

A Mathematical Analysis of Refuge, Migration and Immigration Effects on Prey-Predator Dynamics

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Abstract

The analysis of the dynamic predator-prey model with functional response is proposed in this paper. A mathematical model is constructed through the interaction of one prey and two predator systems to study the influence of prey refuge, migration of the first predator and immigration of the second predator. The model examines the Holling type I functional responses for predation between the prey and predator's. The prey refuge component is essential for managing the predator species and achieving prey-predator balances within the habitat. The system investigates the mortality rate, effects of migration and immigration, positiveness, boundedness, and asymptotic behaviour at every potential equilibrium point for the species survival in the ecology. Consequently, the stability study of the coexistence of the three species is carried out for positive interior equilibrium points both at a local and global level using the Routh Hurwitz criterion and the Lyapunov function. Furthermore, numerical simulations are performed to analyze the behavior of the system.

Keywords: Predator, Prey, Holling Type I Functional Response, Refuge, Migration, Immigration, Routh Hurwitz Criterion, Lyapunov Function.

1. Introduction

In dynamic ecological systems, predator-prey interactions play a pivotal role in driving population dynamics, significantly influencing the abundance, distribution, and behavior of species. These interactions are characterized by oscillatory patterns, where changes in predator and prey populations reciprocally affect each other. The abundance of one species can profoundly influence the population dynamics of the other. This delicate balance involves predators regulating prey populations while prey influencing the growth and survival of predators. Such complex interplay leads to diverse and fascinating outcomes, including population cycles, trophic cascades, and species coexistence. The ecological system consists of numerous interactions among organisms within a habitat. Key factors such as predation, intra-specific competition, migration, immigration, and prey refuge are essential in shaping population dynamics and maintaining ecosystem stability.

Predation is a vital part of nature where predators hunt and eat prey. These interactions affect the populations and evolutionary changes of species over time. It puts pressure on prey to develop ways to survival strategies, such as hiding or running faster. Concurrently, predators improve their skills at catching prey, leading to a constant race for survival between them. Predation also leads to fluctuations in prey numbers. When prey populations increase, predator populations also grow due to the abundance of food. As the number of predators rises, the prey population decreases because of higher predation pressure. The decrease in prey numbers results in a subsequent reduction in predator numbers due to food scarcity, allowing the prey population to recover, and the cycle restarts. This cycle helps keep the environment balanced. Besides population dynamics, predation influences how prey behaves. Prey may adapt their

foraging strategies and hiding places to avoid predators. This adaptive behavior enhances their survival and prevents one species from dominating, which promotes the overall health and resilience of the ecosystem. These interactions can be influenced by various factors, including the availability of resources, habitat structure, and environmental conditions.

Intra-specific competition, or competition within a species, also plays a significant role. Individuals of the same species compete for the same resources, such as food, shelter, and mate. This competition can limit population growth and lead to natural selection, where only the fittest individuals survive and reproduce, thereby enhancing the adaptive capacity of the species. Migration and immigration add further complexity to these dynamics, as species may relocate to new areas in search of resources, breeding opportunities, or more favorable living conditions, which can introduce new predator-prey relationships and alter existing ones. The arrival of new individuals can lead to increased competition for resources, both within and between species, affecting population growth and survival rates. Prey refuge is another critical aspect of predator-prey interactions. Refuges provide prey species with areas where they can avoid predators, reducing predation pressure and allowing prey populations to sustain themselves. These refuges can be physical spaces, such as burrows or dense vegetation, or temporal, such as adopting behaviors that help prey avoid predators active during daylight hours.

In this discussion, investigating the significance of each factor within biological contexts leads to the introduction of differential equations that capture their effects on population dynamics. Differential equations serve as powerful tools for modeling ecological systems, allowing quantification of the effects of migration, immigration, predation, intra-specific competition, and prey refuge on species abundance and distribution over time. By integrating these factors into ecological models, researchers gain understanding of the complex interactions between species and their environment. Scientists dedicate significant research efforts to understanding predator-prey systems within ecological contexts, exploring the intricate dynamics between predators and prey, and examining various aspects such as population dynamics, behavioral interactions, prey refuge and evolutionary adaptations [2, 8, 19, 22, 31]. Recent studies highlight the significance of prey refuges in predator-prey dynamics, uncovering complex behaviors such as unbounded solutions, multiple equilibria, and limit cycles [6]. Evaluation of prey-predator models that integrate Holling type II and III responses with prey refuges aim to provide mathematical view of biological dynamics, revealing multiple limit cycles around positive equilibria influenced by refuge [15,16,30,36]. Notably, Zhihui Ma et al. analyze a predator-prey system with Holling type function response and prey refuge, highlighting the influence of functional response shape and noting prey refuge's destabilizing effect under certain conditions [37]. Mussa Amos Stephano et al. highlight how the influence of refuge prey on system stability varies according to the functional response type. The observation is that as predator populations rise, an increase in refuge-seeking prey diminishes the risk of prey extinction. This emphasizes the crucial function of prey refuges in controlling predator populations and maintaining ecological balance in ecosystems [23].

Additionally, Debaldev Jana et al. study the impacts of prey refuge usage on prey-predator dynamics, exploring interspecific competition and refuge patch positioning through mathematical and numerical analyses [4]. Longxing Qi et al. present a model considering Allee effect and prey carrying capacity, revealing Hopf bifurcation dynamics dependent on these factors, while Ghosh, Sahoo, and Poria investigate how additional food for predators affects prey-predator dynamics, emphasizing stability behaviors and Hopf bifurcation conditions [17,18]. Md. Manarul Haque et al. inspect a prey refuge and independent harvesting, revealing refuge's stabilizing effect and optimal harvesting policies, whereas Odhiambo Francis et al. analyze a multi-species system with prey refuge and Holling type III response, emphasizing refuge's role in enhancing dynamic stability [21,25].

Furthermore, Robert M May's works explore into the dynamics of model ecosystems, exploring stability, complexity, and the presence of multiple stable states [27,28]. Yuming Chen and Fengqin Zhang propose a delayed predator-prey model with migration, revealing backward bifurcation and delay-dependent stability of positive equilibrium, while Gang Zhu et al. investigate a predator-prey system with delayed dynamics, uncovering fixed point and Hopf bifurcations linked to critical time delays [34,35,9]. Figen Kangalgil et al. explore a discrete-time predator-prey system with prey immigration, analyzing fixed points and bifurcations, whereas Naveed Ahmad Khan et al. analyze a prey-predator system with immigrant prey, introducing a novel algorithm for solution approximation [7,24]. Gui-Quan Suna et al. examine spatial patterns in a predator-prey model with migration and diffusion, revealing migration's significant influence on pattern formation [10].

Moreover, Debasis Mukherjee investigates the influence of immigration and refuge on a three-species system, demonstrating that increasing refuges enhance system stability, whereas Banani Roy explores a prey-predator model with a reserved predator region, analyzing equilibria and local stability [5,3]. Alebraheem J introduces an autonomous predator-prey model, analyzing dynamic behaviors and stability, while Salih Djilali study the impact of fractional time derivative memory on temporal behavior [1,29]. Xiangdong Xie et al. and Qin Yue propose modified Leslie-Gower predator-prey models, analyzing equilibrium structure, stability, and the influence of prey refuge [26,32]. Ultimately, Maruthai Selvaraj Surendar et al. analyze stability, bifurcation, and spatial pattern dynamics, demonstrating non-convergence of populations in the Turing domain [20]. Finally, Ying Huang et al. and Hafizul Molla et al. inspect the predator-prey models with Allee effect and prey refuge, determining stability conditions and identifying complex system dynamics [14,33]. Gunasekaran M et al. explore stability analysis of ecological models with various prey-predator dynamics, contributing to understanding ecosystem behaviors [11-13].

2. Structural Analysis of the Mathematical Model

The proposed ecological model depicts the dynamics of three interacting species: prey (u), migration of the first predator (v) and immigration of the second predator (w). Within this framework, interspecific predation occurs between the different groups of species. The model utilizes the Holling Type I functional response to communicate the species interaction in the ecology. The growth of the prey species is influenced by predation pressure from the first predator, as well as by the presence of refuge areas and natural interactions with the second predator. Notably, there is no direct contact between the predators. Additionally, the model incorporates intraspecific competition among the same group of predators and takes into account the natural growth rate of immigrating predators. Furthermore, the model examines the effects of predator death rate, migration, immigration on the habitat and all the equilibrium point elucidating how these factors impact the environment naturally. By integrating these components, the model provides a deeper understanding the complex dynamics of predator-prey interactions and their implications for ecosystem stability and functioning. The presented model constructs the set of equations represents a system of ordinary differential equations describing the dynamics of three interacting variables (u), (v), and (w) over time (t). This system can be expressed in a standard form, which is a common way to organize and present systems of differential equations. The standard form for a system of three first-order differential equations is:

$$\begin{aligned}\frac{du}{dt} &= X(u, v, w, t) \\ \frac{dv}{dt} &= Y(u, v, w, t) \\ \frac{dw}{dt} &= Z(u, v, w, t)\end{aligned}$$

Rephrase the given equations using a conventional format.

$$\left. \begin{aligned}\text{Equation for (u): } \frac{du}{dt} &= u\left(r - \frac{u}{k}\right) - b(1 - \rho)uv - cuw - hu \\ \text{Equation for (v): } \frac{dv}{dt} &= -ev^2 + fb(1 - \rho)uv - m_1v - d_1v \\ \text{Equation for (w): } \frac{dw}{dt} &= sw - lw^2 + guw + m_2w - d_2w\end{aligned}\right\} \dots(2.1)$$

Here, $X(u, v, w, t)$, $Y(u, v, w, t)$ and $Z(u, v, w, t)$ represent the functions defining the rate of change of u , v and w respectively, with respect to time t . The parameters $r, k, b, \rho, c, h, e, f, m_1, d_1, s, l, g, m_2$, and d_2 are constants that affect the behavior of the system.

