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Complementary nil D- Eccentric Domination in Graphs

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A subset D of the vertex set $V(G)$ of a graph $G(V,E)$ is said to be a dominating set if every vertex not in D is adjacent to at least one vertex in D. A dominating set D is said to be an D- eccentric dominating set if for every $v \in V - D$, there exists at least one D- eccentric vertex u of v in D. A D- eccentric dominating set D of G is complementary nil D- eccentric dominating set if the set D is a D- eccentric dominating set and $V - D$ is not a D- eccentric dominating set. The minimum of the cardinality of the complementary nil D- eccentric dominating set of G is called the complementary nil D- eccentric domination number $\gamma_{\text{cned}}^{D}(G)$. In this paper, we obtain some bounds for $\gamma_{\text{cned}}^{D}(G)$.

Keywords: Dominating set, Eccentric dominating set, D- Eccentric dominating set, D- Eccentric domination number, complementary nil dominating set, complementary nil D- eccentric dominating set and its number.

1. Introduction

In 1962 O. Ore[1] proposed a new idea dominating set and domination number. In 1972, F. Harary [2] introduced the Graph theory concepts. In 1998 T. W. Haynes et al., [3] deliberated various dominating parameters. Chelvam TT, Chellathurai SR [4] introduced the concept complementary nil domination number in graphs in 2009. In 2010, V.R. Kulli [5] introduced the Theory of domination in graphs. In 2010 T. N. Janakiraman et al., [6] illustrated eccentric domination in graphs. In 2013 L. N. Varma et al., [7] determined D- Distance in graphs. Mohamed Ismayil and Ismail Mohideen [8] introduced the concept complementary nil domination in fuzzy graphs in 2014. Mohamed Ismayil and Muthupandiyan [9] initiated the concept complementary nil g-eccentric domination in fuzzy graphs in 2020. In 2021, Prasanna A. and N. Mohamedazarudeen [10] initiated the concept of D-Eccentric domination in graph. In 2022, Prasanna A. and N. Mohamedazarudeen [11] initiated the concept of Detour D- Eccentric domination in graph. Prasanna A. and N. Mohamedazarudeen [12] stated the concept of connected D- Eccentric domination in graphs in 2024.

In this article, the concept of complementary nil D- eccentric dominating set and its numbers are found. The bounds for complementary nil D- eccentric domination numbers are found. Theorems related to complementary nil D- eccentric dominating set and its number are stated and proved.

2. Preliminaries

Definition 2.1[1]: For any graph $G\ (V, E)$,V is the vertex set of G or V(G) and E is the edge set of G or E(G). The cardinality $|V|$ of the vertex set V is called order of G and the cardinality $|E|$ of the edge set E is called size of G. Let u, v $\in$ V be two vertices of G. The standard or usual distance d (u, v) between u and v is the length of the shortest u – v path in G.

Definition 2.2[10]: If u ,v are any two vertices of a connected graph G, then the D- length of a u – v path s is defined as $l^D(s) = d(u\ ,v) + \deg(u) + \deg(v) + \sum \deg(w)$ where sum runs over all intermediate vertices w of s. The D- distance $d^D(u, v) = \min\{\, l^D(s)\,\}$, where the minimum is taken over all u – v paths in G.

Definition 2.3[10]: For a vertex v, each vertex at a D- distance $e^D(v)$ from v is a D- eccentric vertex of v. D- eccentric set of a vertex v is defined as $E^D(v) =\{\, u \in V \,/\, d^D(v) = e^D(v)\,\}$ or any vertex u for which $d^D(u ,v) = e^D(v)$ is called D- eccentric vertex of v and a vertex u is said to be D- eccentric vertex of G if it is the D- eccentric vertex of some vertex.

Definition 2.4[10]: The D- eccentricity of a vertex v is defined by $e^D(v) = \max\{d^D(u,v)/\ u \in V\}$

Definition 2.5[11]: The D- radius, defined and denoted by $r^D(G) = \min\{\ e^D(v): v \in V\ \}$.The D-diameter, defined and denoted by $d^D(G) = \max\{\ e^D(v): v \in V\ \}$.

Definition 2.6[10]: The vertex v in G is a D- central vertex if $r^D(G) = e^D(v)$ and the D- center $C^D(G)$ is the set of all central vertices. The D- central sub graph $< C^D(G) >$ is induced by the center.

