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EXISTENCE OF POSITIVE SOLUTIONS OF NON LINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract:
In this paper we study the existence of positive solutions of a non linear fractional order delay differential equation of the form
$L(D)y(t) = f\big(t, y(t), y(t-\tau)\big), y(0) = 0, 0 < t < 1, \tau > 0,$
where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - -a_{n-2}D^{s_{n-2}} - \cdots - -a_1 D^{s_1}$
$0 < s_1 < s_2 < \cdots < s_n < 1, a_j > 0$ forall $j = 1,2, .. n-1$ and $\tau > 0$ is a constant delay.
Also, D^{s_j} is the standard Riemann-Liouville fractional derivative and
$f: [0,1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function. In addition to this we also prove that, if the condition $a_j > 0$ is relaxed then the equation we have considered has a unique solution which may not necessarily be positive.
Key words: Riemann-Liouville fractional order derivatives, delay, Banach space, normal cone, completely continuous operator, existence of positive solution.

1. Introduction

Many authors [6,12&15] have investigated the existence of positive solution of ordinary differential equations. Recent analysis shows that in science and engineering the dynamics of many systems can be described more accurately by using fractional order differential equations. In [2,3,4,7,9,13&17] we can infer that the authors investigated the existence of positive solution of fractional order differential equations. But the fractional order delay differential equations [5,16&18] are often more effective to describe the natural phenomenon than those equations without delay.

In this paper we consider the more general fractional order differential equation with constant delay of the form

$$L(D)y(t) = f\big(t, y(t), y(t-\tau)\big), y(0) = 0, 0 < t < 1, \tau > 0. \qquad (1.1)$$

Where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - -a_{n-2}D^{s_{n-2}} - \cdots - -a_1 D^{s_1}$
$0 < s_1 < s_2 < \cdots < s_n < 1, a_j > 0$ for all $j = 1,2, .. n-1$ and $\tau > 0$ is a constant delay.
Also, D^{s_j} is the standard Riemann-Liouville fractional derivative and
$f: [0,1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function.
We provide the conditions for f and $a_j{}'s$ for which the equation (1.1) has a unique positive solution. In addition to this we also prove that, if the condition $a_j > 0$ is relaxed, then the equation we have considered has a unique solution which may not necessarily be positive.

2. Preliminaries

Let B be a real Banach space with a cone K. K introduces a partial order $\leq$ in B as follows [10]
$a \leq b \; if \; b - a \in K$

Definition 2.1: For $a, b \in$ B, the order interval $< a, b >$ is defined as [10]

$< a, b >= \{c \in B: a \leq c \leq b\}$ (2.1)

Definition 2.2: A cone K is called normal, if there exists a positive constant δ such that $f, g \in$ K and $\theta < f < g \Rightarrow \|f\| \leq \delta\|g\|$ (2.2)
where θ denotes the zero element of K.

Theorem 2.1: [10] Let K be a normal cone in a partially ordered Banach space B. Let F be increasing on the segment $< x_0, y_0 >$ into itself. That is,
$Fx_0 \geq x_0$ and $Fy_0 \leq y_0$. (2.3)
Also, if we assume that F is compact and continuous, then F has atleast one fixed point $x^* \in < x_0, y_0 >$.

Theorem 2.2: (Banach fixed point theorem) [10]. Let K be a closed subspace of a Banach space B. Let F be a contraction mapping with Lipschitz constant $k(< 1)$ from K to K itself. Then F has a unique fixed point $x^* \in K$.
Moreover, if x_0 is an arbitrary point in K and $\{x_n\}$ is defined by $x_{n+1} = Fx_n$ where $n = 0,1,2, ...$ then $\lim_{n\to\infty} x_n = x^* \in K$ and $d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_1, x_0)$.