This system extends the above model by incorporating a refuge protecting ρu of the prey, where $\rho \in [0,1)$ is constant. This leaves $(1 - \rho)u$ of the prey available to the first level predator, and modifying system (2.1) accordingly yields the system:

Now, let's rewrite the given equations in standard form:

$$\left. \begin{aligned}\frac{du}{dt} &= u\left(r - \frac{u}{k}\right) - b(1 - \rho)uv - cuw - hu \\ \frac{dv}{dt} &= -ev^2 + fb(1 - \rho)uv - m_1v - d_1v \\ \frac{dw}{dt} &= sw - lw^2 + guw + m_2w - d_2w\end{aligned}\right\} \dots(2.2)$$

To determine the stability for the following assumption

$$\left. \begin{aligned} \frac{1}{k} &= a, & h &= qE, & fb &= n \\ \frac{du}{dt} &= ru - au^2 - b(1-\rho)uv - cuw - qEu \\ \frac{dv}{dt} &= -ev^2 + n(1-\rho)uv - m_1v - d_1v \\ \frac{dw}{dt} &= sw - lw^2 + guw + m_2w - d_2w \end{aligned} \right\} \dots(2.3)$$

These equations describe the rates of change of the variables u , v , and w with respect to time t , under the condition that $u(t) \geq 0$, $v(t) \geq 0$, $w(t) \geq 0$. The system operates under certain initial conditions and parameter values.

3. The Mathematical Symbolism of the Equation System

The system of differential equations describes the dynamics of an ecological model with three interacting species: prey (u), first predator (v), and second predator (w).

- (i) The prey population, governed by logistic growth, is influenced by its intrinsic growth rate (r), carrying capacity (k), intra-specific competition (a), interactions with both predators (modulated by parameters b and c), prey refuge parameter ($1 - \rho$), where ρ lies in the interval $[0,1)$ as well as external factors such as harvesting (qE).
- (ii) The first predator's population dynamics are affected by its intra-specific competition (e), predation on prey (n), migration (m_1) and mortality (d_1).
- (iii) The second predator's population growth is influenced by its interaction with prey (g), natural growth (s), intra-specific competition (l), migration (m_2), and mortality (d_2).
- (iv) The parameters ρ , r , s , and qE represent ecological parameters, while a , b , c , e , n , l , g , m_1 , m_2 , d_1 and d_2 are coefficients governing species interactions and vital rates.

4. Equilibrium state:

- (i). Trivial equilibrium state $Eq_1(0,0,0)$ all the species extinct state.
- (ii). Semi trivial equilibrium state $Eq_2\left(\frac{(r-qE)}{a}, 0, 0\right)$ prey species is livelihood
- (iii). Semi trivial equilibrium state $Eq_3\left(0, \frac{-(m_1+d_1)}{e}, 0\right)$ first predator is survival
- (iv). Semi trivial equilibrium state $Eq_4\left(0, 0, \frac{(s+m_2-d_2)}{l}\right)$ second predator is survival
- (v). Planar equilibrium state $Eq_5\left(\frac{(r-qE)e+b(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2}, \frac{(r-qE)n(1-\rho)-a(m_1+d_1)}{ae+bn(1-\rho)^2}, 0\right)$ prey and first predator exists state.
- (vi). Planar equilibrium states $Eq_6\left(0, \frac{-(m_1+d_1)}{e}, \frac{(s+m_2-d_2)}{l}\right)$ both predators exist state.
- (vii). Planar equilibrium state $Eq_7\left(\frac{(r-qE)l-c(s+m_2-d_2)}{al+cg}, 0, \frac{(r-qE)g+a(s+m_2-d_2)}{al+cg}\right)$ prey and second predator exists state.
- (viii). All the species exist $Eq_8(u^*, v^*, w^*)$

$$u^* = \frac{le(r-qE) + lb(1-\rho)(m_1+d_1) - ce(s+m_2-d_2)}{ale + lbn(1-\rho)^2 + ceg},$$

$$v^* = \frac{ln(r-qE)(1-\rho) - la(m_1+d_1) - cn(1-\rho)(s+m_2-d_2) - cg(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg},$$

$$w^* = \frac{(r-qE)eg + ae(s+m_2-d_2) + nb(1-\rho)^2(s+m_2-d_2) + gb(1-\rho)(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg}$$

5. Positivity and Boundedness of the system

Lemma: 5.1

All the solutions of $(u(t), v(t), w(t))$ of system (2.3) with initial value $(u_0, v_0, w_0) \in \mathbb{R}_+^3$ remains positive for all $t \geq 0$.

Proof:

The right-hand sides of systems are continuous and satisfy a Lipschitz condition on $\mathbb{R}_+^3$. Hence, the solution $(u(t), v(t), w(t))$ of system (2.3) exists and are unique on the interval $[0, T)$ where $0 < T < \infty$.

Positivity of $u(t)$:

From the first equation of system (2.3),

$$\frac{du}{dt} = ru - au^2 - b(1 - \rho)uv - cuw - qEu$$

The solution is given by

$$u(t) = u(0) \exp \left[\int_0^t \{r - au(\theta) - b(1 - \rho)v(\theta) - cw(\theta) - qE\} d\theta \right] > 0$$

Since $u(0) > 0$, the exponential term is always positive. Thus, $u(t) > 0$ for all $t > 0$.

Positivity of $v(t)$:

From the second equation of the system (2.3),

$$\frac{dv}{dt} = -ev^2 + n(1 - \rho)uv - m_1v - d_1v$$

The solution is given by

$$v(t) = v(0) \exp \left[\int_0^t \{-ev(\theta) + n(1 - \rho)u(\theta) - m_1 - d_1\} d\theta \right] > 0$$

Since $v(0) > 0$, the exponential term is always positive. Thus, $v(t) > 0$ for all $t > 0$.

Positivity of $w(t)$:

From the third equation of the system (2.3),

$$\frac{dw}{dt} = sw - lw^2 + guw + m_2w - d_2w$$

The solution is given by

$$w(t) = w(0) \exp \left[\int_0^t \{s - lw(\theta) + gu(\theta) + m_2 - d_2\} d\theta \right] > 0$$

Since $w(0) > 0$, the exponential term is always positive. Thus, $w(t) > 0$ for all $t > 0$.

Since $u(t) > 0$, $v(t) > 0$, $w(t) > 0$ for all $t \geq 0$. The interior of $\mathbb{R}_+^3$ is an invariant set of system (2.3).

Lemma: 5.2

The set $D = \{(u, v, w) \in \mathbb{R}_+^3 : 0 \leq \delta = pu + v + w \leq \frac{\tau}{\eta}\}$ is positively invariant and serves as the region of attraction for all solution originating in the interior of the positive orthant, where

- $\eta > 0$ is a constant such that $\eta < \min(d_1, d_2, m_1)$, and
- $\tau = \frac{p(r+\eta)^2}{4a} + \frac{(s+m_2+\eta)^2}{4l} + \frac{\eta^2}{4e}$.

