The Edge Steiner Global Domination Number of a Graph

[1]J. Suja, [2]V.Sujin Flower,

[1]Register Number 20223042092017, Research Scholar,
Department of Mathematics, Holy Cross College, Nagercoil - 629 001, India.
email: jesinsuja@gmail.com
[2]Assistant Professor, Department of Mathematics,
Holy Cross College, Nagercoil - 629 001, India. email: sujinflower@gmail.com
Affiliated to Manonmaniam Sundaranar University, Abishekapatti,
Tirunelveli-627 012, Tamil Nadu, India.

Abstract

Let G be a connected graph. A set of vertices W in G is called an *edge Steiner global dominating set* or simply a (ES, GD)-set if W is both a Steiner set and a global dominating set of G. The minimum cardinality of a edge Steiner global dominating set of G is its *edge Steiner domination number* and is denoted by $\overline{\gamma}_{se}(G)$. Some general properties satisfied by this concept are studied. The detour monophonic global domination numbers of certain standard graphs are determined. Connected graphs of order n with the edge Steiner global domination number 2 or n are characterized. It is shown that for every two integers $a, b \geq 2$ with $2 \leq a \leq b$, there is a connected graph G such that $s_e(G) = a$ and $\overline{\gamma}_{se}(G) = b$, where $s_e(G)$ is the edge Steiner number of a graph.

Keywords: edge Steiner global domination number, edge Steiner number, domination number, global domination number, distance, Steiner distance.
AMS Subject Classification: 05C69, 05C12

1. Introduction

The *undirected graph* $G = (V, E)$ discussed in this paper is simple and connected. The *order* and *size* are denoted by n and m respectively. The *neighbors* of a vertex x are the vertices that are adjacent to x, it is denoted by $N(x)$. The *degree of a vertex* x in a graph G is $deg_G\ (x)$=|$N(x)$|. The minimum and maximum degree of vertices in G are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex $x \in V$ in a connected graph G is said to be a *universal vertex* of G if $deg_G(x) = n - 1$. For any set W of vertices of G, the *induced subgraph* $G[W]$ is the maximal subgraph of G with vertex set. Thus the two vertices of W are adjacent $G[W]$ if and only if they are adjacent in G. A vertex $x \in G$ is said to be extreme if the $G[N(x)]$ is complete.

The $u - v$ *geodesic* represents the shortest path between the vertices u and v in G. The length of the shortest path between any two vertices u and v is the *distance* betwe-
en the corresponding vertices and is represented as $d(u, v)$. For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each such subgraph is a tree and is called a *Steiner tree with respect to* W or a *Steiner* W*-tree*. It is to be noted that $d(W) = d(u, v)$, when $W = \{u, v\}$. The set of all vertices of G that lie on some Steiner W-tree is denoted by $S(W)$. If $S(W) = V$, then W is called a *Steiner set* for

G. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a *s-set* of G and this cardinality is the *Steiner number* $s(G)$ of G. A set $W \subseteq V(G)$ is called an *edge Steiner set* of G if every edge of G is contained in a Steiner W-tree of G. The *edge Steiner number* $s_e(G)$ of G is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_e(G)$ is a edge Steiner set of G. These concepts were studied in [1,2,4-14]

A set $D \subseteq V$ of vertices in a graph G is called a *dominating set* if every vertex $v \in V$ is either an element of D or is adjacent to an element of D. A subset $D \subseteq V$ is called a *global dominating set* in G if D is a dominating set of both G and $\overline{G}$. The *global domination number* $\bar{\gamma}(G)$ is the minimum cardinality of a minimal global dominating set in G. These concepts were studied in [3,15]. In this paper we studied the concept of the edge Steiner domination number of a graph. The following theorems are used in sequel.

Theorem 1.1. [12] Each extreme vertex of a connected graph G belongs to every edge Steiner set of G.

