difference schemes for first order ODEs

Consider a first-order ordinary differential equation [17-21]:

$$\begin{cases} y'(x) = f(x, y(x)), & \text{for } x \in [a, b] \\ y(a) = \alpha \in \mathbb{R}. \end{cases}$$

We introduce an equidistant grid points (x_j), $0 \leq j \leq N$ defined as follows:

$$x_k = kh + a, \qquad k = 0, 1, 2, 3, \ldots, N,$$

the spacial step $\Delta x = h$ is defined as follows:

$$h = \frac{b-a}{N}.$$

In general, remark that we have:

$$h = x_{j+1} - x_j, \quad \forall j = 0, \cdots, N.$$

2.2.1 The Forward Euler Method

For more details about this method, we refer the author to (Butcher, 1987; Hairer & Wanner, 1996; Henrici, 1962; Lambert, 1991; Stroud, 1974).

At

the point (x_j), we have:

$$y'(x_j) = f(x_j, y(x_j)) \tag{11}$$

Now, using the Taylor series (Causon & Mingham, n.d.) to approximate $y'(x_j)$ yields:

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{\Delta x} \tag{12}$$

From relations (11) and (12), we deduce that:

$$y(x_{j+1}) = y(x_j) + \Delta x\, f(x_j, y(x_j)) \tag{13}$$

Let us denote: $y_j = y(x_j)$. So, we obtain, From (13), that:

$$y_{j+1} = y_j + \Delta x\, f(x_j, y_j), \quad d = 0, \ldots, N-1$$

and $y_0 = y(x_0) = y(a) = \alpha$. Then, the forward EuLer method is given

by:

$$\begin{cases} y_0 = \alpha \\ y_{j+1} = y_j + \Delta x f(x_j, y_j), \quad d = 0, \ldots, N-1 \end{cases}$$

2.2.2 The backward Euler method

For more details about this method, we refer the author to (Butcher, 1987; Hairer & Wanner, 1996; Henrici, 1962; Lambert, 1991; Stroud, 1974).

At the grid point x_j, we have:

$$y'(x_j) = f(x_j, y(x_j)). \tag{14}$$

Then, we can approximate $y'(x_j)$ by:

$$y'(x_j) \approx \frac{y(x_{j+1}) - y(x_j)}{\Delta x} \tag{15}$$

So, from relations (14) and (15), we get:

$$\frac{y(x_j) - y(x_{j-1})}{\Delta x} = f(x_j, y(x_j))$$

Consequently, we get the following scheme:

$$\begin{cases} y_0 = \alpha, \\ y_j = y_{j-1} + \Delta x f(x_j, y_j), \quad j = 1, \ldots, N. \end{cases} \tag{16}$$

Notice that y_j is the unknown in the system (16). Consequently, (16) is a nonlinear system, that can be solved by Newton's method or fixed-point iteration method.

2.3 Finite difference schemes for second order ODEs

For more details about finite difference schemes for second order ODEs, we refer the author to (Butcher, 1987; Hairer & Wanner, 1996; Henrici, 1962; Lambert, 1991; Stroud, 1974). Consider the interval $\Omega =]a,b[$ and the problem:

$$D: \quad \begin{cases} -u''(\omega) + c(\omega)u(\omega) = f(\omega), & \omega \in \Omega \\ u(a) = \alpha \\ u(b) = \beta \end{cases}$$

where $c(\omega) > 0$ for all $\omega \in \overline{\Omega}$.
Consider the sequence of points $(x_q)_{0 \le q \le N}$:

$$x_q = qh + a, \qquad q = 0,1,2,3,\ldots,N, \qquad h = (b-a)/N.$$

At each points x_q, numerical value of the solution is given by:

$$u_q \approx u(x_q), \qquad q = 1,\ldots,N-1.$$

Assume that the boundary condition satisfies:

$$u(x_0) = \alpha, \qquad u(x_N) = \beta.$$

The unknown of the discrete problem is the vector; $u_h = (u_1, u_2, \ldots, u_{N-1}) \in \mathbb{R}^{N-1}$.
By Taylor series (Causon & Mingham, n.d.), we have:

$$\begin{aligned} u(x_{q+1}) &= u(x_q + h), \\ &= u(x_q) + hu'(x_q) + \frac{h^2}{2}u''(x_q) + \frac{h^3}{6}u^{(3)}(x'_q) + \theta_1(h^4). \end{aligned} \tag{17}$$