Proof:

Let $\delta = pu + v + w$ and consider the time derivative:

$$\frac{d\delta}{dt} = p \frac{du}{dt} + \frac{dv}{dt} + \frac{dw}{dt}$$

$$\begin{aligned}
&= p[ru - au^2 - b(1 - \rho)uv - cuw - qEu] + [-ev^2 + n(1 - \rho)uv - m_1v - d_1v] \\
&\quad + [sw - lw^2 + guw + m_2w - d_2w] \\
\frac{d\delta}{dt} + \eta\delta &= p[ru - au^2 - qEu] + [-ev^2 - m_1v - d_1v] + [sw - lw^2 + m_2w - d_2w] + \eta p(u) + \eta v + \eta w \\
\frac{d\delta}{dt} + \eta\delta &= \left[-pa \left(u - \frac{(r + \eta) - qE}{2a} \right)^2 - e \left(v - \frac{\eta - (m_1 + d_1)}{2e} \right)^2 - l \left(w - \frac{(s + m_2 - d_2) + \eta}{2l} \right)^2 \right. \\
&\quad \left. + \frac{p((r + \eta) - qE)^2}{4a} + \frac{(\eta - (m_1 + d_1))^2}{4e} + \frac{((s + m_2 - d_2) + \eta)^2}{4l} \right] \\
\frac{d\delta}{dt} + \eta\delta &= \left[\frac{p((r + \eta) - qE)^2}{4a} + \frac{(\eta - (m_1 + d_1))^2}{4e} + \frac{((s + m_2 - d_2) + \eta)^2}{4l} \right]
\end{aligned}$$

The maximum value of the quadratic terms is given by

$$\begin{aligned}
\frac{d\delta}{dt} + \eta\delta &\leq \tau \text{ (constant)} \\
\text{where } \tau &= \frac{p(r + \eta)^2}{4a} + \frac{\eta^2}{4e} + \frac{(s + m_2 + \eta)^2}{4l}.
\end{aligned}$$

Which is linear differential equation in δ , by solving it gives $\delta \leq \frac{\tau}{\eta} + Ce^{-\eta t}$ at $t = 0$ & $\delta = 0$. As a result $C = -\frac{\tau}{\eta}$. Therefore, $\delta \leq \frac{\tau}{\eta}(1 - e^{-\eta t})$ and since $\delta \geq 0$, it follows that $0 \leq \delta \leq \frac{\tau}{\eta}(1 - e^{-\eta t})$. As $t \rightarrow \infty$, $\delta \leq \frac{\tau}{\eta}$ ensuring δ remains bounded for all $t \geq 0$. Hence all solutions of the system are bounded.

6. Steady State Analysis and Feasibility

To investigate the dynamic equilibrium states of the model under consideration, the analysis involves examining the Jacobian matrix at various points, which is indicated by

$$J = \begin{bmatrix} r - 2au - b(1 - \rho)v - cw - qE & -b(1 - \rho)u & -cu \\ n(1 - \rho)v & -2ev + n(1 - \rho)u - m_1 - d_1 & 0 \\ gw & 0 & s - 2lw + gu + m_2 - d_2 \end{bmatrix}$$

This matrix facilitates understanding the stability and characteristics of the equilibrium points. By analyzing the resultant deviation matrices, each equilibrium state within the model can be effectively characterized.

Proposition: 6.1 The trivial equilibrium point $Eq_1 (0,0,0)$ is inherently unstable under conditions where $r > qE$ and $s + m_2 > d_2$, given the positive growth rate of each species.

Proof: The jacobian matrix evaluated at the trivial equilibrium point Eq_1 is expressed as

$$J(Eq_1) = \begin{bmatrix} r - qE & 0 & 0 \\ 0 & -(m_1 + d_1) & 0 \\ 0 & 0 & s + m_2 - d_2 \end{bmatrix}$$

The eigen values of the above matrix $\lambda_1 = r - qE$, $\lambda_2 = -(m_1 + d_1)$ and $\lambda_3 = s + m_2 - d_2$. Since growth rate of each species exceed their respective mortality rates and harvesting rate. It is noted that second eigen values is negative, while the other two eigen values are produce positive if $r > qE$ and $s + m_2 > d_2$. Therefore, the jacobian matrix at $J(Eq_1)$ is unstable.

Proposition: 6.2 The semi trivial equilibrium state in which prey exist equilibrium point $Eq_2 \left(\frac{r - qE}{a}, 0, 0 \right)$ is unstable. If $n(1 - \rho)(r - qE) > a(m_1 + d_1)$ and $s + m_2 > d_2$.

Proof: The jacobian matrix at the semi trivial equilibrium point Eq_2 for the prey existence state is characterized by

$$J(Eq_2) = \begin{bmatrix} -(r - qE) & \frac{-b(1 - \rho)(r - qE)}{a} & \frac{-c(r - qE)}{a} \\ 0 & \frac{n(1 - \rho)(r - qE) - a(m_1 + d_1)}{a} & 0 \\ 0 & 0 & \frac{a(s + m_2 - d_2) + g(r - qE)}{a} \end{bmatrix}$$

The eigen values of the above matrix can be expressed as $\lambda_1 = -(r - qE)$, $\lambda_2 = \frac{n(1 - \rho)(r - qE) - a(m_1 + d_1)}{a}$ and $\lambda_3 = \frac{a(s + m_2 - d_2) + g(r - qE)}{a}$. Since $\lambda_1 < 0$, $\lambda_2 > 0$, $\lambda_3 > 0$. As a result, the equilibrium of the system $J(Eq_2)$ is unstable, with presence of saddle points.

Proposition: 6.3 The semi trivial equilibrium state where first predator exists at equilibrium point $Eq_3 \left(0, \frac{-(m_1 + d_1)}{e}, 0\right)$ is inherently unstable under the condition $r > qE$, $\rho < 1$ and $s + m_2 > d_2$.

Proof: The Jacobian matrix at the semi-trivial equilibrium point Eq_3 where the first predator exists is determined by

$$J(Eq_3) = \begin{bmatrix} \frac{e(r - qE) + b(1 - \rho)(m_1 + d_1)}{e} & 0 & 0 \\ -\frac{n(1 - \rho)(m_1 + d_1)}{e} & (m_1 + d_1) & 0 \\ 0 & 0 & s + m_2 - d_2 \end{bmatrix}$$

The eigen values of the above jacobian matrix $\lambda_1 = \frac{e(r - qE) + b(1 - \rho)(m_1 + d_1)}{e}$, $\lambda_2 = (m_1 + d_1)$ and $\lambda_3 = s + m_2 - d_2$. it observed that all the eigen values are positive if $r > qE$, $\rho < 1$ and $s + m_2 > d_2$. So, $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_3 > 0$. Consequently, the steady state of the system $J(Eq_3)$ is unstable.

Proposition: 6.4 The semi trivial equilibrium state where the second predator exists at equilibrium point $Eq_4 \left(0, 0, \frac{(s + m_2 - d_2)}{l}\right)$ is stable, if $\frac{l(r - qE) - c(s + m_2 - d_2)}{l} < 0$. Otherwise it is unstable.

Proof: The Jacobian matrix of the system at the semi-trivial equilibrium point Eq_4 , where the second predator exists, is identified by

$$J(Eq_4) = \begin{bmatrix} \frac{l(r - qE) - c(s + m_2 - d_2)}{l} & 0 & 0 \\ 0 & -(m_1 + d_1) & 0 \\ \frac{g(s + m_2 - d_2)}{l} & 0 & -(s + m_2 - d_2) \end{bmatrix}$$

The eigen value of the matrix are $\lambda_1 = \frac{l(r - qE) - c(s + m_2 - d_2)}{l}$, $\lambda_2 = -(m_1 + d_1)$ and

$\lambda_3 = -(s + m_2 - d_2)$. It's observed that the root λ_2 , λ_3 are negative.

If, $\lambda_1 < 0$ (ie., $\frac{l(r - qE) - c(s + m_2 - d_2)}{l} < 0$), it follows that $\lambda_2 < 0$ and $\lambda_3 < 0$. Consequently, the semi trivial state of the system $J(Eq_4)$ is asymptotically stable.

If, $\lambda_1 > 0$ (ie., $\frac{l(r - qE) - c(s + m_2 - d_2)}{l} > 0$) it follows that $\lambda_2 < 0$ and $\lambda_3 < 0$. consequently, the semi trivial state of the system $J(Eq_4)$ is unstable and saddle point exists.

Proposition: 6.5 The planar equilibrium state at the second predator extinct state, given by $Eq_5 \left(\frac{(r - qE)e + b(1 - \rho)(m_1 + d_1)}{ae + bn(1 - \rho)^2}, \frac{n(r - qE)(1 - \rho) - a(m_1 + d_1)}{ae + bn(1 - \rho)^2}, 0\right)$ is unstable.