Theorem 1.2. [12] For the complete graph $G = K_n (n \geq 2), s_e(G) = n$.

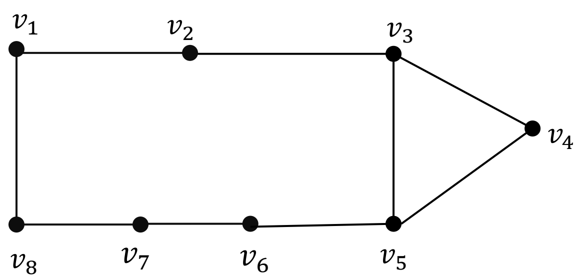
Theorem 1.3. [12] For the complete bipartite graph $G = K_{r,s}$ $(2 \leq r \leq s), s_e(G) = r$

Theorem 1.4. [3] For the cycle $G = C_n (n \geq 6), \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

2. The Edge Steiner Global Domination Number of a Graph

Definition 2.1. Let G be a connected graph. A set of vertices W in G is called an *edge Steiner global dominating set* or simply a $(ES,)$- set if W is both a Steiner set and a global dominating set of G. The minimum cardinality of a edge Steiner global dominating set of G is its *edge Steiner domination number* and is denoted by $\overline{\gamma}_{se}(G)$. A (ES,GD) - setof size $\overline{\gamma}_{se}(G)$ is said to be a $\overline{\gamma}_{se}$*-set.*

Example 2.2. For the graph G given in Figure 2.1, $W = \{v_1, v_4, v_6, v_8\}$ is a $\overline{\gamma}_{se}$- set of G so that $\overline{\gamma}_{se}(G) = 4$.



G

Figure 2.1

Observation 2.3.

(i) Each extreme vertex of a connected graph G belongs to every $(ES,)$-set of G.

(ii) Each universal vertex of a connected graph G belongs to every $(ES,)$-set of G.

(iii) If G is a connected graph of order n, then $2 \leq max\{s_e(G), \gamma(G)\} \leq \overline{\gamma}_{se}(G) \leq n$.

(iv) For the complete graph $G = K_n (n \geq 2), \overline{\gamma}_{se}(G) = n$

(v) For the star graph $G = K_{1,n-1} (n \geq 3),\ \overline{\gamma}_{se}(G) = n$

Theorem 2.4. Let G be a connected graph of order $n \geq 2$. Then $\overline{\gamma}_{se}(G) = 2$ if and only if there exist a (ES, GD) -set $W = \{u, v\}$ of G such that $d(u, v) \leq 3$.

Proof. Suppose $\overline{\gamma}_{se}(G) = 2$. Let $W = \{u, v\}$ be a $(ES,)$-set of G. Suppose that $d(u, v) \geq 4$. Then the diametrical path contains at least three internal vertices. Therefore $\overline{\gamma}_{se}(G) \geq 3$, this is a contradiction. Therefore $d(u, v) \leq 3$. The converse is clear. ■

Theorem 2.5. If G is a connected graph of order $n \geq 3$ containing a vertex v of degree $n \geq 1$, then all the neighbors of v belong to every edge Steiner set of G.

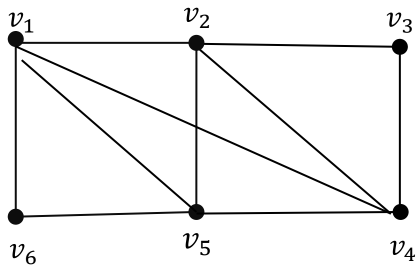
Proof. Let v be a vertex of degree $n \geq 1$ and $v_1, v_2, \dots, v_{n-1}$ be the neighbors of v in G. Let W be an edge Steiner set of G. Suppose $v_1 \notin W$. Then the edge vv_1 lies on a Steiner W-tree of G, say T. Since $v_1 \notin W$, v_1 is not an end-vertex of T. Let T' be a tree obtained from T by removing the vertex v_1 in T and joining all the neighbors of v_1 other than v in T to v. Then T' is a Steiner W-tree such that $|V(T')| = |V(T)| - 1$, which is a contradiction to T a Steiner W-tree. ■

Theorem 2.6. Let G be a connected graph of order $n \geq 3$. If G contains a vertex of degree $n - 1$, then $\overline{\gamma}_{se}(G) = n$.