In the same way, we get:

$$\begin{aligned} u(x_{q-1}) &= u(x_q - h), \\ &= u(x_q) - hu'(x_q) + \frac{h^2}{2}u''(x_q) - \frac{h^3}{6}u^{(3)}(x_q) + \theta_2(h^4) \end{aligned} \tag{18}$$

Then, from equations (17) and (18), we deduce that:

$$u''(x_q) = u(x_{q+1}) - 2u(x_q) + u(x_{q-1})h^{-2} + \theta_1(h^3) + \theta_2(h^3).$$

As $\theta_1(h^3)+\theta_2(h^3)$ tends to 0 when h tends to 0, then we can write the following approximation:

$$\begin{aligned} u''(x_j) &\approx \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} \\ &\approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} \end{aligned}$$

Finally, the discrete problem, (associated to the continuous problem D), is given by:

$$D_h : \begin{cases} -\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} + c(x_j)u_j = f(x_j) & j = 1,\ldots,N-1 \\ u_0 = \alpha \\ u_N = \beta \end{cases}$$

The problem D_h can be transformed into algebric linear system as follows:

$$A_h U_h = b_h, \tag{19}$$

where A_h is a $(N-1)\times(N-1)$ tridiagonal matrix defined by:

$$A_h = \begin{pmatrix} 2+h^2c(x_1) & -1 & 0 \\ & \ddots & \ddots & \\ -1 & 2+h^2c(x_2) & -1 \\ & \ddots & \ddots & \\ 0 & -1 & 2+h^2c(x_{N-1}) \end{pmatrix}$$

and the second member $U_h, b_h \in \mathbb{R}^{N-1}$:

$$\begin{aligned} U_h &= (u_1,\cdots,u_{N-1})^T, \\ b_h &= \left(h^2 f(x_1)+\alpha, h^2 f(x_2),\ldots..,h^2 f(x_{N-2}), h^2 f(x_{N-1})+\beta\right)^T. \end{aligned}$$

This formulation poses a question regarding whether a systemic solution exists (Trefethen & Bau, 1997).

Definition 2. *(*Faddeev & Faddeeva, 1963; Golub & Van Loan, 1989; Stewart, 1973; Trefethen & Bau, 1997) *Let $A = (a_{i,j})_{1\le i,\, j\le n}$ be a square matrix real or complex of order n. The matrix A is said to have a strictly dominant diagonal, if:*

$$|a_{i,i}| > \sum_{j\neq i} |a_{i,j}|.$$

From the definition of the function $c(x)$, it's clear that the matrix A_h is a strictly dominant diagonal matrix. Then, A_h is invertible. Consequently, the linear system has a unique solution.

Remark 2. *(*Faddeev & Faddeeva, 1963; Golub & Van Loan, 1989; Stewart, 1973; Trefethen & Bau, 1997*) The linear system can be solved by several methods. Direct methods such as the Cramer's method, the Gauss elimination method, the Gauss-Jordan method, the LU method, etc... Iterative methods such as the Gauss-Seidel method, the Jaccobi method, etc...*

2.4 Finite difference schemes for the Heat equation

Consider a time-dependent boundary value problem governed by the heat equation (Larrouturou, 1998; Rappaz, 1998; Raviart, n.d.) posed in a bounded domain: $\Omega =]a,b[$

Find $u : \overline{\Omega} \times [0,T] \longrightarrow \mathbb{R}$ such that

$$(P): \begin{cases} \frac{\partial u}{\partial t}(x,t) - \nu\frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), & \text{on} \quad \Omega\times[0,T], \\ u(x,0) = u^0(x), & \text{on} \quad \Omega, \\ u(a,t) = g_1(t), u(b,t) = g_2(t), & \text{on} \quad [0,T], \end{cases} \tag{20}$$

where $\nu \geq 0$, $f(x,t)$ is a given source term.