Definition 2.3: The left sided Riemann-Liouville fractional integral [11,13&14] of a function f of order α is defined as

$$I_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \alpha > 0, x > a. \quad (2.4)$$

Definition 2.4: The left sided Riemann-Liouville fractional derivative [11,13&14] of a function f of order α is defined as

$$D_{a+}^{\alpha} f(x) = \frac{d^n}{dx^n}[I_{a+}^{n-\alpha} f(x)],\ n-1 \leq \alpha < n, n \in N \quad (2.5)$$

Here, we denote $I_{0+}^{\alpha} f(x)$ and $D_{0+}^{\alpha} f(x)$ as $I^{\alpha} f(x)$ and $D^{\alpha} f(x)$. Also, $I_{a+}^{\alpha} f(x)$ and $D_{a+}^{\alpha} f(x)$ refer to $I_a^{\alpha} f(x)$ and $D_a^{\alpha} f(x)$.
If the fractional order derivative $D_a^{\alpha} f(x)$ is integrable, then [13]

$$I_a^{\alpha}\,(D_a^{\beta} f(x)) = I_a^{\alpha-\beta} f(x) - \left[I_a^{1-\beta} f(x)\right]_{x=a} \frac{(x-a)^{1-\alpha}}{\Gamma(\alpha)},\ 0 \leq \beta \leq \alpha < 1 \quad (2.6)$$

If f is continuous on $[a, b]$ then $\left[I_a^{1-\beta} f(x)\right]_{x=a} = 0$ and equation (2.6) refers to

$$I_a^{\alpha}\,(D_a^{\beta} f(x)) = I_a^{\alpha-\beta} f(x),\ 0 \leq \beta \leq \alpha < 1 \quad (2.7)$$

3. Existence of Positive solution

Here we discuss the conditions under which the fractional order delay differential equation mentioned below has a positive solution.

$$L(D)y(t) = f\big(t, y(t), y(t-\tau)\big), y(0) = 0, 0 < t < 1, \tau > 0, \quad (3.1)$$

where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - -a_{n-2}D^{s_{n-2}} - \cdots - -a_1 D^{s_1}$
$0 < s_1 < s_2 < \cdots < s_n < 1, a_j > 0$ forall $j = 1,2,..n-1$ *and* $\tau > 0$ is a constant delay.
Also, D^{s_j} is the standard Riemann-Liouville fractional derivative and
$f: [0,1] \times [0, +\infty) \to [0, +\infty)$ is a given continuous function.
Let us denote $Y = C[0,1]$ the Banach space of all continuous functions on [0,1] endowed with the super norm. That is

$$\|y\|_C = \|y_0\| + \|y\| = \|y\| = \sup\{|y(t)|: 0 \leq t \leq 1\},\ y \in Y \quad (3.2)$$

Let K be the cone $K = \{y \in Y: y(t) \geq 0, 0 \leq t \leq 1\}$ (3.3)

By (2.6) and (2.7), equation (3.1) is equivalent to the integral equation

$$y(t) = \sum_{j=1}^{n-1} a_j I^{s_n - s_j} y(t) + I^{s_n} f(t, y(t), y(t-\tau)) \quad (3.4)$$

Lemma 3.1: The operator $F: K \to K$ defined as

$$F\big(y(t)\big) = \sum_{j=1}^{n-1} a_j I^{s_n - s_j} y(t) + I^{s_n} f\big(t, y(t), y(t-\tau)\big) \quad (3.5)$$

is continuous and completely continuous.
Proof: It is trivial that in view of continuity of f, the operator $F: K \to K$ is continuous.
Lemma 3.2: Let $G \subset K$ be bounded. That is there exists a positive constant l such that
$\|y\| \leq l$, for all $y \in G$ (3.6)
then $\overline{F(G)}$ is compact. i.e., F maps bounded sets into equicontinuous sets of K.
Proof:
Let $L = \max\{1 + f(t, y(t), y(t-\tau)): 0 \leq t \leq 1 \text{ and } 0 \leq y \leq l\}$ (3.7)
For $y \in G$ we have by (3.5)

$$|F(y(t))| \leq \sum_{j=1}^{n-1} a_j I^{s_n - s_j}|y(t)| + I^{s_n} f(t, y(t), y(t-\tau))$$

$$\leq \sum_{j=1}^{n-1} \frac{l a_j t^{s_n - s_j}}{\Gamma(s_n - s_j + 1)} + \frac{1}{\Gamma(s_n)} \int_0^t (t-s)^{s_n - 1} f(s, y(s), y(s-\tau)) ds$$

$$\leq \sum_{j=1}^{n-1} \left[\frac{l a_j}{\Gamma(s_n - s_j + 1)} + \frac{L}{\Gamma(s_n + 1)}\right] t^{s_n - s_{n-1}}$$

