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A Reliability Stationary Distribution of Semi -Markov Model Involving Partial Product Process

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We examine a semi-Markov process that simulates the upkeep and repair of a steady-state repairable system. Both the operating and repair times have general distributions and are independent random variables. Both internal and external factors may contribute to failures. While some malfunctions can be fixed, others cannot. Our approach takes into account that the system is not as good as new after a repairable breakdown. When a system fails and cannot be fixed, a new one is installed. We assume a Poisson process governs the occurrence of external breakdowns. Additionally, there is a maximum of N repairs, and regardless of the type of failure, a new system is installed in its place. They follow partial product processes since multiplicative rates impact operational and repair timeframes. The stationary distribution is computed for this system.
Keywords: Markov process, semi-Markov process, geometric process, partial product process, reliability

1. Introduction

We examine a single unit repairable system that has sporadic failures that may or may not be repairable. The operational and repair timeframes are thought to be random variables with broad distributions. The systems can then be represented as continuous-time Markov chains under specific exponential-time assumptions. When operational and maintenance time probability distributions follow general distributions, this is no longer the case. The system can then be modelled using a semi-Markov process.

The literature on models for repairable systems that take into consideration the impact of several repairs on reliability is then briefly reviewed. Ravichandran [1] examined a class of simple models of redundant repairable systems in which a semi-Markov process was caused by stochastic behaviour. The author provided a number of simple instances. Lam [2] determined the Markov system failure rate and applied it to systems that could be repaired. Neuts et al. [3] regarded as a repairable system that had a policy of repair N (the system is replaced at the first failure after N repairs), phase-type dispersed operating and repair timeframes, and various failure types. In the present paper, the availability of a repairable system and the frequency of faults are examined using a semi-Markov model under the conditions given in Reference [3]. This paper extends the study of previous authors in several ways: (a) because the operating and repair times follow broad distributions, take into consideration a more general model; (b) this general system is vulnerable to both internal and external malfunctions; (c) since geometric processes dictate subsequent operational and repair durations, the model can take into account both dependability growth and decline; (d) for the semi-Markov stochastic process, the availability and failure rate are explicitly computed and (e) every discovered formula is computationally implemented and applied to a numerical example.

The definition of semi-Markov process (SMP) can be seen in Cinlar [4]. A SMP is a stochastic process $[Z(t), t \geq 0]$ assuming values on a finite set S can be interpreted in the following way: in times t_n, transitions occur, the successive occupied states X_n induce a discrete Markov chain on S; and the stay in any state of S is a random variable depending on the current state as well as the state to be visited.

Let $\{X_n, n = 1,2,..\}$ be a sequence of non-negative independent random variables and let $G(x)$ be the distribution function of X_1. Then – is called a partial product process, if the distribution function of X_{i+1} is $G(\beta_k x)$ $(k = 1,2,...)$ where $\beta_k > 0$ are constants and $\beta_k = \beta_0\beta_1\beta_2 \dots \beta_{k-1}$.

We examine a system with a generic distribution that runs for arbitrary periods of time. The system is subjected to external failures which occur according to a Poisson process of rate λ_1, and can be repairable and non-repairable. The necessary repair time has a general distribution if it is repairable. Additionally, internal problems are possible and cannot be fixed.

There are two options in the model. First, the system is not as good as new after repairs, and a parameter that gauges the system's deterioration is used to discount the upcoming operating periods. Repair times lengthen as a result of this degradation. This circumstance is described by a geometric process whose parameter can be higher or less than one. The geometric process is stochastically decreasing if the parameter is larger than one. Conversely, the geometric process is increasing stochastically if the parameter is less than one. At the next failure, a new copy takes its place if the system survives long enough to perform N repairs. When a system fails and cannot be fixed, a new one is installed right away.

The paper is organized as follows. In Section 2 the semi-Markov model is built and the main functions associated with the process are calculated. The stationary distribution of the semi-

Markov process is determined in Section 3.

2. THE MODEL

We examine a system that is prone to both internal and external failures. At first, the system is brand-new and functional. The state of the system is up when it is operating and down when it is being repaired. Both the repair and operation times are arbitrary.

2.1. ASSUMPTIONS OF THE MODEL

Regarding the replacement policy and the operating and repair times, we assume the following.

Assumption 1. External failures occur according to a Poisson process of rate λ: Bernoulli trials determine whether these failures are repairable (with probability p) or non-repairable (with probability $q = 1 - p$). Ageing can potentially cause the system to malfunction. This failure is always irreparable.

Assumption 2. Repairable failures are attended to by a repairman. The system resumes functionality after repair, albeit an operational aspect affects the operating hours. A repair factor also affects subsequent repairs. When a system fails and cannot be fixed, a new one must be installed. We make the assumption that every failure happens on its own.