To solve numerically the problem (20), we first need to consider a set of points in $D = \Omega \times [0,T]$ as follows:

$$x_j = jh + a, \quad j = 0,1,2,3,\ldots,N, \quad h = (b-a)/N,$$

$$t_n = n\Delta t, \quad n = 0,1,2,3,\ldots,M, \quad \Delta t = T/M.$$

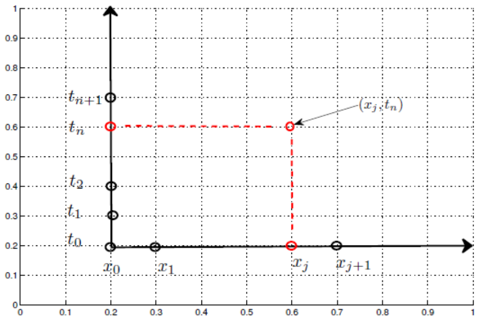

Figure 1: Finite difference grid.

The partial derivative

$$u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

is always approximated by central difference quotient at the grid point (x_j, t_n):

$$u_{xx}(x_j,t_n) \approx \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2}, \tag{21}$$

where we assume that:

$$u_j^n = u(x_j,t_n).$$

2.4.1 Explicit schem (ES)

The term u_t can be approximated by a Forward difference quotient, (LeVeque, n.d.; LeVeque, 1992), in time at the grid point (x_q,t_n):

$$u_t(x_q,t_n) \approx \frac{u_q^{n+1} - u_q^n}{\Delta t},$$

then, the corresponding finite difference scheme to problem (20), using the identity (21), at grid point (x_q,t_n) takes the following form:

$$\Delta t^{-1}\left(u_q^{n+1}-u_q^n\right)-\nu\Delta x^{-2}\left(u_{q+1}^n-2u_q^n+u_{q-1}^n\right)=f\left(x_q,t_n\right), \tag{22}$$

or equivalently: for all $q=1,\ldots,N-1$ and $n=1,\ldots,M$

$$u_q^{n+1}=\lambda u_{q-1}^n+(1-2\lambda)u_q^n+\lambda u_{q+1}^n+\Delta t f\left(x_q,t_n\right),$$

where $\lambda=\nu\Delta t/(\Delta x)^2$.

At the time $t_0=0$, the initial condition is given by:

$$u_q^0=u^0\left(x_q\right),\quad q=0,1,2,3,\ldots,N.$$

The boundary conditions are:

$$u_0^n=g_1\left(t_n\right),\quad u_N^n=g_2\left(t_n\right),\quad n=0,\ldots,M.$$

Thus, we can solve explicitly the equation (22).

2.4.2 Implicit scheme (IS)

A backward difference quotient is used to approximate the term u_t, (LeVeque, n.d.; LeVeque, 1992), at the point (x_q,t_n):

$$u_t\left(x_q,t_n\right)\approx\frac{u_q^n-u_q^{n-1}}{\Delta t}$$

In this case, we have:

$$u_{xx}\left(x_q,t_n\right)\approx\frac{u_{q+1}^n-2u_q^n+u_{q-1}^n}{(\Delta x)^2}$$

Then, the corresponding difference equation to the problem (20) at grid points (x_q,t_n) is:

$$\Delta t^{-1}\left(u_q^n-u_q^{n-1}\right)-\nu(\Delta x)^{-2}\left(u_{q+1}^n-2u_q^n+u_{q-1}^n\right)=f\left(x_q,t_n\right) \tag{23}$$

or equivalently: for all $q=1,2,3,\ldots,N-1$ and for all $n=1,\ldots,M$

$$-\lambda u_{q-1}^n+(1+2\lambda)u_q^n-\lambda u_{q+1}^n=u_q^{n-1}+\Delta t f\left(x_q,t_n\right),$$

Where $\lambda=\nu\frac{\Delta t}{(\Delta x)^2}$.

Let $U^n=\left(u_1^n,u_2^n,\ldots,u_{N-1}^n\right)^T\in\mathbb{R}^{N-1}$

be the unknown vector. Then, the equations (23) can be written in matrix form:

$$AU^n = U^{n-1} + b^n,$$

where A is a square tridiagonal matrix of order $N-1$ and is given by the following form:

$$A = \begin{pmatrix} 1+2\lambda & -\lambda & 0 \\ -\lambda & \ddots & -\lambda \\ 0 & -\lambda & 1+2\lambda \end{pmatrix}$$

and the second member $b^n \in \mathbb{R}^{N-1}$ is defined by:

$$b^n = \left(\lambda u_0^n + \Delta t f(x_1, t_n), \Delta t f(x_2, t_n), \ldots, \lambda u_N^n + \Delta t f(x_{N-1}, t_n)\right).$$

2.4.3 Some practical remarks

In the presence of so many possible methods for solving problem (Stroud, 1974), the question of choosing between all these schemes arises? We will see that the numerical analysis results of the next sections will allow us to propose some criteria for choosing between all these schemes. Some of these criteria already appear, since it is clear that the implementation of an implicit schema is more costly than that of an explicit schema, because it requires the resolution of a linear system at each time step; it is also clear that the use of a three-level schema will require more space in memory to store the information necessary for calculating the new values u^{n+1}.