Hence, $\|F(y(t))\| \leq \sum_{j=1}^{n-1} \left[\frac{l a_j}{\Gamma(s_n - s_j + 1)} + \frac{L}{\Gamma(s_n + 1)}\right]$ (3.8)

And so, $F(G)$ is bounded.
Now, we show that $F(G)$ is equicontinuous as follows.
Let $y \in G, 0 \leq t_1 \leq t_2 \leq 1$.
Consider,

$$|F(y(t_1)) - F(y(t_2))| = \left|\sum_{j=1}^{n-1} a_j I^{s_n - s_j}|y(t_1)| + I^{s_n} f(t_1, y(t_1), y(t_1 - \tau)) - \sum_{j=1}^{n-1} a_j I^{s_n - s_j}|y(t_2)| + I^{s_n} f(t_2, y(t_2), y(t_2 - \tau))\right|$$

$$\leq \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j)} \left|\int_0^{t_1} (t_1 - s)^{s_n - s_j - 1} y((s) ds - \int_0^{t_2} (t_2 - s)^{s_n - s_j - 1} y((s) ds\right|$$
$$+ \left|\frac{1}{\Gamma(s_n)} \int_0^{t_1} (t_1 - s)^{s_n - 1} f(s, y(s), y(s-\tau)) ds - \frac{1}{\Gamma(s_n)} \int_0^{t_2} (t_2 - s)^{s_n - 1} f(s, y(s), y(s-\tau)) ds\right|$$

$$\leq \sum_{j=1}^{n-1} \frac{l a_j}{\Gamma(s_n - s_j)} \left\{\int_0^{t_1} [(t_1 - s)^{s_n - s_j - 1} - (t_2 - s)^{s_n - s_j - 1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{s_n - s_j - 1} ds\right\}$$
$$+ \frac{1}{\Gamma(s_n)} \left\{\int_0^{t_1} [(t_1 - s)^{s_n - 1} - (t_2 - s)^{s_n - 1}] f(s, y(s), y(s-\tau)) ds + \int_{t_1}^{t_2} (t_2 - s)^{s_n - 1} f(s, y(s), y(s-\tau)) ds\right\}$$

$$\leq \sum_{j=1}^{n-1} \frac{l a_j}{\Gamma(s_n - s_j)} \left\{\int_0^{t_1} [(t_1 - s)^{s_n - s_j - 1} - (t_2 - s)^{s_n - s_j - 1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{s_n - s_j - 1} ds\right\}$$
$$+ \frac{L}{\Gamma(s_n)} \left\{\int_0^{t_1} [(t_1 - s)^{s_n - 1} - (t_2 - s)^{s_n - 1}] ds + \int_{t_1}^{t_2} (t_2 - s)^{s_n - 1} ds\right\}$$

Integrating we get

$$\leq 2l\sum_{j=1}^{n-1}\frac{a_j(t_2-t_1)^{s_n-s_j}}{\Gamma(s_n-s_j+1)}+\frac{2L(t_2-t_1)^{s_n}}{\Gamma(s_n)}$$
$$\leq \rho^{s_n-s_{n-1}}\left[2l\sum_{j=1}^{n-1}\frac{a_j}{\Gamma(s_n-s_j+1)}+\frac{2L}{\Gamma(s_n)}\right]$$
$$< \epsilon$$

where $\rho = \left[\epsilon(2l\sum_{j=1}^{n-1}\frac{a_j}{\Gamma(s_n-s_j+1)}+\frac{2L}{\Gamma(s_n)})^{-1}\right]^{\frac{1}{s_n-s_{n-1}}}$

Hence $F(G)$ is equicontinuous. So, Arzela-Ascoli's theorem implies that $\overline{F(G)}$ is compact.

Theorem 3.1: Let $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ be continuous and $f(t,y(t),y(t-\tau))$ increasing for each $0\leq t\leq 1$. Let there exists υ_0,ω_0 satisfying

$$L(D)\upsilon_0 \leq f(t,\upsilon_0(t),\upsilon_0(t-\tau)) \quad (3.9)$$
$$L(D)\omega_0 \geq f(t,\omega_0(t),\omega_0(t-\tau)) \quad (3.10)$$

Then (3.1) has a positive solution.