Proof: The Jacobian matrix at the planar equilibrium point where the prey and the first predator coexist, denoted as Eq_5 , is determined by

$$J(Eq_5) = \begin{bmatrix} -\left(\frac{(r-qE)ae+ab(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2}\right) & -\left(\frac{(r-qE)be(1-\rho)+b^2(1-\rho)^2(m_1+d_1)}{ae+bn(1-\rho)^2}\right) & -\left(\frac{(r-qE)ec+bc(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2}\right) \\ \left(\frac{(r-qE)n^2(1-\rho)^2-an(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2}\right) & \left(\frac{-(r-qE)en(1-\rho)+ae(m_1+d_1)}{ae+bn(1-\rho)^2}\right) & 0 \\ 0 & 0 & \frac{(s+m_2-d_2)(ae+bn(1-\rho)^2)+(r-qE)ge+gb(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2} \end{bmatrix}$$

The characteristic equation of the above matrix $J(Eq_5)$ is define by

$$\lambda_{1,2}^2 - \text{trace } J(u, v)\lambda_{1,2} + \det J(u, v) = 0 \text{ and}$$

$$\lambda_3 = \frac{(s+m_2-d_2)(ae+bn(1-\rho)^2)+(r-qE)ge+gb(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2}$$

The eigen values of this equation are

$$\lambda_1 = \frac{1}{2}[-(\text{trace } (u, v)) + \sqrt{(\text{trace } (u, v))^2 - 4(\det (u, v))}] < 0,$$

$$\lambda_2 = \frac{1}{2}[-(\text{trace } (u, v)) - \sqrt{(\text{trace } (u, v))^2 - 4(\det (u, v))}] < 0 \text{ and}$$

$$\lambda_3 = \frac{(s+m_2-d_2)(ae+bn(1-\rho)^2)+(r-qE)ge+gb(1-\rho)(m_1+d_1)}{ae+bn(1-\rho)^2} > 0.$$

Where $[\text{trace } (u, v)^2 - 4(\det (u, v))] > 0$, $-\text{trace } (u, v) < 0$ and

$$\sqrt{(\text{trace } (u, v))^2 - 4(\det (u, v))} < \text{trace } (u, v).$$

Where $(\text{trace } (u, v)) = \left(\frac{-(r-qE)(ae+en(1-\rho)+(m_1+d_1)(ae-ab(1-\rho)))}{ae+bn(1-\rho)^2}\right) < 0$ and

$$(\det(u, v)) = \left(\frac{\left(\left(-(r-qE)ae - ab(1-\rho)(m_1+d_1) \right) \left(-(r-qE)en(1-\rho) + ae(m_1+d_1) \right) \right) + \left(\left((r-qE)be(1-\rho) + b^2(1-\rho)^2(m_1+d_1) \right) \left((r-qE)n^2(1-\rho)^2 - an(1-\rho)(m_1+d_1) \right) \right)}{ae+bn(1-\rho)^2} \right) > 0$$

Given that $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 > 0$ (always), the system of planar equilibrium point $J(Eq_5)$ is unstable, indicating the presence of a saddle point.

Proposition: 6.6 The planar equilibrium state where the prey washed out, this state denoted as $Eq_6 \left(0, \frac{-(m_1+d_1)}{e}, \frac{(s+m_2-d_2)}{l}\right)$ is unstable.

Proof: The Jacobian matrix at the planar equilibrium point where the first and second predator coexist, identified as Eq_6 , is characterized by

$$J(Eq_6) = \begin{bmatrix} \frac{el(r-qE)+bl(1-\rho)(m_1+d_1)-ce(s+m_2-d_2)}{el} & 0 & 0 \\ -\frac{n(1-\rho)(m_1+d_1)}{e} & (m_1+d_1) & 0 \\ \frac{g(s+m_2-d_2)}{l} & 0 & -(s+m_2-d_2) \end{bmatrix}$$

The characteristic polynomial of the above matrix is

$$\left[\frac{el(r-qE)+bl(1-\rho)(m_1+d_1)-ce(s+m_2-d_2)}{el} - \lambda_1 \right] [(m_1+d_1) - \lambda_2][-(s+m_2-d_2) - \lambda_3] = 0$$

The eigen values corresponding of this characteristic's polynomial are

$$\lambda_1 = \frac{el(r-qE)+bl(1-\rho)(m_1+d_1)-ce(s+m_2-d_2)}{el}, \quad \lambda_2 = (m_1 + d_1) \quad \text{and} \quad \lambda_3 = -(s+m_2 - d_2).$$

It is observed that one eigen value ($\lambda_3 < 0$) is negative, while the other two eigen values are positive ($\lambda_1 > 0$, $\lambda_2 > 0$). Therefore, the steady state of the system J(Eq₆) is unstable, indicating the presence of saddle points.

Proposition: 6.7 The planar equilibrium state of the first predator is washed out, represented as $Eq_7 \left(\frac{(r-qE)l-c(s+m_2-d_2)}{al+cg}, 0, \frac{(r-qE)g+a(s+m_2-d_2)}{al+cg} \right)$ is stable if $\lambda_2 < 0$. otherwise it is unstable.

Proof: The Jacobian matrix at the planar equilibrium point where the prey and the second predator coexist, denoted as Eq_7 , is determined by

$$J(Eq_7) = \begin{bmatrix} \frac{-al(r-qE)+ac(s+m_2-d_2)}{al+cg} & \frac{-b(1-\rho)(l(r-qE)-c(s+m_2-d_2))}{al+cg} & \frac{-c(l(r-qE)-c(s+m_2-d_2))}{al+cg} \\ 0 & \frac{n(1-\rho)(l(r-qE)-c(s+m_2-d_2))-(al+cg)(m_1+d_1)}{al+cg} & 0 \\ \frac{g((r-qE)g+a(s+m_2-d_2))}{al+cg} & 0 & -\left(\frac{al(s+m_2-d_2)+gl(r-qE)}{al+cg}\right) \end{bmatrix}$$

The characteristic equation of the above matrix J(Eq₇) is

$$\lambda_{1,3}^2 - \text{trace } J(u,v)\lambda_{1,3} + \det J(u,v) = 0 \quad \text{and}$$

$$\lambda_2 = \frac{n(1-\rho)(l(r-qE)-c(s+m_2-d_2))-(al+cg)(m_1+d_1)}{al+cg}$$

The eigen values of this equation is

$$\lambda_1 = \frac{1}{2} \left[-(\text{trace } (u,v)) + \sqrt{(\text{trace } (u,v))^2 - 4(\det (u,v))} \right] < 0,$$

$$\lambda_3 = \frac{1}{2} \left[-(\text{trace } (u,v)) - \sqrt{(\text{trace } (u,v))^2 - 4(\det (u,v))} \right] < 0 \quad \text{and}$$

$$\lambda_2 = \frac{n(1-\rho)(l(r-qE)-c(s+m_2-d_2))-(al+cg)(m_1+d_1)}{al+cg}$$

Where $[\text{trace } (u,v)^2 - 4(\det (u,v))] > 0$, $-\text{trace } (u,v) < 0$ and

$$\sqrt{(\text{trace } (u,v))^2 - 4(\det (u,v))} < \text{trace } (u,v).$$

Where $(\text{trace } (u,v)) = \left(\frac{-al(r-qE)+ac(s+m_2-d_2)-al(s+m_2-d_2)-gl(r-qE)}{al+cg} \right) < 0$ and

Where $(\text{trace } (u,v)) = \left(\frac{-(r-qE)(al+gl)+(s+m_2-d_2)(ac-al)}{al+cg} \right) < 0$ and

$$\det(u,v) = \left(\frac{\left((-al(r-qE)+ac(s+m_2-d_2))(-al(s+m_2-d_2)-gl(r-qE)) \right) + \left(c(l(r-qE)-c(s+m_2-d_2)) \right) \left(g((r-qE)g+a(s+m_2-d_2)) \right)}{al+cg} \right) > 0$$

Since $\lambda_1 < 0$, $\lambda_3 < 0$ and $\lambda_2 < 0$ (if $(n(1-\rho)(l(r-qE)-c(s+m_2-d_2))-(al+cg)(m_1+d_1)) < 0$). Therefore, the system of equilibrium point J(Eq₇) is asymptotically stable. Otherwise, it is unstable (if $\lambda_2 > 0$) and saddle point exist.

Proposition: 6.8 The Positive interior equilibrium point $Eq_8(u^*, v^*, w^*)$ is defined by

$$u^* = \left(\frac{le(r-qE) + lb(1-\rho)(m_1+d_1) - ce(s+m_2-d_2)}{ale + lbn(1-\rho)^2 + ceg} \right),$$

$$v^* = \left(\frac{\ln(r-qE)(1-\rho) - la(m_1+d_1) - cn(1-\rho)(s+m_2-d_2) - cg(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg} \right),$$

$$w^* = \left(\frac{(r-qE)eg + ae(s+m_2-d_2) + nb(1-\rho)^2(s+m_2-d_2) + gb(1-\rho)(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg} \right) \text{ this equilibrium is locally asymptotically stable If } x_1 > 0, x_3 > 0, \text{ and } (x_1x_2 - x_3) > 0.$$