2.5 Discrete maximum principle

In this section we will study the two schemes (22) and (23) from the previous section. We will see that the convergence of these two schemes is based on a principle property of the discrete maximum.

2.5.1 Consistency and precision

Let's begin the analysis of explicit (22) by defining the important notions of consistency, precision and truncation error.

We give the following definition using the explicit scheme (22), (for the system (20)), to fix the ideas, but the reader will easily transpose it to other schemes.

Definition 3. (Causon & Mingham, n.d.; Larrouturou, 1998; Rappaz & Picasso, 1998; Raviart &Thomas, n.d.; LeVeque, n.d.; LeVeque, 1992; Crank & Nicolson, 1947; Teukolsky, 2000). *We say that the scheme:*

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \nu \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{(\Delta x)^2} \tag{24}$$

defines a consistent approximation of the operator $\frac{\partial}{\partial s} - \nu \frac{\partial^2}{\partial \omega^2}$ *if, for any function* $v = v(\omega, s)$, *(sufficiently regular), the difference:*

$$\left(\frac{v(\omega, s+\Delta s) - v(\omega, s)}{\Delta s} - \nu \frac{v(\omega+\Delta\omega, s) - 2v(\omega, s) + v(\omega-\Delta\omega, s)}{(\Delta\omega)^2}\right) - (v_s - \nu v_{\omega\omega})(\omega, s)$$

(25)

tends towards 0 *when* $\Delta\omega$ *and* Δs *tend towards* 0 *independently.*

The difference (25) is called the truncation error of the scheme, (for function v).

We further say that scheme (24) defines an approximation precise to order p in space and to order q in time for $\frac{\partial}{\partial t} - \nu \frac{\partial^2}{\partial x^2}$ *operator*

if and only if, for any regular function v, the truncation error (25) tends towards 0 *like* $O(\Delta x^p + \Delta t^q)$ *when* Δx *and* Δt *tend towards* 0 *independently.*

Table 1: Table of some numerical finite difference schemes.

Scheme	Truncation error
Explicit (22)	O(Δx2 + Δt)
Implicit (23)	O(Δx2 + Δt)

To clarify this definition, let us verify the following Lemma.

Lemma 2. (Causon & Mingham, n.d.; Larrouturou, 1998; Rappaz & Picasso, 1998; Raviart &Thomas, n.d.; LeVeque, n.d.; LeVeque, 1992; Crank & Nicolson, 1947; Teukolsky). *The explicit scheme (23) is consistent, precise to order two in space and one in time, for solving numerically the one-dimensional heat equation* $u_t - \nu u_{xx} = 0$.

Proof It is enough to write Taylor expansions [22]. For any function v of class C^4, we easily obtain:

$$\frac{v(x,t+\Delta t)-v(x,t)}{\Delta t} - \nu\frac{v(x+\Delta x,t)-2v(x,t)+v(x-\Delta x,t)}{(\Delta x)^2} = (v_t - \nu v_{xx})(x,t) + \frac{\Delta t}{2} v_{tt}(x,t+\theta_t\Delta t) - \frac{\nu\Delta x^2}{24}\left(v_{xxxx}(x+\theta_x^-\Delta x,t) + v_{xxxx}(x+\theta_x^+\Delta x,t)\right), \quad (26)$$

Where $0 \le \theta_t \le 1$ and $-1 \le \theta_x^- \le 0 \le \theta_x^+ \le \Delta x.$

Note that all these schemes are consistent, precise to order one or two in space and time, with the exception of the Dufort-Frankel scheme which is inconsistent.

3. Numerical examples

In the following, let $\Omega = (a,b) = (0,1)$.