Definition 2.7[11]: The D- peripheral of G, $P^D(G) = d^D(G) = e^D(G)$. V is a D- peripheral vertex if $e^D(v) = d^D(G)$. The D- periphery $P^D(G)$ is the set of all peripheral vertices. The D- peripheral sub graph $< P^D(G) >$ is induced by the periphery.

Definition 2.8[5]: A set $D \subseteq V(G)$of vertices in a graph G = (V, E) is called a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D.

Definition 2.9[4] A dominating set $D \subseteq V(G)$ of a graph G is a complementary nil dominating set if the set D is a dominating set and $V - D$ is not a dominating set..

Definition 2.10[6] A set $S \subseteq V(G)$ is known as an eccentric point set of G if for every $v \in V - S$ there exist at least one eccentric vertex u in S such that $u \in E(v)$.

Definition 2.11[10] A set $S \subseteq V(G)$ is known as an D- eccentric point set of G if for every $v \in V - S$ there exist at least one D-eccentric vertex u in S such that $u \in E(v)$.

Definition 2.12[6] A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and also for every v in $V - D$ there exist at least one eccentric point of v in D.

Definition 2.13[10] In a dominating set $D \subseteq V$ of a graph G (V, E) if there exists at least one D-eccentric vertex u of v in D for every $v \in V - D$ then it is called a D- eccentric dominating set.

Definition 2.14[2]: The neighbourhood N(u) of a vertex u is the set of all vertices adjacent to u in G. $N[v] = N(v) \cup \{v\}$ is called the closed neighbourhood of v.

Definition 2.15[11]: A subgraph that has the same vertex set as G is called linear factor the degree of all vertices is one.

In this paper, only non trivial simple connected undirected graphs are considered and for all the other undefined terms one can refer [1, 2].

3. Complementary Nil D- Eccentric Dominating Set in Graphs

Definition 3.1: Let $D \subseteq V(G)$ be a set of vertices in a graph $G(V, E)$, then D is said to be a complementary nil D- eccentric dominating set (CNDED-set) of G if the set D is a D- eccentric dominating set and $V - D$ is not a D- eccentric dominating set. A complementary nil D-eccentric dominating set D of G is called minimal complementary nil D-eccentric dominating set, if no proper subset D' of D is a complementary nil D- eccentric dominating set of G. The minimum cardinality of a minimal complementary nil D- eccentric dominating set of D is called the complementary nil D- eccentric domination number and is denoted by $\gamma_{cned}^{D}(G)$ and simply denoted by γ_{cned}^{D}. The maximum cardinality of a minimal complementary nil D-eccentric dominating set is called the upper complementary nil D- eccentric domination number and is denoted by $\Gamma_{cned}^{D}(G)$ and simply denoted by Γ_{cned}^{D}.

Note 3.1: The minimum complementary nil D- eccentric dominating set is denoted by γ_{cned}^{D} - set.

Example 3.1: Consider the graph given in the following figure 1.

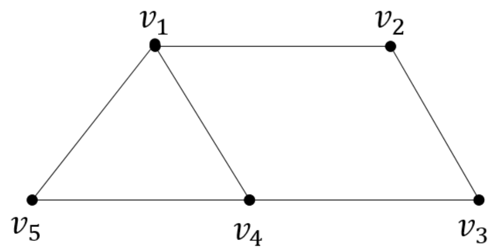

Figure.1 Complementary nil D- eccentric dominating set.

From the graph given in figure 1, the following points are observed.

The eccentricity and eccentric set of a vertices v_1, v_2, v_3, v_4 and v_5 are $e(v_1) = e(v_2) = e(v_3) = e(v_4) = e(v_5) = 2$ and $E(v_1) = \{v_3\}, E(v_2) = \{v_4, v_5\}$, $E(v_3) = \{v_1, v_{5,}\}, E(v_4) = \{v_2\}$ and $E(v_5) = \{v_2, v_3\}$ respectively.

The D-eccentricity and D-eccentric set of a vertices v_1, v_2, v_3, v_4 and v_5 are $e^D(v_1) = 9, e^D(v_2) = 8, e^D(v_3) = 9, e^D(v_4) = 9$, and $e^D(v_5) = 9$ and $E^D(v_1) = \{v_3\}$, $E^D(v_2) = \{v_{4,}\}$, $E^D(v_3) = \{v_{5,}\}$, $E^D(v_4) = \{v_2\}$ and $E^D(v_5) = \{v_2, v_3\}$ respectively. Hence,

(i) Dominating set is $D = \{v_2, v_4\}$ and $\gamma(G) = 2$.

