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A Modish Glance of Integer Solutions to Non-homogeneous Cubic Diophantine Equation with Three Unknowns $5\,(x^2+y^2)=13\,z^3$

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Abstract
The main thrust of thispaper is focused on obtaining varieties of choices of solutions in integers to the third degree Diophantine equation with three unknowns $5(x^2+y^2)=13z^3$.Various choices of solutions in integers are obtained by reducing it to the equation which is solvable, through employing suitable transformations and applying the factorization method.
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Introduction

One of the interesting areas of Theory of Numbers is the steady of diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that Diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. Obviously, diophantine equations are plenty. The theory of diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results, this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated. There is no well unified body of knowledge concerning general methods. A diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability, to exhibit all integers satisfying the requirements set forth in the problem. The successful completion of presenting solutions in integers satisfying the requirements set forth in the problem add to the improvement of number theory as they offer good applications in the field of graph theory, Modular theory , coding and cryptography, Engineering, Music and so on. Integers have frequentlytake part ina pivotal role in the evolution of the Natural Sciences. The theory of integers provides answers to real world problems.

It is well- known that diophantine equalization, homogeneous or not homogeneous have aroused the integers of many mathematicians. It is worth to observe that cubic diophantine equations fall in to the theory of Eliptical curves which are used in Cryptography. In particular, one may refer [1-10] for the third degree equations with three and four unknowns.

The present paper aims in illustrating different choices of solutions in integers to an interesting ternary not- homogeneous cubic equation $5(x^2+y^2)=13z^3$ through employing elementary algebraic methods. The fascinating observations will be useful to the researchers in this field.

Technical procedure

The not-homogeneous third degree diophantine equalization with three unknowns to be solvedis

$$5\,(x^2+y^2)=13z^3 \qquad (1)$$

Different approaches of obtaining different solutions in integer to (1) areexhibited under.

Approach 1

The substitution

$$x=k\,y \qquad (2)$$

In (1) leads to

$$5(k^2+1)\;y^2=13z^3 \qquad (3)$$

It is observed that (3) is satisfied by

$$y=5*13^2*(k^2+1)\,t^{3s}\,,z=5*13*(k^2+1)\,t^{2s} \qquad (4)$$

Using (4) in (2), we get

$$x=5*13^2*k\,(k^2+1)\,t^{3s} \qquad (5)$$

Thus, (4) & (5) represent the integer solutions to (1).

Approach 2

Taking $x=5*13^2*w\,,z=65*w$ (6)

in (1), it leads to

$$y^2=25*169^2*(w-1)\,w^2 \qquad (7)$$

The R.H.S. of (7) is a perfect square when

$$w=(s^2+1) \qquad (8)$$

Employing (8) in (6) & (7), the respective solutions inintegers to (1) are given by

$$x = 5*13^2*(s^2+1), y = 5*13^2*s(s^2+1)\ , z = 65*(s^2+1)$$

Approach 3

Taking

$$x = 13^2*w, z = 65*w \tag{9}$$

in (1), it leads to

$$y^2 = 169^2*(25w-1)w^2 \tag{10}$$

The R.H.S. of (10) is a perfect square when

$$w = (25n^2+14n+2) \tag{11}$$

Employing (11) in (9) & (10), the respective solutions in integers to (1) are given by

$x = 13^2(25n^2+14n+2)$,
$y = 13^2(25n^2+14n+2)(25n+7)$,
$z = 65\ (25n^2+14n+2)$

Approach 4

Assume

$$z = a^2+b^2 = (a+ib)(a-ib) \tag{12}$$

Express the integers 5 &13 as below:

$$5 = (2+i)(2-i) \tag{13}$$

and

$$13 = (2+3i)(2-3i) \tag{14}$$

Substituting (12), (13) and (14) in (1) and using factorization,

$$(2+i)(x+iy) = (2+3i)(a+ib)^3 \tag{15}$$

On comparision in (15) and solving for x, y ,we have

$$5x = 7(a^3 - 3ab^2) - 4(3a^2b - b^3),$$
$$5y = 4(a^3 - 3ab^2) + 7(3a^2b - b^3) \quad (16)$$

As we require thesolutions in integers, replacing a by $5A$ and

b by $5B$ in (12) & (16), respective solutions in integers to (1) are given by

$$x = 5^2[7(A^3 - 3AB^2) - 4(3A^2B - B^3)],$$
$$y = 5^2[4(A^3 - 3AB^2) + 7(3A^2B - B^3)],$$
$$z = 5^2(A^2 + B^2)$$

Note:

It is worth mentioning that the integers 5 &13 may also be written as

$$5 = (1+2i)(1-2i),\ 13 = (3+2i)(3-2i)$$

The repeating the above producer we obtain a new set of Solutions to (1).

Approach 5

Introduction of the transformations

$$x = 1690\,w^2, z = 130\,w^2 \quad (17)$$

in (1) leads to

$$y^2 = 1690^2(2w^2 - 1)w^4 \quad (18)$$

Assume

$$\alpha^2 = 2w^2 - 1 \quad (19)$$

Which is the well-known Pellian equationusing the standardprocedure,the general solution (α_{n+1}, w_{n+1}) to (19) is given by

$$w_{n+1} = \frac{\sqrt{2}f_n + g_n}{2\sqrt{2}},\ \alpha_{n+1} = \frac{f_n + \sqrt{2}g_n}{2} \quad (20)$$

Thus, from (17) & (18), respective solutions in integers to (1) aregiven by

$$x_{n+1} = 1690\,(\frac{\sqrt{2}f_n + g_n}{2\sqrt{2}})^2\ ,$$

$$y_{n+1} = 1690\,(\frac{\sqrt{2}f_n + g_n}{2\sqrt{2}})^2\,(\frac{f_n + \sqrt{2}g_n}{2}),$$

$$z_{n+1} = 130\,(\frac{\sqrt{2}f_n + g_n}{2\sqrt{2}})^2$$

Approach 6

Taking

$$x = 5kw\,, z = 5w \qquad (21)$$

in (1), it leads to

$$y^2 = 5^2 * (13\,w - k^2)\,w^2 \qquad (22)$$

The R.H.S. of (22) is a perfect square when

$$w = (13\,n^2 + 140nk + 2k^2) \qquad (23)$$

Employing (23) in (21) & (22), respective solutions in integers to (1) are given by

$$x = 5k\,(13\,n^2 + 10nk + 2k^2)\ ,$$
$$y = 5\,(13\,n^2 + 10nk + 2k^2)\,(13n + 5k)\,,$$
$$z = 5\ (13\,n^2 + 10nk + 2k^2)$$

Conclusion:

The ternary not homogeneous cubic represented by $5\left(x^2 + y^2\right) = 13z^3$ is analyzed for obtaining varies choices of solutions in integers. Thereaders of this paper may consider other forms of cubic equations with three or more variables for determining their solutions in integers.

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