Example 1. *Consider a first-order differential equation:*

$$\begin{aligned} y'(x) &= f(x, y(x)), \\ y(0) &= \alpha, \end{aligned}$$

where the exact solution:

$$y(x) = \sin(\pi x) + x^2,$$

the initial condition $y_0 = 0$, *and the second member*

$$f(x, y(x)) = \pi\cos(\pi x) + 2x.$$

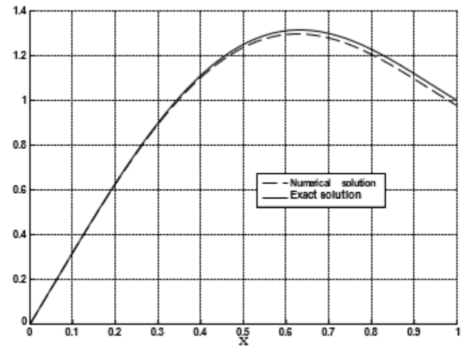

Figure 2: The exact solution and the numerical solution obtained by forward

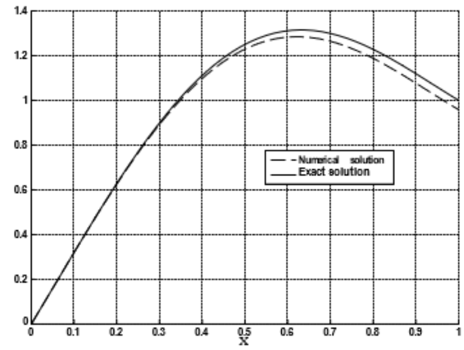

Figure 3: The exact solution and the numerical so- solution obtained by Rung-Kutta method.

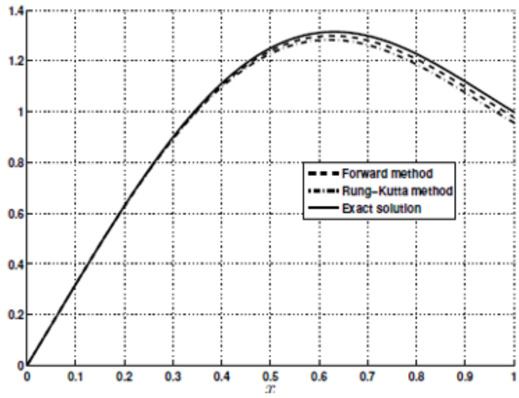

Figure4. The exact solution and the numerical solutions obtained by forward method and Rung- Kutta method

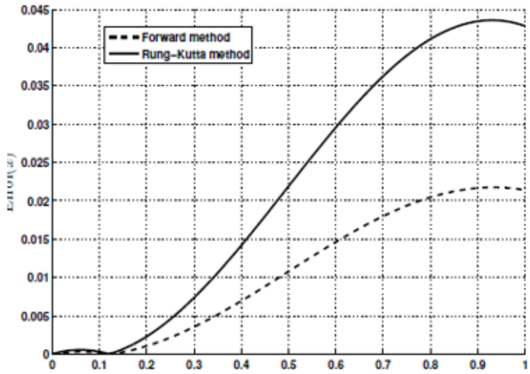

Figure 5: The error between the exact solution and the numerical solution obtained by different methods.

Figures 2, 3, and 4 depict both the exact solution and the numerical solutions yielded by various numerical methods. Additionally, Figure 5 illustrates the error between the exact solution and the numerical solutions. The comprehensive analysis of these figures reveals that all employed numerical methods provide an outstanding approximation of the exact solution.

Example 2. *Consider the Dirichlet boundary problem:*

$$\begin{aligned} -u''(\omega) + c(\omega)u(\omega) &= f(\omega), \\ u(0) &= \alpha, \\ u(1) &= \beta \end{aligned}$$

where $u(0) = 1$ *and* $u(1) = 0$*. The exact solution:*

$$u(\omega) = \cos(\pi\omega) + \omega^3,$$

$$c(\omega) = \omega,$$

and the source term:

$$f(\omega) = \omega^3 c(\omega) - 6\omega + c(\omega) + \pi^2 \cos(\pi\omega).$$

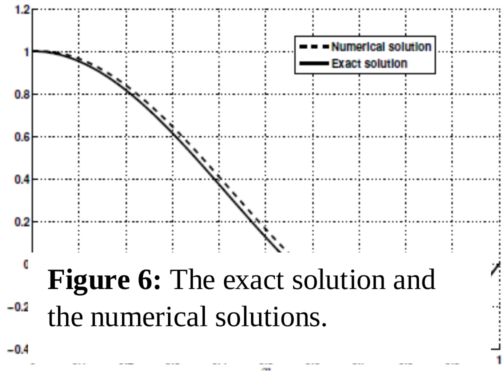

Figure 6: The exact solution and the numerical solutions.