Proof. Let v be a vertex of degree $n \geq 1$. Then by Theorem 2.5, v is a subset of every edge Steiner global dominating set of G. Let $v_1, v_2, \dots, v_{n-1}$ be the neighbors of v in G. Then by Theorem 2.5, $v_1, v_2, \dots, v_{n-1}$ belong to every edge Steiner global dominating set of G. Hence it follows that $V(G)$ is the unique edge Steiner set of G so that $\overline{\gamma}_{se}(G) = n$.

■

Remark 2.7. The converse of Theorem 2.6 need not be true. For the graph G given in Figure 2.2, $W = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ is an edge Steiner global dominating set of G so that $\overline{\gamma}_{se}(G) = 6 = n$ and G has no vertex of degree $n - 1$.



G

Figure 2.2

Theorem 2.8. For the path $G = P_n (n \geq 2)$, $\overline{\gamma}_{se}(G) = \begin{cases} \left\lceil \frac{n+2}{3} \right\rceil & if\ n \geq 5 \\ 2\ if\ n = 2,3\ or\ 4 \end{cases}$

Proof. Let P_n be $v_1, v_2, \dots, v_n$. If $n = 2$,3 or 4, then $\{v_1, v_n\}$ is a (ES, GD) - set of G so that $\overline{\gamma}_{se}(G) = 2$. Let $n \geq 5$. Let W be a $\overline{\gamma}_{se}$- set of P_{n-4}. Then $W \cup \{v_1, v_n\}$ is a $\overline{\gamma}_{se}$ -set of G so that $\overline{\gamma}_{se}(G) = \left\lceil \frac{n-4}{3} \right\rceil + 2 = \left\lceil \frac{n+2}{3} \right\rceil$.

■

Theorem 2.9. For the cycle $G = C_n (n \geq 6)$, $\overline{\gamma}_{se}(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proof. For the cycle C_n $(n \geq 6)$, every dominating set of G is a (ES, GD) - set of G and so $\overline{\gamma}_{se}(C_n) \leq \gamma(C_n)$. Then it follows from Theorem 1.4, $\overline{\gamma}_{se}(C_n) = \gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$. ■