Proof:

To prove this we need to consider the fixed point of the operator F. By Lemma 3.1 the operator $F:K\to K$ is completely continuous. Let $y_1,y_2\in K$, and $y_1\leq y_2$. As F is non-decreasing, we have

$$F(y_1(t)) = \sum_{j=1}^{n-1}a_jI^{s_n-s_j}y_1(t)+I^{s_n}f(t,y_1(t),y_1(t-\tau))\leq F(y_2(t)) \quad (3.11)$$

which implies F is an increasing operator.

Also, by our assumption $F\upsilon_0\geq\upsilon_0$ and $F\omega_0\leq\omega_0$ (3.12)

Hence by lemma 3.1 and lemma 3.2, $F:(\upsilon_0,\omega_0)\to(\upsilon_0,\omega_0)$ is completely continuous and a compact operator. Also, we have K is a normal cone and F is compact and continuous. Therefore, by theorem 2.1, F has a fixed point $y^*\in(\upsilon_0,\omega_0)$ which is the required positive solution of (3.1).

Theorem 3.2: Let $f:[0,1]\times[0,+\infty)\to[0,+\infty)$ be continuous and $f(t,y(t),y(t-\tau))$ increasing for each $0\leq t\leq 1$. If

$$0<\lim_{y\to+\infty}f(t,y(t),y(t-\tau))<+\infty,\ \text{for each } 0\leq t\leq 1 \quad (3.13)$$

Then equation (3.1) has a positive solution.

Proof:

Assume that, there exist non-negative constants A,B such that

$f(t,y(t),y(t-\tau))\leq A$ for all $0\leq t\leq 1, y\geq B$. (3.14)

Let $C=max\{f(t,y(t),y(t-\tau)):0\leq t\leq 1 \text{ and } 0\leq y\leq B\}$ (3.15)

Then we have $f\leq A+C$ forall $y\geq 0$ (3.16)

Consider the equation

$$L(D)\omega(t)=A+C,\omega(0)=0,0<s_1<s_2<\cdots<s_n<1,0\leq t\leq 1 \quad (3.17)$$

where $L(D)=D^{s_n}-a_{n-1}D^{s_{n-1}}-a_{n-2}D^{s_{n-2}}-\cdots--a_1D^{s_1}$ and $a_j>0$ forall $j=1,2,..n-1$. Using (2.6) and (2.7) the solution of equation (3.17) is equivalent to the solution of the following integral equation

$$\omega(t)=\sum_{j=1}^{n-1}a_jI^{s_n-s_j}\omega(t)+I^{s_n}(A+C) \quad (3.18)$$

But $\omega(t)\geq\sum_{j=1}^{n-1}a_jI^{s_n-s_j}\omega(t)+I^{s_n}f(t,\omega(t),\omega(t-\tau))=F(\omega(t))$ (3.19)

Now, for $\omega(t)\equiv 0, F(\omega(t))=I^{s_n}f(t,\omega(t),\omega(t-\tau))\geq\omega(t)$ (3.20)

Hence by theorem 3.1, equation (3.1) has a positive solution.

Theorem 3.3: The following fraction order differential equation has a non-negative solution.

$L(D)y(t) = gy(t) + c,\ y(0) = 0,\ 0 \le t \le 1$ and $g, c \ge 0.$ (3.21)
where $L(D) = D^{s_n} - a_{n-1}D^{s_{n-1}} - -a_{n-2}D^{s_{n-2}} - \cdots - -a_1 D^{s_1}$
$0 < s_1 < s_2 < \cdots < s_n < 1, a_j > 0\ forall\ j = 1,2, .. n-1\ and\ D^{s_j}$ is the standard Riemann-Liouville fractional derivative and
Proof: Equation (3.21) is equivalent to the integral equation

$$y(t) = \sum_{j=1}^{n-1} a_j I^{s_n - s_j} y(t) + I^{s_n}(gy(t) + c) \quad (3.22)$$

Let $T: K \to K$ be defined as

$$T\big(y(t)\big) = \sum_{j=1}^{n-1} a_j I^{s_n - s_j} y(t) + I^{s_n}(gy(t) + c) \quad (3.23)$$