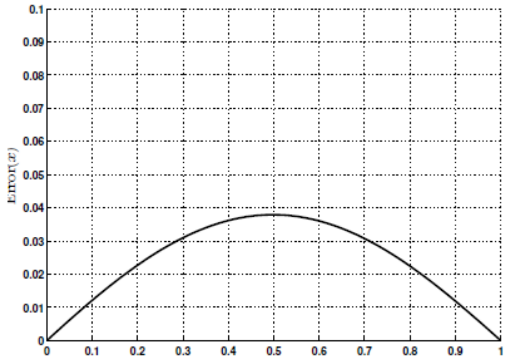

Figure 7: The error between the exact solution and the numerical solution

In Figure 6, we plotted the exact solution and the numerical solution obtained by different numerical method. Also, in Figure 7, we plotted the error between the exact solution and the numerical solution. We can deduce from all those figures that the numerical method gives an excellent approximation of the exact solution.

Example 3. *Consider the heat equation:*

$$\frac{\partial u}{\partial t}(x,t) - \nu \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t), \tag{27}$$

$$u(x,0) = u^0(x), \tag{28}$$

$$u(0,t) = g_1(t), \tag{29}$$

$$u(1,t) = g_2(t). \tag{30}$$

The exact solution of the system was chosen as follows:

$$u(x,t) = (1+t)^2 x[1+\sin(2\pi x)],$$

and consequently, we get:

$$\begin{aligned} u_0(x) &= x(1+\sin(2\pi x)), \\ g_1(t) &= 0, \quad g_2(t) = (1+t)^2, \\ f(x,t) &= 2(1+t)x(1+\sin(2\pi x)) - 4\pi\nu(1+t)^2[\cos(2\pi x) - \pi x \sin(2\pi x)], \qquad \nu = 0.002. \end{aligned}$$

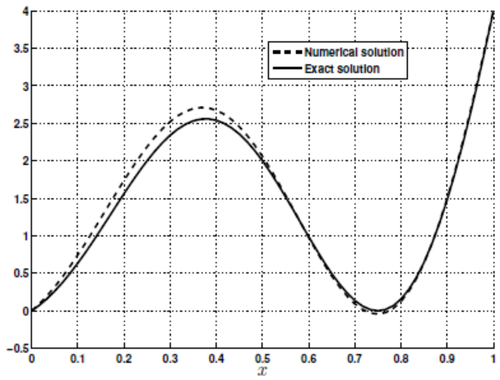

Figure 8: The exact solution and the numerical solution obtained by explicit scheme.

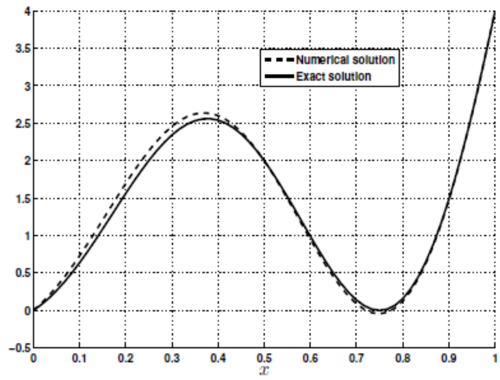

Figure 9: The exact solution and the numerical solution obtained by implicit scheme.