Proof: let us assume that prey and both predator's species coexist at equilibrium point Eq_8 is defined by

$$J(Eq_8) = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}$$

Where

$$a_{11} = \left(\frac{-ael(r-qE) - abl(1-\rho)(m_1+d_1) + ace(s+m_2-d_2)}{ael + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{12} = -b(1-\rho) \left(\frac{el(r-qE) + bl(1-\rho)(m_1+d_1) - ce(s+m_2-d_2)}{ael + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{13} = -c \left(\frac{el(r-qE) + bl(1-\rho)(m_1+d_1) - ce(s+m_2-d_2)}{ael + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{21} = n(1-\rho) \left(\frac{\ln(r-qE)(1-\rho) - la(m_1+d_1) - cn(1-\rho)(s+m_2-d_2) - cg(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{22} = \left(\frac{(ael + ceg)(m_1+d_1) - eln(r-qE)(1-\rho) + cen(1-\rho)(s+m_2-d_2)}{ael + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{31} = g \left(\frac{(r-qE)eg + ae(s+m_2-d_2) + nb(1-\rho)^2(s+m_2-d_2) + gb(1-\rho)(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg} \right),$$

$$a_{33} = - \left(\frac{(r-qE)egl + ael(s+m_2-d_2) + nbl(1-\rho)^2(s+m_2-d_2) + gbl(1-\rho)(m_1+d_1)}{ale + lbn(1-\rho)^2 + ceg} \right)$$

The characteristic equation of the jacobian matrix $J(Eq_8)$ is expressed as

$$\lambda^3 + x_1\lambda^2 + x_2\lambda + x_3 = 0$$

Where,

$$x_1 = (a_{11} + a_{22} + a_{33})$$

$$x_2 = (a_{22}a_{33}) + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{22} - a_{12}a_{21})$$

$$x_3 = a_{11}a_{22}a_{33} - a_{12}(a_{21}a_{33}) + a_{13}(-a_{22}a_{31})$$

The result of the Routh-Hurwitz criterion indicates that the coexistence steady state, when it exists as a positive interior equilibrium with negative real parts, occurs if and only if $x_1 > 0, x_3 > 0, (x_1x_2 - x_3) > 0$ and $x_3(x_1x_2 - x_3) > 0$. Consequently, if all three eigenvalues are negative, the equilibrium point $J(Eq_8)$ is always asymptotically stable. Otherwise, it is unstable.

7. Global Stability Analysis

This section aims to demonstrate the global stability of all equilibrium states in a three-species ecosystem model. A Lyapunov function, used to analyze system stability, will be employed to establish that the equilibrium states remain stable across different scenarios, providing clear perspective into the dynamics of the ecosystem.

Theorem: 7.1 The planar equilibrium state Eq_5 (u^*, v^*) is globally asymptotically stable.

Proof: Apply the Lyapunov function to analyze stability of the interior equilibrium points Eq_5 in the following manner.

$$L(u^*, v^*) = \left\{ u - u^* - u^* \ln \left(\frac{u}{u^*} \right) \right\} + \left\{ v - v^* - v^* \ln \left(\frac{v}{v^*} \right) \right\}$$

By differentiating L with respect to t

$$\begin{aligned}\frac{dL}{dt} &= \frac{du}{dt} \left[1 - \frac{u^*}{u}\right] + \frac{dv}{dt} \left[1 - \frac{v^*}{v}\right] \\ &= [u - u^*][r - au - b(1 - \rho)v - qE] + [v - v^*][-ev + n(1 - \rho)u - m_1 - d_1]\end{aligned}$$

Substituting $r - qE = au^* + b(1 - \rho)v^*$ and $-m_1 - d_1 = ev^* - n(1 - \rho)u^*$

$$\frac{dL}{dt} \leq - \left[[u - u^*]^2 \left(a + \frac{(b-n)(1-\rho)}{2} \right) + [v - v^*]^2 \left(e + \frac{(b-n)(1-\rho)}{2} \right) \right]$$

Which can be written as of $-\alpha^T X \alpha$. where $\alpha^T = (u - u^* \quad v - v^*)$, $X = \begin{pmatrix} a & \frac{(b-n)(1-\rho)}{2} \\ \frac{(b-n)(1-\rho)}{2} & e \end{pmatrix}$

Hence, the planer equilibrium state Eq_5 is globally asymptotically stable if $\frac{dL}{dt} < 0$. This condition is satisfied if the matrix X is positive definite. In this case, it is essential that all the principal minors of the matrix X are positive.

Theorem: 7.2 The planar equilibrium state $Eq_6 (v^*, w^*)$ is globally asymptotically stable.

Proof: Apply the Lyapunov function to analyze the interior equilibrium points Eq_6 as follows

$$L(v^*, w^*) = \left\{ v - v^* - v^* \ln \left(\frac{v}{v^*} \right) \right\} + \left\{ w - w^* - w^* \ln \left(\frac{w}{w^*} \right) \right\}$$

By differentiating L with respect to t

$$\begin{aligned}\frac{dL}{dt} &= \frac{dv}{dt} \left[1 - \frac{v^*}{v}\right] + \frac{dw}{dt} \left[1 - \frac{w^*}{w}\right] \\ &= [v - v^*][-ev - m_1 - d_1] + [w - w^*][s - lw + m_2 - d_2]\end{aligned}$$

Substituting $-m_1 - d_1 = ev^*$ and $s + m_2 - d_2 = lw^*$

$$\frac{dL}{dt} \leq -[[v - v^*]^2 (e) + [w - w^*]^2 (l)]$$

Which can be written as $-\beta^T Y \beta$. Where $\beta^T = (v - v^* \quad w - w^*)$, $Y = \begin{pmatrix} e & 0 \\ 0 & l \end{pmatrix}$

Therefore, the Planer equilibrium state Eq_6 is globally asymptotically stable when $\frac{dL}{dt} < 0$. This condition is satisfied if the matrix Y is positive definite. In this case, it is essential that all the principal minors of the matrix Y are positive.

Theorem: 7.3 The planar equilibrium state $Eq_7 (u^*, w^*)$ is globally asymptotically stable.

Proof: Applying the Lyapunov function to analyze the interior equilibrium points Eq_7 yields the following

$$L(u^*, w^*) = \left\{ u - u^* - u^* \ln \left(\frac{u}{u^*} \right) \right\} + \left\{ w - w^* - w^* \ln \left(\frac{w}{w^*} \right) \right\}$$

By differentiating L with respect to t

$$\begin{aligned}\frac{dL}{dt} &= \frac{du}{dt} \left[1 - \frac{u^*}{u}\right] + \frac{dw}{dt} \left[1 - \frac{w^*}{w}\right] \\ &= [u - u^*][r - au - cw - qE] + [w - w^*][s - lw + gu + m_2 - d_2]\end{aligned}$$

Substituting $r - qE = au^* + cw^*$ and $s + m_2 - d_2 = lw^* - gu^*$

$$\frac{dL}{dt} \leq - \left[[u - u^*]^2 \left(a + \frac{(c-g)}{2} \right) + [w - w^*]^2 \left(l + \frac{(c-g)}{2} \right) \right]$$

Which can be written as of $-\gamma^T Z \gamma$. where $\gamma^T = (u - u^* \quad w - w^*)$, $Z = \begin{pmatrix} a & \frac{(c-g)}{2} \\ \frac{(c-g)}{2} & l \end{pmatrix}$

Hence, the planar equilibrium state Eq_7 is globally asymptotically stable if $\frac{dL}{dt} < 0$. This condition is satisfied if the matrix Z is positive definite. In this case, it is essential that all the principal minors of the matrix Z are positive.

Theorem: 7.4 The positive interior equilibrium state Eq_8 (u^*, v^*, w^*) is shown to be globally asymptotically stable.

Proof: The Lyapunov function is applied to analyze the stability of the positive interior equilibrium points Eq_8 as detailed below

$$L(u^*, v^*, w^*) = \left\{ u - u^* - u^* \ln \left(\frac{u}{u^*} \right) \right\} + \left\{ v - v^* - v^* \ln \left(\frac{v}{v^*} \right) \right\} + \left\{ w - w^* - w^* \ln \left(\frac{w}{w^*} \right) \right\}$$

By differentiate L with respect to t

$$\begin{aligned} \frac{dL}{dt} &= \frac{du}{dt} \left[1 - \frac{u^*}{u} \right] + \frac{dv}{dt} \left[1 - \frac{v^*}{v} \right] + \frac{dw}{dt} \left[1 - \frac{w^*}{w} \right] \\ &= [u - u^*][r - au - b(1 - \rho)v - cw - qE] + [v - v^*][-ev + n(1 - \rho)u - m_1 - d_1] \\ &\quad + [w - w^*][s - lw + gu + m_2 - d_2] \end{aligned}$$

Substituting $r - qE = au^* + b(1 - \rho)v^* + cw^*$, $-m_1 - d_1 = ev^* - n(1 - \rho)u^*$
and $s + m_2 - d_2 = lw^* - gu^*$

$$\begin{aligned} \frac{dL}{dt} &\leq - \left[[u - u^*]^2 \left(a + \frac{(b - n)(1 - \rho) + (c - g)}{2} \right) + [v - v^*]^2 \left(e + \frac{(b - n)(1 - \rho)}{2} \right) \right. \\ &\quad \left. + [w - w^*]^2 \left(l + \frac{(c - g)}{2} \right) \right] \end{aligned}$$

Which can be written as $-\mu^T D \mu$.

$$\text{Where } \mu^T = (u - u^* \quad v - v^* \quad w - w^*), \quad D = \begin{pmatrix} a & \frac{(b-n)(1-\rho)}{2} & \frac{(c-g)}{2} \\ \frac{(b-n)(1-\rho)}{2} & e & 0 \\ \frac{(c-g)}{2} & 0 & l \end{pmatrix}$$

Hence, the equilibrium state Eq_8 is globally asymptotically stable when $\frac{dL}{dt} < 0$. This stability is guaranteed if the matrix D is positive definite, which requires that all of its principal minors of the matrix D are positive.