Theorem 2.10. For the fan graph $G = W_n = K_1 + P_{n-1}, (n \geq 5), \overline{\gamma}_{se}(G) = n.$
Proof: This follows from Theorem 2.6. ■
Theorem 2.11. For the wheel $G = W_n = K_1 + C_{n-1}$ $(n \geq 4), \overline{\gamma}_{se}(G) = n.$
Proof: This follows from Theorem 2.6. ■
Theorem 2.12. For the complete bipartite graph $G = K_{r,s}(G)(1 \leq r \leq s), \overline{\gamma}_{se}(G) = r + s$
Proof.
Case(i). Let $r = s = 1$. Then $G = K_2$. By Observation 2.4 (iv), $\overline{\gamma}_{se}(G)$=2.
Case(ii). Let $r = 1, s \geq 2$. Then $G = K_{1,s}$. By Observation 2.4 (v), $\overline{\gamma}_{s}(G) = r + 1$.
So let $2 \leq r \leq s$. Let X= $\{x_1, x_2, \dots, x_r\}$ and $Y = \{y_1, y_2, \dots, y_s\}$ be the two bipartite set of G. Then X, Y and $V(G)$ are the only three edge Steiner dominating sets of G. Let W be a $\overline{\gamma}_{se}(G)$- set of G. If $W = X$, then y_j $(1 \leq j \leq s)$ is not dominated by any elements of W in G. If $W = Y$, then xi (1≤i≤r) is not dominated by any elements of W in G. Therefore W is not a (ES,D) set of G. Hence it follow that W=(G) is the unique $\overline{\gamma}_{se}(G)$- set of G so that $\overline{\gamma}_{se}(G)$= r+s. ■
Theorem 2.13. If G is a connected graph with $\Delta(G) \leq n - 2$ such that it has a minimum cut-set of G consisting of i independent vertices, then $\overline{\gamma}_{se}(G) \leq p - i$.
Proof. Since $\Delta(G) \leq n - 2, G$ is non complete and G has no universal vertices. Therefore $1 \leq k(G) \leq n - 2$. Let $U = \{u_1, u_2, \dots, u_i\}$ be a minimum cut set of G. Let $G_1, G_2, \dots, G_r (r \geq 2)$ be the components of $G - U$ and let $W = V(G) - U$. Then every vertex $u_s (1 \leq s \leq i)$ is adjacent to at least one vertex of G_j for every $j (1 \leq j \leq r)$. Therefore W is a dominating set of G. Since G has no universal vertices, W is a dominating set of $\overline{G}$. Hence W is a global dominating set of G. We prove that W is an edge Steiner set of G. Let uv be any edge of G. If uv lies on one of the G_t for any $(1 \leq t \leq r)$, say uv belongs to G_1. Let T_1 be a spanning tree in G_1 that contains uv. Now, let T_j $(2 \leq j \leq r)$ be a spanning tree in G_j $(2 \leq j \leq r)$. Let T be the tree obtained from $T_1, T_2, T_3, \dots, T_r$ by joining the edges $u_j v_1, u_j v_2, u_j v_3, \dots, u_j v_r$, for some $v_k \in V(G_k)$ $(1 \leq k \leq r)$ and any u_j belonging to U. Then clearly T is a Steiner W-tree of G containing the edge uv. If uv is an edge that joins a vertex of U to a vertex of G_j $(1 \leq j \leq r)$ for convenience, let us assume that $uv = u_j v_1$, where $v_1 \in V(G_1)$. Let T_j $(1 \leq j \leq r)$ be a spanning tree in $G_j (1 \leq j \leq r)$. Let T' be the tree obtained from $T_1, T_2, T_3, \dots, T_r$, the edge $u_j v_1$, by joining the edges $u_j v_2, u_j v_3, \dots, u_j v_r$, for some $v_k \in V(G_k)(2 \leq k \leq r)$. Then clearly T' is a Steiner W-tree of G containing the edge $u_j v_1$. Thus W is an edge Steiner set of G. Therefore W is an edge Steiner global dominating set of G and so $\overline{\gamma}_{se}(G) \leq |V(G) - U| = p - i$. ■
Theorem 2.14. Let G be a connected graph of order $n \geq 2$. If every vertex of G is either an extreme vertex or a universal vertex, then $\overline{\gamma}_{se}(G) = n$.
Proof. The result follows from Observation 2.3 (i) and (ii). ■
Remark 2.15. The converse of Theorem 2.14 need be true. For the complete bipartite graph $G = K_{r,s}, (2 \leq r \leq s), \overline{\gamma}_{se}(G) = r + s = n$. But there is no vertex v such that v is either an extreme vertex or a universal vertex of G.
Open Problem 2.16. Characterize connected graphs of order $n \geq 2$ with edge Steiner global domination number n.
Theorem 2.17. There is no connected graph of order $n \geq 4$ with $\overline{\gamma}_{se}(G) = n - 1$.