Assumption 3. Let U_n be the lifetime of the system after its $(n-1)$st repair. If $F(.)$ is the distribution function of the lifetime of a new item, the probability distribution, $F_n(.)$ of U_n is given by

$$F_n(x) = F\left(\beta_0^{2^{n-1}} x\right),\ \ \beta_0 > 0, x > 0 \qquad \dots (1)$$

We assume that $F(.)$ has a finite non-null mean. Its density is denoted by $f(.)$; and it is bounded.

Assumption 4. Let D_n be the repair time of the system after its n^{th} failure. If $G(.)$ is the distribution function of the repair time for the first time, the probability distribution of D_n is given by

$$D_n(x) = G\left(\alpha_0^{2^{n-1}} x\right),\ \ \alpha_0 > 0, x > 0 \ \dots (2)$$

We assume that $G(.)$ has a finite non-null mean.

Assumption 5. Sequences $\{U_n, n \geq 1\}$ and $\{D_n, n \geq 1\}$ g are independent.

Assumption 6. The item is replaced with a new, identical one after the next failure if it has already undergone repairs. A new system replaces the old one instantly. The item is replaced with a new, identical one after the next failure if it has already undergone repairs. A new system replaces the old one instantly.

The operational time when the system is new, U_1; has the distribution function $F(x) = F_1(x)$; and the first repair time, D_1; has the distribution function $G(x) = G_1(x)$:

The repair model that we consider includes different repair policies, depending on the factors β_0, α_0. The ideal repair occurs when $\beta_0 = 1$; $\alpha_0 = 1$ (the system is as good as new after repair), the successive operational and repair times form renewal processes. The imperfect repair occurs when $\beta_0 > 1$; $\alpha_0 \leq 1$ (the next operational time decreases and prolong the next non-ideal repair time before the system is replaced). It is of interest to say that the reliability growth ($\beta_0 < 1$) can be studied under the model. Other policies also can be included in this model depending on the factor values without additional calculations.

2.2. SEMI-MARKOV KERNEL

First, we specify the states of the system. Let $\{Z(t), t \geq 0\}$ be the stochastic process with right continuous sample functions which represents the state of the system at time t; state space is denoted by $S = \{0,1,2,\dots,N\}$. At time 0 the system just enters to state 0. The system follows a particular semi-Markov model in which the holding times in states do not depend on the next transition; that is, the staying time in states depends only on the current state and not on the state to be visited.

Let $0 = t_0 < t_1 < t_2 < \cdots$, be the transition epochs and the $\tau_n = t_{n+1} - t_n, n \geq 0$; the sojourn times in the successive states. The state entered into at time t_n is denoted by $X_n = Z(t_n)$; so $\{X_n, n \geq 0\}$ is the embedded Markov chain of the SMP.
The semi-Markov kernel, $Q(x) = (Q_{ij}(x))$, is defined for $i, j = 0,1,\dots,N, x \geq 0$, by
$Q_{ij}(x) = P\{X_{n+1} = j, \tau_n \leq i |\, X_n = i\} = P\{X_{n+1} = j | X_n = i\}\, P\{\, \tau_n \leq x | X_n = i\}$ … (3)

As $Q_{ij}(x)$ does not depend on n, the transition probabilities are time-homogeneous. Next, we explicitly calculate the entries of the kernel $Q(x)$. If the system is in state $i, i = 0,1,\dots,N-1$; a transition to state 0 occurs if a non-repairable failure happens, while a repairable failure results in a transition to state $i+1$:
The expression $Q_{i0}(x) = P\{X_{n+1} = 0, \tau_n \leq x |\, X_n = i\}, i = 0,1, \dots N-1, x \geq 0$, is the probability that the item has completed i repairs by time t_n; and that at t_{n+1}, the item is new, and this interval of time is $\leq x$: This can occur in only two ways during the time x, at some infinitesimal interval $(u, u+du)$; an external non-repairable failure or an internal failure occurs; the corresponding probability is

$$Q_{i0}(x) = \int_0^x \lambda q\, e^{-\lambda x}\,(1 - F(\beta_0^2 u))\, du + \int_0^x e^{-\lambda x}\, dF(\beta_0^2 u) \quad \dots (4)$$

The expression $Q_{ij+1}(x) = P\{X_{n+1} = i+1, \tau_n \leq x |\, X_n = i\}\; i = 0,1,2 \dots, N-1, x \geq 0$, is the conditional probability that between t_n and t_{n+1}, a repairable failure occurs, given that the system enters state i at time t_n. Moreover, in the time remaining until x; the repair is completed. That event has the probability

$$Q_{i,i+1}(x) = \int_0^x \lambda p e^{-\lambda u}\,(1 - F(\beta_0^2 u) du + \; G(\alpha_0^2 (x-u) du \quad \dots (5)$$

From the state N; the only possible transition is to state 0, since if the system has been repaired N times it is replaced by a new one at the next failure. The corresponding probability is

$$Q_{N0}(x) = \int_0^x \lambda\, e^{-\lambda u}\, \left(1 - F(\beta_0^{2^N} u)\right) du + \int_0^x e^{-\lambda u}\, dF(\beta_0^{2^N} u) \quad \dots (6)$$

$Q_{ij}(x) = 0$ in all other cases. The matrix $Q(x)$ therefore has the form

$$Q(x) = \begin{pmatrix} Q_{00}(x) & Q_{01}(x) & 0 & \cdots & 0 \\ Q_{10}(x) & 0 & Q_{12}(x) & \cdots & 0 \\ Q_{20}(x) & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ Q_{N0}(x) & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \dots (7)$$

The sojourn time distribution function in state i is

$$Q_i(x) = \sum_{j=0}^{N} Q_{ij}(x) = Q_{i0}(x) + Q_{i,i+1}(x), \qquad i = 0, \dots, N-1$$

$$Q_N(x) = Q_{N0}(x) \quad \dots (8)$$

It is routinely verified that $Q_i(\infty) = 1$.