8. Bionomic Equilibrium

The bionomic equilibrium represents the integration of biological and economic equilibrium. The biological equilibrium is characterized by $\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = 0$, indicating that the total revenue from selling the harvested biomass equal to the total cost of harvesting. Achieving this balance signifies that the bionomic equilibrium is attained

Let C_1 = the harvesting cost per unit effort of prey species.

P_1 = price per unit biomass of prey species.

Then the economic rent (or) net revenue at any time t is given by

$$B(u, v, w, E) = (P_1 q u - C_1) E$$

The bionomic equilibrium $((u)_\infty, (v)_\infty, (w)_\infty, (E)_\infty)$ is the positive solution of

$$\frac{du}{dt} = \frac{dv}{dt} = \frac{dw}{dt} = B = 0$$

$$\begin{aligned} ie \quad ru - au^2 - b(1 - \rho)uv - cuw - qEu &= 0 \\ -ev^2 + n(1 - \rho)uv - m_1v - d_1v &= 0 \\ sw - lw^2 + guw + m_2w - d_2w &= 0 \end{aligned} \quad \dots(8.1)$$

$$B = (P_1qu - C_1)E = 0$$

To determine the bionomic equilibrium, we consider the following two cases.

- (i) If $C_1 > P_1qu$ then the cost is greater than revenue for the prey species. ie., prey species is not harvested.
- (ii) If $C_1 < P_1qu$ then the cost is less than the revenue. ie., if the net revenue is positive then the system will be in operation.

$$\begin{aligned} \text{From equation (8.1)} \quad (u)_\infty &= C_1/P_1q \\ (v)_\infty &= \frac{1}{e} \left[\frac{C_1 n(1-\rho) - (m_1 + d_1)P_1q}{P_1q} \right] \\ (w)_\infty &= \frac{1}{l} \left[\frac{gC_1 + (s + m_2 - d_2)P_1q}{P_1q} \right] \end{aligned}$$

By substituting $((u)_\infty, (v)_\infty, (w)_\infty, (E)_\infty)$ in equation (8.1)

$$(E)_\infty = \frac{1}{q} [r - a(u)_\infty - b(1-\rho)(v)_\infty - c(w)_\infty]$$

$$\text{ie., } (E)_\infty = \frac{1}{q} \left[r - \frac{aC_1}{P_1q} - b(1-\rho) \left(\frac{C_1 n(1-\rho) - (m_1 + d_1)P_1q}{eP_1q} \right) - c \left(\frac{gC_1 + (s + m_2 - d_2)P_1q}{lP_1q} \right) \right]$$

It is observe that $(E)_\infty > 0$, if $r > \left(\frac{aC_1}{P_1q} + b(1-\rho)(v)_\infty + c(w)_\infty \right)$... (8.2)

Thus, the non-trivial bionomic equilibrium point $((u)_\infty, (v)_\infty, (w)_\infty, (E)_\infty)$ exist if the condition (8.2) holds.

9. Optimal harvesting policy

In this, the objective is to maximize the present value J of continuous time stream of revenues given by

$$J = \int_0^\infty e^{-\beta t} (P_1qu - C_1) E(t) dt \quad \dots (9.1)$$

Where β denotes the instantaneous annual rate of discount, $e^{-\beta t}$ is depreciation, q is the catchability coefficient of the biomass of (u) and P_1 is the selling price of unit biomass of (u) . The aim is to maximize equation (9.1) subject to the state equation of the given model by applying pontryagin's maximum principle.

The hamiltonian for the problem is given by

$$\begin{aligned} H = e^{-\beta t} [(P_1qu - C_1)E] &+ \lambda_1 [ru - au^2 - b(1-\rho)uv - cuw - qEu] \\ &+ \lambda_2 [-ev^2 + n(1-\rho)uv - m_1v - d_1v] + \lambda_3 [sw - lw^2 + guw + m_2w - d_2w] \end{aligned}$$

Where λ_1 , λ_2 , and λ_3 are the adjoint variable.

By using pontryagin's maximum principle.

$$\frac{\partial H}{\partial E} = 0; \quad \frac{d\lambda_1}{dt} = -\left(\frac{\partial H}{\partial u}\right); \quad \frac{d\lambda_2}{dt} = -\left(\frac{\partial H}{\partial v}\right); \quad \frac{d\lambda_3}{dt} = -\left(\frac{\partial H}{\partial w}\right)$$

The control variable $E(t)$ is subjected to the constraints $0 \leq E(t) \leq (E)_{max}$.

And the switching function is given by

$$\begin{aligned} \frac{\partial H}{\partial E} = \varphi(t) &= e^{-\beta t} (P_1qu - C_1) - \lambda_1 qu \\ \lambda_1 &= e^{-\beta t} \left(P_1 - \frac{C_1}{qu} \right) \end{aligned} \quad \dots (9.2)$$

Now the aim is to find an optimal equilibrium $((u)_\infty, (v)_\infty, (w)_\infty)$ to maximize the hamiltonian H . since the hamiltonian H is linear in the control variable E . The optimal control can be extreme control (or) the singular control, thus we have

$$(E) = (E)_{max} \text{ where } \varphi(t) > 0, \quad \lambda_1 e^{\beta t} < \left(P_1 - \frac{C_1}{qu} \right)$$

$$(E) = (E)_{\min}, \text{ where } \varphi(t) < 0, \lambda_1 e^{\beta t} > \left(P_1 - \frac{C_1}{qu}\right)$$

$$(E) = 0, \quad \text{where } \varphi(t) = 0, \lambda_1 e^{\beta t} = \left(P_1 - \frac{C_1}{qu}\right)$$

By pontryagin's maximum principle, the adjoint equations are,

$$\begin{aligned} \frac{d\lambda_1}{dt} &= -\left(\frac{\partial H}{\partial u}\right) \\ \frac{d\lambda_1}{dt} &= (\lambda_1 au - \lambda_2(n(1-\rho)v) - \lambda_3 gw - e^{-\beta t} P_1 qE) \end{aligned} \quad \dots(9.3)$$

$$\begin{aligned} \frac{d\lambda_2}{dt} &= -\left(\frac{\partial H}{\partial v}\right) \\ \frac{d\lambda_2}{dt} &= (\lambda_1 b(1-\rho)u + \lambda_2 ev) \end{aligned} \quad \dots(9.4)$$

$$\begin{aligned} \frac{d\lambda_3}{dt} &= -\left(\frac{\partial H}{\partial w}\right) \\ \frac{d\lambda_3}{dt} &= (\lambda_1 cu + \lambda_3 lw) \end{aligned} \quad \dots(9.5)$$

From equation (9.2) and (9.4)

$$\begin{aligned} \frac{d\lambda_2}{dt} - \lambda_2 ev &= B_1 e^{-\beta t} \\ \text{where } B_1 &= \left(P_1 - \frac{C_1}{qu}\right) b(1-\rho)u \end{aligned}$$

the above equation is linear in λ_2 and its solution

$$\lambda_2 = \frac{-B_1}{\beta + ev} e^{-\beta t} \quad \dots(9.6)$$

Now the equation (9.2) and (9.5) then we have

$$\begin{aligned} \frac{d\lambda_3}{dt} - \lambda_3 lw &= B_2 e^{-\beta t} \\ \text{where } B_2 &= \left(P_1 - \frac{C_1}{qu}\right) cu \end{aligned}$$

the above equation is linear in λ_3 and its solution

$$\lambda_3 = \frac{-B_2}{\beta + lw} e^{-\beta t} \quad \dots(9.7)$$

Now the equation (9.2), (9.6) and (9.7) substitute in equation (9.3) then we obtain

$$\begin{aligned} \frac{d\lambda_1}{dt} - \lambda_1 au &= B_3 e^{-\beta t} \\ \text{where } B_3 &= \left(\frac{B_1 n(1-\rho)v}{\beta + ev} + \frac{B_2 gw}{\beta + lw} - P_1 qE\right) \end{aligned}$$

the above equation is linear in λ_1 and its solution

$$\lambda_1 = \frac{-B_3}{\beta + au} e^{-\beta t} \quad \dots(9.8)$$

From equation (9.2) and (9.8), we obtain a singular path

$$e^{-\beta t} \left(P_1 - \frac{C_1}{qu}\right) = \frac{-B_3}{\beta + au} e^{-\beta t}$$

$$\text{Which gives } F(u^*) = \left(P_1 - \frac{C_1}{qu^*}\right) + \frac{B_3}{\beta + au^*} \quad \dots(9.9)$$

There exist unique positive roots $u^* = (u)_{\beta}$, if $F(u^*) = 0$ in the interval $0 \leq u^* \leq K_1$.