Therefore by lemma 3.1, T is completely continuous.
Consider the case $c > 0$. Let

$$B_R = \left\{ y(t) \in C[0,\delta], y(t) \ge 0: \left\| y - \frac{ct^{s_n}}{\Gamma(s_n+1)} \right\| \le B \right\} \quad (3.24)$$

be a convex bounded and closed subset of the Banach space $C[0,\delta]$, where

$$\delta < mini \left[\left(\frac{B\Gamma(s_n+1)}{2Ec} \right)^{\frac{1}{\Gamma(s_n - s_{n-1})}}, \left(\frac{1}{2E} \right)^{\frac{1}{\Gamma(s_n - s_{n-1})}} \right] \quad (3.25)$$

Where $E = \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j + 1)} + \frac{g}{\Gamma(s_n+1)}$ (3.26)

Now, for all $y \in B_R$, we have

$$\left| T\big(y(t)\big) - \frac{ct^{s_n}}{\Gamma(s_n+1)} \right| \le \|y\| \left[\sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n - s_j + 1)} t^{s_n - s_j} + \frac{g}{\Gamma(s_n+1)} t^{s_n} \right]$$

$$\le t^{s_n - s_j} E t^{s_n - s_j - 1}$$

Since $\|y\| \le \frac{ct^{s_n}}{\Gamma(s_n+1)} + B \le \frac{c\delta^{s_n}}{\Gamma(s_n+1)} + B \le \frac{c}{\Gamma(s_n+1)} + B$
We get

$$\left| T\big(y(t)\big) - \frac{ct^{s_n}}{\Gamma(s_n+1)} \right| \le E\left(\frac{c}{\Gamma(s_n+1)} + B \right) \delta^{s_n - s_{n-1}} \le \frac{1}{2}B + \frac{1}{2}B = B. \quad (3.27)$$

So, we have $T(B_R) \subseteq B_R$. Similar to the proof of Lemma 3.2, it can be seen that $T(B_R)$ is equicontinuous. Let $\{y_n\}$ be a bounded sequence in B_R. Then, $\{T(y_n)\} \subset T(B_R)$. Hence, $\{T(y_n)\}$ is equicontinuous. Since $y_n \in C[a,b]$, Arzela-Ascoli thorem [1,8&10] implies that $\{T(y_n)\}$ has a convergent subsequence. Therefore, T: $B_R \to B_R$ is compact. Hence by Schauder fixed point theorem [10] it has a fixed point, which is the required non-negative solution of (3.21).
As above we can prove the result for the case $c = 0$.
Using theorem 3.1 and theorem 3.3 it is easy to prove the following existence theorem.
Theorem 3.4: Let $f: [0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and $f\big(t, y(t), y(t-\tau)\big)$ increasing for each $0 \le t \le 1$. If

$$0 \le \lim_{y \to \infty} \max_{0 \le t \le 1} \frac{f\big(t, y(t), y(t-\tau)\big)}{y(t)} < +\infty \quad (3.28)$$

Then equation (3.1) has a positive solution.
Examples:

1. For $f\big(t, y(t), y(t-\tau)\big) = t\ arctan(t - \tau)$ exists a positive solution since it satisfies the condition required in theorem 3.2.
2. For $f\big(t, y(t), y(t-\tau)\big) = t\ In(1 + t - \tau)$ exists a positive solution since it satisfies the condition required in theorem 3.4.

4. Existence of unique solution

Here we give conditions for $f\ and\ a_j's$, which provides the unique positive solution to equation (3.1)

Theorem 4.1: Let $f: [0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and Lipschitz with respect to the second variable with Lipschitz constant L. Let $a_j's$ satisfy the following conditions:

(i) $a_j > 0$ forall $j = 1,2,..n-1$

(ii) $0 < \frac{L}{\Gamma(s_n+1)} + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n-s_j+1)} < 1$

Then equation (3.1) has a unique positive solution.

Proof: As per the previous section equation (3.1) is equivalent to equation (3.4). For $u, v \in K$, we have

$|F(u(t)) - F(v(t))| \leq \sum_{j=1}^{n-1} a_j I^{s_n-s_j}|u(t)-v(t)| + LI^{s_n}|u(t)-v(t)|$

$\leq \|u-v\| \left\{\sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n-s_j+1)} t^{s_n-s_j} + \frac{L}{\Gamma(s_n+1)} t^{s_n}\right\}$, where F is given in (3.5). Hence

$$\|Fu - Fv\| \leq \left[\sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n-s_j+1)} + \frac{L}{\Gamma(s_n+1)}\right] \|u-v\|. \tag{4.1}$$

By theorem 2.2, F has a unique fixed point in K, which is the unique positive solution of the equation (3.1)

In the next theorem we exclude the condition that $a_j > 0$ fora ll $j = 1,2,..n-1$ and study the equation (3.1). Using Banach fixed point theorem for $F: C[0,1] \to C[0,1]$, we can exhibit the following result.