(ii) Eccentric dominating set is $D = \{v_1, v_2\}$ and $\gamma_{ed}(G) = 2$.

(iii) D-Eccentric dominating set is $D = \{v_3, v_4\}$ and $\gamma_{ed}^D(G) = 2$.

(iv) Complementary nil D-Eccentric dominating set is $D = \{v_3, v_4, v_5\}$ and $\gamma_{ed}^D(G) = 3$.

Since, $V - D = \{v_1, v_2\}$ is not a D- eccentric dominating set.

(v) Upper complementary nil D-Eccentric dominating set is $D = \{v_1, v_2, v_3\}$ and $\Gamma_{ed}^D(G) = 3$, since $V - D = \{v_4, v_5\}$ is not a dominating set and D- eccentric dominating set.

Remark 3.1: Let D be a minimum complementary nil dominating set of a graph G and S be a minimum D- eccentric vertex set of G. Then clearly $D \cup S$ is a complementary nil D- eccentric dominating set of G.

Results 3.1:

(i) For any connected graph G, $\gamma(G) \leq \gamma_{cned}^D(G) \leq \Gamma_{cned}^D$.

(ii) Every complementary nil D- eccentric dominating set is a dominating set but the converse is not true.

(iii) If $r^D(G) = d^D(G)$, then $\gamma(G) = \gamma_{cned}^D(G)$.

(iv) If a connected graph G has more than one pendent vertex, then the complementary nil D-eccentric dominating set contains at least two pendent vertices.

(v) Every superset of a complementary nil D- eccentric dominating set is also a complementary nil D- eccentric dominating set .

(vi) The subset of a complementary nil D- eccentric dominating set need not be a complementary nil D- eccentric dominating set.

(vii) In a tree every complementary nil D- eccentric dominating set contains at least one pendent vertex.

(viii) Complete graph has no CNDED - set and its number.

Definition 3.2: Let $\mathrm{S} \subseteq \mathrm{V(G)}$ of a graph $\mathrm{G(V, E)}$, then $\mathrm{u} \in \mathrm{S}$ is said to be a D- eccentric enclave of S if $\mathrm{E^D[u]} \subseteq \mathrm{S}$ or $\mathrm{N[u]} \subseteq \mathrm{S}$.

Note 3.2: Enclave of a CNDED - set is not unique.

Lemma 3.1 Let $\boldsymbol{D}$ be a CNDED - set of a graph $\boldsymbol{G}$. Then $\boldsymbol{D}$ contains at least one enclave.

Proof. Let $\boldsymbol{D}$ be a CNDED -set of a graph $\boldsymbol{G}$. By the definition of CNDED - set, $\boldsymbol{V} - \boldsymbol{D}$ is not a $\boldsymbol{D}$- eccentric dominating set, which implies that there exists a vertex $\boldsymbol{v} \in \boldsymbol{S}$ such that $\boldsymbol{v}$ is not adjacent to any vertex in $\boldsymbol{V} - \boldsymbol{S}$ and so $\boldsymbol{N}[\boldsymbol{v}] \subseteq \boldsymbol{D}$.

Proposition 3.1: Let $\boldsymbol{G}$ be a graph and $\boldsymbol{D}$ be a minimum CNDED - set. If $\boldsymbol{u}$ and $\boldsymbol{v}$ are two enclaves of $\boldsymbol{D}$, then $\boldsymbol{N}[\boldsymbol{u}] \cap \boldsymbol{N}[\boldsymbol{v}] \neq \emptyset$ and $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent.