Proof. Suppose that there exists a connected graph of order $n \geq 4$ with $\overline{\gamma}_{se}(G) = n-1$. Let $W = V(G) - \{v\}$ be a $\overline{\gamma}_{se}$- set of G. Then $G[W]$ is connected. Then any Steiner W- tree contains elements of W only. Let $vx \in E(G)$. Then the edge vx does not lie in any Steiner W- tree of G. Which is a contradiction to W a $\overline{\gamma}_{se}$- set of G. Therefore there is no connected graph of order $n \geq 4$ with $\overline{\gamma}_{se}(G) = n-1$.. ∎

Theorem 2.18. For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected graph G such that $s_e(G) = a$ and $\overline{\gamma}_{se}(G) = b$.

Proof. For $a = b$, let $G = K_a$. Then by Theorem 1.2 and Observation 2.3 (iv), $s_e(G) = \overline{\gamma}_{se}(G) = a$. So, let $2 \leq a < b$. Let $G = K_{a,b-a}$ $(2 \leq a < b)$. Then by Theorem 1.3, $s_e(G) = a$ and by Theorem 2.12, $\overline{\gamma}_{se} = a + b - a = b$. ∎

3. Conclusion

In this paper, we defined the edge Steiner global domination number of a graph and studied some of its general properties. We will extend this concept to minimal and forcing concepts in graphs.

References

[1] F. Buckley and F. Harary, Distance in Graphs, *Addition-Wesley, Redwood City, CA*, 1990.

[2] Bijo S. Anand, Manoj Changat, Iztok Peterin and Prasanth G. NarasimhaShenoi, Some Steiner concepts on lexicographic products of graphs, *Discrete Mathematics, Algorithms and Applications*, 06 (04), (2014) 1450060.

[3] Y. Caro and R. Yuster, Dominating a family of graphs with small connected graphs, *Combin.Probab.Comput*.9(2000), 309-313.

[4] G. Chartrand, Oellermann, Ortred, Tian Song, Song Ling,Zou and Hung Kin, Steiner distance in graphs, *Caspopis pro pestovani Mathematiky*, 114, (1989) 399-410.

[5] G. Chartrand and P. Zhang,The Steiner number of a graph, *Discrete Mathematics* 242, (2002,) 41 - 54.

[6] J. John and M. S. Malchijah Raj, The upper restrained Steiner number of a graph, *Discrete Mathematics Algorithms and Applications*, 12 (1),(2020) 2050004.

[7] J. John, G. Edwin and P. Arul Paul Sudhahar, The Steiner domination number of a graph, *International Journal of Mathematics and Computer Applications*, Vol.3, 3 (2013), 37-42.

[8] M. S. Malchijah Raj and J. John, The restrained edge Steiner number of a graph, *Journal of Applied Science and Computations*, 6(2), (2019), 1 - 8.

[9] M. S. Malchijah Raj and J. John, The forcing restrained Steiner number of a graph, *Technology International Journal of Engineering and Advanced*, 8, (2019), 1799 – 1803.

[10] M. Perumalsamy, P. Arul Paul Sudhahar, J. John and R.Vasanthi, Edge fixed Steiner number of a graph, *International Journal of Mathematical Analysis*, 11, (2017), 771 - 785.

[11] M. Perumalsamy, P. Arul Paul Sudhahar, R.Vasanthi and J. John, The forcing edge fixed Steiner number of a graph, *Journal of Statistics and Management Systems*, 22, (2019), 1 - 10.

[12] A. P. Santhakumaran and J. John, Edge Steiner Number of a Graph, *Journal of Discrete Mathematical Sciences and Cryptography*, 10 (3),(2007), 415-432.
[13] A. P. Santhakumaran and J. John, The Upper Steiner Number of a Graph, *Graph Theory Notes (of New York LIX*, 2010), 9 - 14.
[14] A. P. Santhakumaran and J. John, The forcing Steiner number of a graph, *Discussiones Mathematicae Graph Theory*, 31, (2011), 171 - 181.
[15] D. Stalin and J. John, Edge Geodetic Dominations in Graphs, *International Journal of Pure and Applied Mathematics*, 116(22),(2017), 31-40 .