2.3. THE EMBEDDED MARKOV CHAIN

We evaluate the Laplace–Stieltjes transforms (LST)

$$\varphi_{ij}(s) = \int_0^\infty e^{-sx} dQ_{ij}(x), \quad s > 0$$

The LST of the distribution functions F and G will be denoted by ϕ and ψ, respectively. These play a prominent role in the following section. It is well known that the transition matrix $P = (p_{ij})$ of the embedded Markov chain $\{X_n, n \geq 0\}$ can be obtained by taking limits: $p_{ij} = \lim_{x\to\infty} Q_{ij}(x) = Q_{ij}(\infty) = \lim_{s\to 0} \varphi_{ij}(s)$.

Upon integration by parts, we get

$$\varphi_{i0}(s) = \phi((s+\lambda)/\beta_0^{2^i}) + \lambda q \frac{1-\phi\left((s+\lambda)/\beta_0^{2^i}\right)}{s+\lambda}, i = 0,1,2, \dots N-1 \quad \dots. (9a)$$

For $i = N$; we similarly get

$$\varphi_{N0}(s) = \phi((s+\lambda)/\beta_0^{2^N}) + \lambda \frac{1-\phi\left((s+\lambda)/\beta_0^{2^N}\right)}{s+\lambda} \quad \dots. (9b)$$

For $i = 0, 1, \dots, N-1$; we get

$$\varphi_{i,j+1}(s) = \lambda p \frac{1-\phi\left(s+\lambda/\beta_0^{2^i}\right)}{s+\lambda} \psi(s/\alpha_0^{2^i}) \quad \dots. (10)$$

In the transition matrix of the embedded Markov chain the repair factor is not involved:

$$P = \begin{pmatrix} q + p\phi(\lambda) & p - p\phi(\lambda) & & & 0 \\ q + p\phi(\lambda/\beta_0) & & p - p\phi(\lambda/\beta_0) & & \cdots \\ q + p\phi(\lambda/\beta_0^2) & & & p - p\phi(\lambda/\beta_0^2) & \cdots \\ \cdots & & & & \cdots \\ 1 & & & & \cdots \end{pmatrix} \dots (11)$$

3. STATIONARY DISTRIBUTION

The stationary distribution $p = \{p_j, j \in S\}$ of an irreducible and ergodic SMP $\{Z(t), t \geq 0\}$ with state space S is defined by

$$p_j = \lim_{t\to\infty} P\{Z(t) = j | Z(0) = i\}, j \in S$$

As is well known [5], this stationary probability vector is given by

$$p_j = \frac{\pi_j \eta_j}{\sum_{k\in S} \pi_k \eta_k}, \ j \in S \quad \dots (12)$$

with η_j being the expected holding time in state j; and $\pi = (\pi_j, \ j \in S\}$, the stationary distribution of the embedded Markov chain. For calculating these probabilities $p_j, j \in S$; we must first calculate π_j and $\eta_j, j \in S$.

3.1. STATIONARY DISTRIBUTION OF THE EMBEDDED MARKOV CHAIN

In the embedded Markov chain the state space is $S = \{0,1,2,\dots,N\}$; thus $\pi = (\pi_0, \pi_1, \dots, \pi_N)$, with $\pi P = \pi, \pi_j \geq 0$ for all j, and $\pi e = 1, e$ being an $(N+1)$-

dimensional vector. The equations resulting of the formula $\pi P = p$ are

$$\pi_i = p(1-\phi(\lambda/\beta_0^{2^{i-1}}))\pi_{i-1},\ i = 1,2,\dots N-1$$

and

$$\pi_0 = \sum_{i=0}^{N-1} \pi_i(q + p\phi(\lambda/\beta_0^{2^i})) + \pi_N$$

By recurrence we get

$$\pi_1 = \pi_0 p^i \prod_{k=0}^{i-1}(1-\phi(\lambda/\beta_0^{2^k})),\ i = 1,2,\dots N \quad \dots (13a)$$

and from the normalization condition $\pi e = 11$; the value of π_0 is calculated:

$$\pi_0 = \left[1 + \sum_{i=1}^{N} p^i \prod_{k=0}^{i-1}(1-\phi(\lambda/\beta_0^{2^k}))\right]^{-1} \quad \dots (13b)$$

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