If the following inequality hold $F(0) < 0$, $F(K_1) > 0$, where $K_1 = \frac{r}{a}$, and $F'(u^*) > 0$, for $(u^*) > 0$.

For $u^* = (u)_\beta$, $v^* = (v)_\beta$, $w^* = (w)_\beta$, and we get

$$(E)_\beta = \frac{1}{q} [r - a(u)_\beta - b(1 - \rho)(v)_\beta - c(w)_\beta]$$

Where

$$(u)_\beta = \frac{C_1}{P_1 q}$$

$$(v)_\beta = \frac{1}{e} \left[\frac{C_1 n(1 - \rho) - (m_1 + d_1) P_1 q}{P_1 q} \right]$$

$$(w)_\beta = \frac{1}{l} \left[\frac{g C_1 + (s + m_2 - d_2) P_1 q}{P_1 q} \right]$$

$$\text{Here } (E)_\beta > 0 \quad \text{if} \quad r > \left(\frac{a C_1}{P_1 q} + b(1 - \rho)(v)_\beta + c(w)_\beta \right)$$

Hence the optimal equilibrium $((u)_\beta, (v)_\beta, (w)_\beta, (E)_\beta)$ is determined, the optimal harvesting effort $(E)_\beta$ can be determined. We note that $\lambda_i(t)e^{-\beta t}$ ($i = 1, 2, 3$) is independent of time in an optimal equilibrium. Hence the equation satisfies the transversality condition at ∞ , indicating that the value remains bounded as $t \rightarrow \infty$.

From equation (9.9), $\left(P_1 - \frac{C_1}{qu^*} \right) = \frac{-B_3}{\beta + au^*} \rightarrow 0$ as $\beta \rightarrow \infty$.

Thus, the net economic revenue is $B(u, v, w, E) = 0$ this implies that an infinite discount rate tends to the net economic revenue tending to zero and the fishery would remain closed.

10. Mathematical Simulation

In this section, numerical simulations conducted using Python are employed to evaluate the stability of the proposed model, perform parameter sensitivity analysis, and investigate the 3D phase plane trajectory to understand its behavior and dynamics.

Simulation of Population Dynamics Model - I

Parameter specification:

$r = 0.5, a = 0.01, b = 0.02, c = 0.01, q = 0.01, E = 1.0, e = 0.1, n = 0.02, m_1 = 0.01,$

$d_1 = 0.01, s = 0.1, l = 0.02, g = 0.02, m_2 = 0.01, d_2 = 0.01, \rho = 0.5$

Initial conditions $u(0) = 1.0, v(0) = 0.50, w(0) = 0.60$

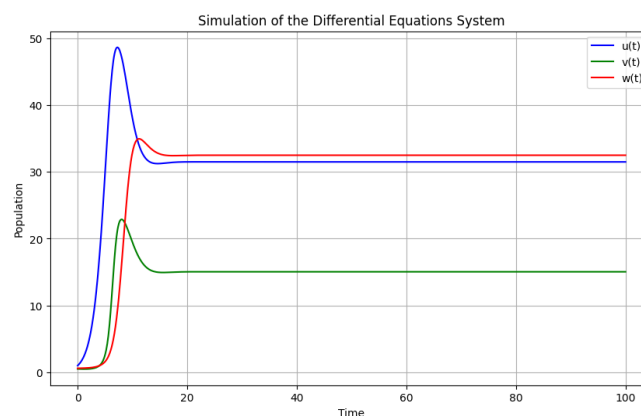


Figure 1: Simulation of the model - I

The graph represents the solution of a system of differential equations over time, depicting the behavior of three populations: prey ($u(t)$), migrating predator ($v(t)$), and immigrating predator ($w(t)$). Initially, all three populations start at low values. The prey population increases rapidly, reaches a peak, and then declines slightly. Similarly, the migrating predator population increases quickly but peaks at a lower value than the prey and then declines. The immigrating predator population increases at a rate similar to the prey initially but levels off more quickly. In the intermediate phase, after their initial peaks, the prey and immigrating predator populations approach a stable value and remain there for the rest of the simulation, while the

migrating predator population decreases and also approaches a stable value, which is lower than that of the prey and immigrating predator populations. Eventually, all three populations reach a steady state where their values remain constant. The prey and immigrating predator populations converging to the same value, while the migrating predator population converging to a different, lower value, suggesting a model where different populations reach equilibrium at distinct levels.

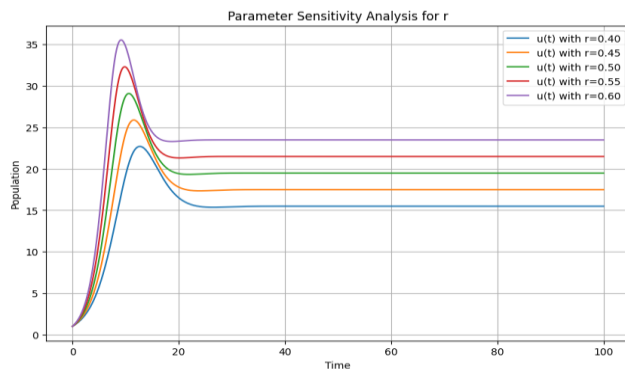


Figure 2: Parameter Sensitivity Analysis for r - I

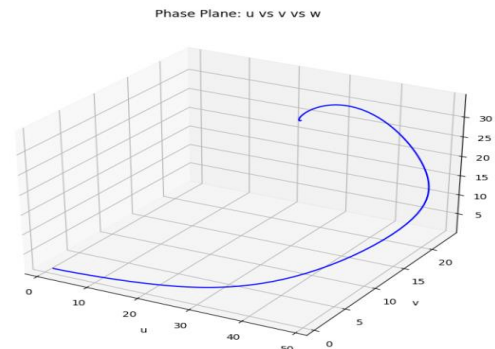


Figure 3: Phase Plane - I

The graph (figure 2) illustrates a parameter sensitivity analysis for population ($u(t)$) over time, with different curves representing various values of parameter (r) (ranging from 0.40 to 0.60). Initially, all populations rapidly increase, with higher (r) values leading to earlier and higher peak populations. After peaking, the populations undergo oscillations before stabilizing. Higher (r) values result in reduced oscillations and higher equilibrium populations. Overall, the parameter (r) significantly influences the speed of growth, peak magnitude, and final stabilized population, with larger (r) values producing faster growth and higher steady-state populations. The phase plane plot shows the trajectory of variables (u), (v), and (w) over time, indicating cyclic behavior and interactions between the variables. The looped trajectory suggests a stable limit cycle, highlighting the system's dynamic nature.

Simulation of Population Dynamics Model - II

Parameter specification:

$r = 0.5, a = 0.1, b = 0.02, c = 0.01, q = 0.01, E = 1.0, e = 0.2, n = 0.05, m1 = 0.1,$
 $d1 = 0.01, s = 0.3, l = 0.05, g = 0.01, m2 = 0.1, d2 = 0.01, \rho = 0.5$

Initial conditions $u(0) = 1.0, v(0) = 1.0, w(0) = 1.0$

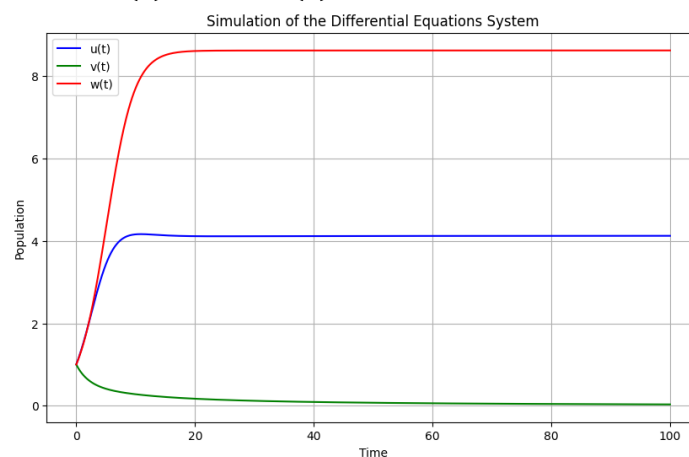


Figure 4: Simulation of the model - II

The graph depicts simulation results from a proposed model illustrating interactions between prey and two types of predators over time. Initially, the prey population ($u(t)$) starts low but rapidly increases to stabilize around 4, suggesting it reaches a stable population level after an initial growth phase. The migrating predator population ($v(t)$) starts at a higher level but quickly declines to near zero, indicating it rapidly decreases and becomes negligible. On the other hand, the immigrating predator population ($w(t)$) starts at a minimal level, increases significantly, and stabilizes around 9, indicating substantial growth to a stable high population. This simulation indicates that after an initial period of dynamic changes, all three

populations reach stable values, with the prey stabilizing at a moderate level, the migrating predator declining to near extinction, and the immigrating predator achieving a high stable population.