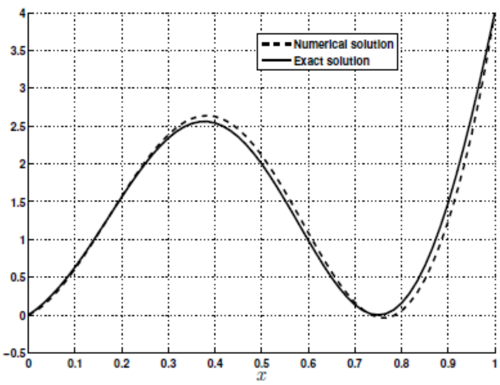

Figure 10: The exact solution and the

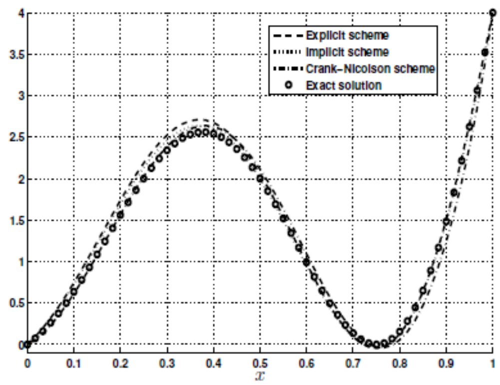

Figure 11: The exact solution and the three

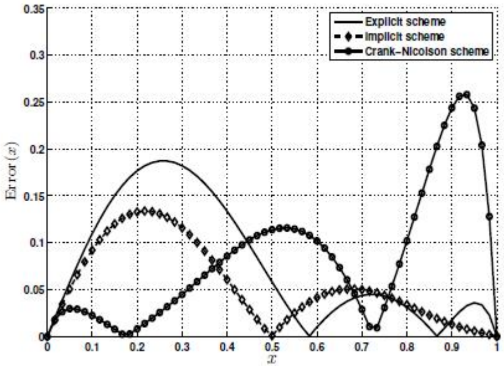

Figure 12: The error between the exact solution and the numerical solutions obtained by different schemes.

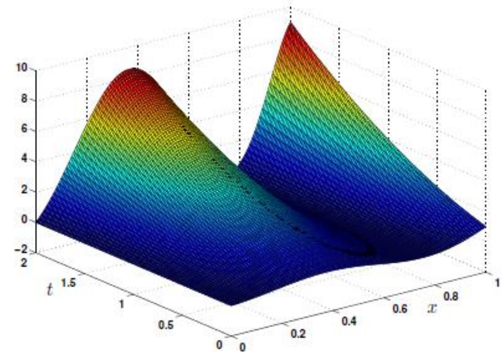

Figure 13: The evolution, in time on [0,2], of the numerical solution obtained by explicit scheme.

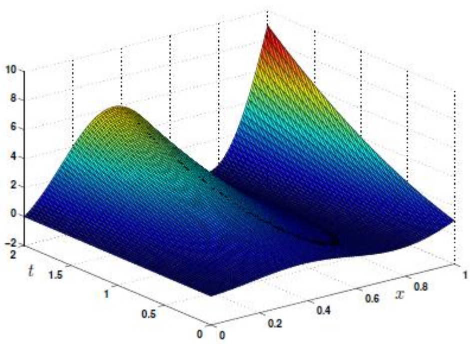

Figure 14: The evolution, in time on [0,2], of the the numerical solution obtained by Crank-Nicolson scheme.

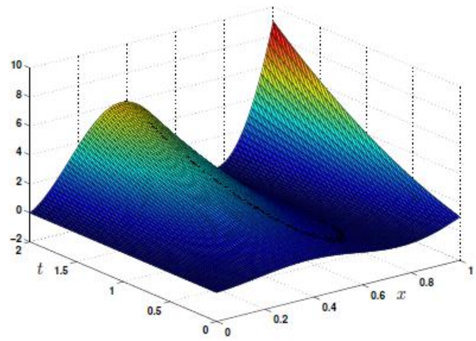

Figure 15: The evolution, in time on [0, 2], of numerical solution obtained by implicit scheme.

In Figures 8, 9, 10 and 11, we plotted the exact solution and the numerical solutions obtained by different numerical methods. Also, in Figure 12, we plotted the error between the exact solution and the numerical solutions. We can deduce from all those figures that all the numerical methods give an excellent approximation of the exact solution. Also, we plotted in Figures 13 to 15 the evolution of the numerical solution in time for $t \in [0,2]$ *given by the above numerical methods.*

In this section, we have achieved several goals. Firstly, we gave a general study of some finite difference schemes for the numerical resolution of certain EPDs and ODEs. Then, we proposed some well-chosen numerical examples to show the effectiveness of these numerical methods. The numerical implementation of the different examples was carried out using Matlab software.

4. Conclusion

In this research, we provided a comprehensive overview of the definition and numerical resolution of partial differential equations (PDEs). We briefly reviewed the most significant categories of PDEs that have been extensively studied across various scientific fields, including mathematics, engineering, medicine, and physics. We achieved several objectives in this work. Firstly, we conducted a general study of some finite difference schemes for the numerical resolution of specific PDEs and ordinary differential equations (ODEs). We proposed well-chosen numerical examples to demonstrate the effectiveness of these numerical methods, and the numerical implementation of different examples was carried out using MATLAB software.

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