Theorem 4.2: Let $f: [0,1] \times [0,+\infty) \to [0,+\infty)$ be continuous and Lipschitz with respect to the second variable with Lipschitz constant L. Let $a_j's$ satisfy the following condition:

$0 < \frac{L}{\Gamma(s_n+1)} + \sum_{j=1}^{n-1} \frac{a_j}{\Gamma(s_n-s_j+1)} < 1$

Then equation (3.1) has a unique solution which may not be necessarily positive.

Proof: By using equations (2.6) and (2.7), equation (3.1) is equivalent to the integral equation

$y(t) = \sum_{j=1}^{n-1} a_j I^{s_n-s_j} y(t) + I^{s_n} f(t, y(t), y(t-\tau))$

We define the operator $F: C[0,1] \to C[0,1]$ as follows

$F(y(t)) = \sum_{j=1}^{n-1} a_j I^{s_n-s_j} y(t) + I^{s_n} f(t, y(t), y(t-\tau))$.

For $u, v \in C[0,1]$,

$\|Fu - Fv\| \leq \left[\sum_{j=1}^{n-1} \frac{|a_j|}{\Gamma(s_n-s_j+1)} + \frac{L}{\Gamma(s_n+1)}\right] \|u-v\|$.

Hence by theorem 2.2, F will have the unique fixed point in $C[0,1]$, which is the unique solution of equation (3.1) but which may not necessarily be a positive value.

Example 4.1:

Consider the equation

$$\left(D^{\frac{1}{2}} - aD^{\frac{1}{3}}\right) y = Ly(t-1) + e^t,\ y(0) = 0, 0 < t < 1. \tag{4.2}$$

Case 1: For $0 \leq a \leq \frac{1}{5}, 0 < L \leq \frac{2}{5}$ the equation (4.2) satisfies the conditions required in theorem 4.1. The corresponding iterated sequence is given by

$y_1 = I^{1/2} e^t$

$y_2 = aI^{\frac{1}{6}} y_1(t) + LI^{\frac{1}{2}} y_1(t-1) + y_1(t)$

$y_3 = aI^{\frac{1}{6}} y_2(t) + LI^{\frac{1}{2}} y_2(t-1) + y_1(t)$

In general,

$y_{n+1} = y_2 = aI^{\frac{1}{6}}y_n(t) + LI^{\frac{1}{2}}y_n(t-1) + y_1(t)\ ,\ n = 1,2,3, \ldots$

where $I^{\alpha}y_1 = t^{\alpha+\frac{1}{2}}E_{1,\alpha+\frac{3}{2}}\ ,\ \alpha > 0,\ y(t) = \lim_{n\to\infty} y_n(t)$ is the unique non-negative solution.

Case 2: For $-\frac{1}{5} \le a \le \frac{1}{5}, 0 < L \le \frac{2}{5}$ the equation (4.2) satisfies the conditions required in theorem 4.2. The corresponding iterated sequence is given by

$y_1 = I^{1/2}e^t$

$y_2 = aI^{\frac{1}{6}}y_1(t) + LI^{\frac{1}{2}}y_1(t-1) + y_1(t)$

$y_3 = aI^{\frac{1}{6}}y_2(t) + LI^{\frac{1}{2}}y_2(t-1) + y_1(t)$

In general,

$y_{n+1} = y_2 = aI^{\frac{1}{6}}y_n(t) + LI^{\frac{1}{2}}y_n(t-1) + y_1(t)\ ,\ n = 1,2,3, \ldots$

where $I^{\alpha}y_1 = t^{\alpha+\frac{1}{2}}E_{1,\alpha+\frac{3}{2}}\ ,\ \alpha > 0,\ y(t) = \lim_{n\to\infty} y_n(t)$ is the unique solution which may not be positive.

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