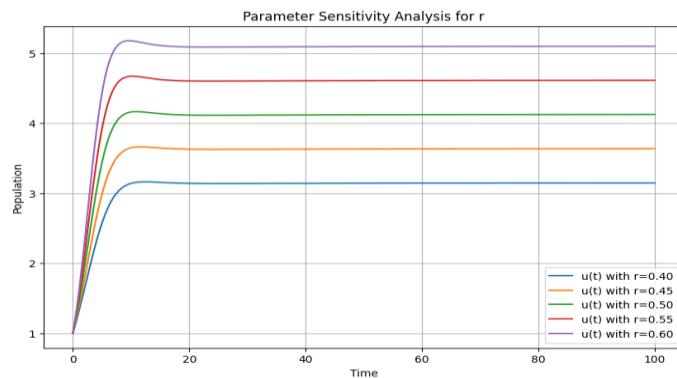


Figure 5: Parameter Sensitivity Analysis for r- II

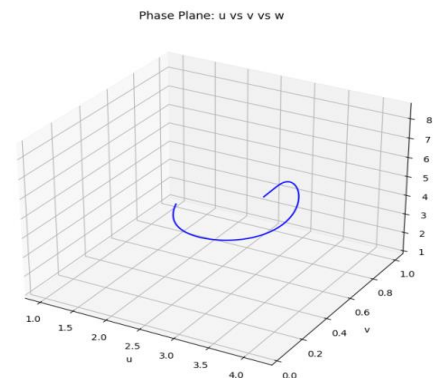


Figure 6: Phase Plane - II

The graph shows a parameter sensitivity analysis for (r) on the prey population ($u(t)$) over time. Different curves represent varying (r) values from 0.40 to 0.60. As (r) increases, the initial growth rate and peak height of ($u(t)$) also increase. Finally, each population stabilizes at a higher constant value, with ($r = 0.40$) stabilizing around 3, and ($r = 0.60$) around 5. This indicates that higher (r) values lead to faster growth and higher equilibrium levels, emphasizing (r)'s role in the prey population dynamics.

The phase plane plot illustrates the dynamic relationship between the populations over time, forming a loop in the phase space that indicates cyclical or oscillatory dynamics among these populations. This cyclical pattern suggests that the populations do not reach a steady state but instead undergo continuous fluctuations. As the prey population increases, there are corresponding changes in the migrating predator population and the immigrating predator population, reflecting their interdependent relationships. The plot shows how the populations evolve over time in a three-dimensional space, highlighting the complex interactions and continuous cycles among the prey and predator populations.

Simulation of Population Dynamics Model - III

Parameter specification:

$r = 0.1, a = 0.01, b = 0.02, c = 0.01, q = 0.01, E = 1.0, e = 0.1, n = 0.02, m1 = 0.01,$
 $d1 = 0.01, s = 0.1, l = 0.02, g = 0.02, m2 = 0.01, d2 = 0.02, \rho = 0.5$

Initial conditions $u(0) = 1.0, v(0) = 1.0, w(0) = 1.0$

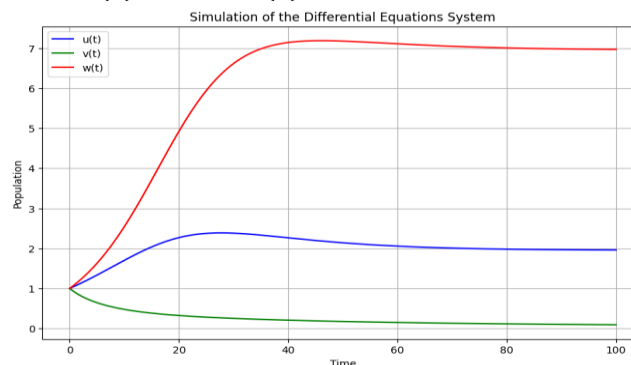


Figure 7: Simulation of the model - III

The graph illustrates the population dynamics of three species over time, each starting with an initial population of different values. The prey population initially increases rapidly, peaking around ($t = 25$), and then stabilizes slightly above its initial level. The migration predator population experiences a rapid initial decline, followed by a slower decrease, eventually stabilizing near zero. The immigration predator population rises quickly, peaking around ($t = 50$), and then stabilizes at a much higher population level. This behavior indicates a stable equilibrium where the populations of the prey and immigration predator remain steady over time, while the migration predator declines significantly, suggesting it is outcompeted.

by the other two. The chosen parameters lead to a system where each species reaches a steady-state population

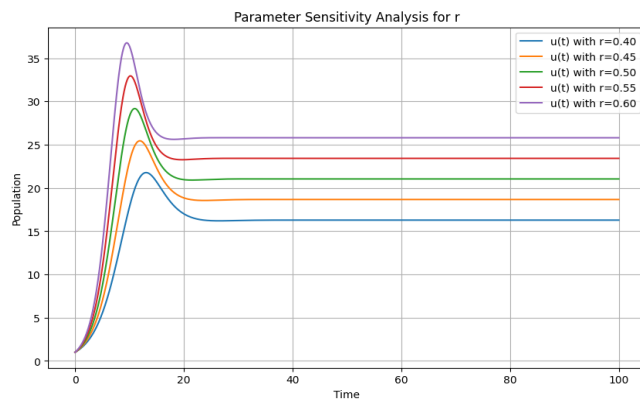


Figure 8: Parameter Sensitivity Analysis for r - III

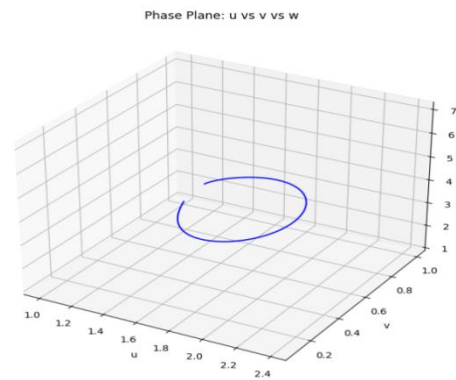


Figure 9: Phase Plane - III

The graph shows a sensitivity analysis of the population model parameter (r) over time, with growth rate values ranging from 0.40 to 0.60. Initially, the population grows rapidly for all (r) values, reaching a peak that occurs earlier and at a lower value for smaller values. After the peak, the population declines and stabilizes, with higher growth rate values leading to higher equilibrium levels. Specifically, ($r = 0.40$) stabilizes around 20, while ($r = 0.60$) stabilizes around 30. This demonstrates that growth rate significantly affects both the peak and stable population levels.

The phase plane plot depicts the relationship between three variables, showing a trajectory that represents their evolution over time. The spiral pattern of the trajectory indicates oscillatory behavior, suggesting that the system does not settle into a fixed point but continues to oscillate, potentially forming a limit cycle where the system's state repeats periodically. The variables are bounded within specific ranges u from approximately 1.0 to 2.4, v from 0.2 to 0.8, and w from 1.0 to 7.0, providing understanding of the limits of the system's behavior. This plot helps visualize how changes in one variable affect the others, revealing the complex, oscillatory dynamics and feedback mechanisms in the system. Understanding these interactions is crucial for comprehending the stability and long-term behavior of the dynamical system.

11. Conclusion

Studying the effects of refuge, migration, and immigration on predator-prey dynamics through mathematical models emphasizes their critical role in shaping ecosystem stability and maintaining balance by managing prey populations and regulating species diversity. The dynamics are influenced by factors such as predation intensity, the availability of prey refuges, and competition among prey individuals. Mathematical models using differential equations offer a robust framework to comprehend the impact of migration, immigration, and other ecological factors on the stability and persistence of predator-prey systems.

Further, the system analysis the equilibrium states that ensure the stability during interactions between prey and individual predators within the ecosystem. The presence of prey refuge can increase the stability of the system and promote coexistence between predators and prey. The migration of the first predator can lead to oscillations in the prey population size, which can make it more difficult for coexistence to occur. The immigration of the second predator can stabilize the system and promote coexistence between predators and prey. The system's stability is affected by changes in parameters such as predation rates, conversion rates, and mortality rates. The stability of equilibrium points is assessed using the Routh-Hurwitz criterion and Lyapunov functions. This analysis helps determine the conditions under which the system will return to equilibrium after disturbances.

However, instability among predators is evident due to a lack of communication between them. The model demonstrates that the ecosystem exhibits significantly greater stability on a global scale compared to a local scale. The system analyzes bionomic equilibrium and optimal harvesting policies for species interactions. The simulation demonstrates that prey stabilize at a moderate level after initial growth, while migrating predators decline and immigrating predators stabilize at a higher level. Increasing parameter (r) accelerates prey growth and raises equilibrium levels. The phase plane plot confirms cyclic dynamics among prey and predators, highlighting their interconnected behavior over time.

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