Proof. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be two enclaves of $\boldsymbol{D}$. Suppose $\boldsymbol{N}[\boldsymbol{u}] \cap \boldsymbol{N}[\boldsymbol{v}] = \emptyset$. Then $\boldsymbol{u}$ is an enclave of $\boldsymbol{D} - \boldsymbol{N}(\boldsymbol{v})$. Clearly $\boldsymbol{D} - \boldsymbol{N}(\boldsymbol{v})$ is a CNDED - set of $\boldsymbol{G}$ and $|\boldsymbol{D} - \boldsymbol{N}(\boldsymbol{v})| < |\boldsymbol{D}| = \boldsymbol{\gamma^{D}_{cnded}}(\boldsymbol{G})$, which is a contradiction to the minimality of $\boldsymbol{D}$. Hence $\boldsymbol{N}[\boldsymbol{u}] \cap \boldsymbol{N}[\boldsymbol{v}] \neq \emptyset$. Suppose $\boldsymbol{u}$ and $\boldsymbol{v}$ are non-adjacent, $\boldsymbol{u} \notin \boldsymbol{N}(\boldsymbol{v})$ and so $\boldsymbol{D} - \{\boldsymbol{v}\}$ contains an enclave $\boldsymbol{u}$ of $\boldsymbol{D} - \{\boldsymbol{v}\}$. Hence $\boldsymbol{D} - \{\boldsymbol{v}\}$ is a CNDED-set, which is a contradiction to the minimality of $\boldsymbol{D}$.

Theorem 3.1. Let $\boldsymbol{D}$ be a CNDED - set of a graph $\boldsymbol{G}$. Then $\boldsymbol{D}$ is minimal if and only if for each vertex $\boldsymbol{u} \in \boldsymbol{D}$ one of the following conditions are satisfied.
(i) $\boldsymbol{u}$ has a private neighbour
(ii) $\boldsymbol{V} - (\boldsymbol{D} - \{\boldsymbol{u}\})$ is a dominating set of $\boldsymbol{G}$.

Proof. Suppose $\boldsymbol{D}$ is minimal. On the contrary if there exists a vertex $\boldsymbol{u} \in \boldsymbol{D}$ such that $\boldsymbol{u}$ does not satisfy any of the given conditions (i) and (ii), then $\boldsymbol{D_1} = \boldsymbol{D} - \{\boldsymbol{u}\}$ is a dominating set of $\boldsymbol{G}$. Also by $\boldsymbol{V} - (\boldsymbol{D} - \{\boldsymbol{u}\})$ is not a dominating set. This implies that $\boldsymbol{D_1}$ is a CNDED -set of $\boldsymbol{G}$, which is a contradiction.

Conversely, suppose that $\boldsymbol{D}$ is a CNDED - set and for each vertex $\boldsymbol{u} \in \boldsymbol{D}$, one of the two stated conditions holds. We show that $\boldsymbol{D}$ is a minimal CNDED - set of $\boldsymbol{G}$. On the contrary, we assume that $\boldsymbol{D}$ is not a minimal CNDED -set. That is, there exists a vertex $\boldsymbol{u} \in \boldsymbol{D}$ such that $\boldsymbol{D} - \{\boldsymbol{u}\}$ is a CNDED -set of $\boldsymbol{G}$. Hence $\boldsymbol{u}$ is adjacent to atleast one vertex in $\boldsymbol{D} - \{\boldsymbol{u}\}$. Also $\boldsymbol{D} - \{\boldsymbol{u}\}$ is a $\boldsymbol{D}$ −eccentric dominating set, every vertex in $\boldsymbol{V} - \boldsymbol{D}$ is adjacent to atleast one vertex in $\boldsymbol{D} - \{\boldsymbol{u}\}$. That is, condition (i) does not hold. Since $\boldsymbol{D} - \{\boldsymbol{u}\}$ is a CNDED -set, $\boldsymbol{V} - (\boldsymbol{D} - \{\boldsymbol{u}\})$ is not a $\boldsymbol{D}$- eccentric dominating set. That is condition (ii) does not hold. Therefore there exists a vertex $\boldsymbol{u} \in \boldsymbol{D}$ such that which does not satisfy conditions (i) and (ii), a contradiction to the assumption.

Theorem 3.2. For any graph $\boldsymbol{G}$, every $\boldsymbol{\gamma^{D}_{cned}}$ – set intersects with every $\boldsymbol{\gamma}$-set of $\boldsymbol{G}$.

Proof. Let $\boldsymbol{D_1}$ be a $\boldsymbol{\gamma^{D}_{cned}}$– set and D be a $\boldsymbol{\gamma}$-set of $\boldsymbol{G}$. Suppose that $\boldsymbol{D_1} \cap \boldsymbol{D} = \boldsymbol{\phi}$, then $\boldsymbol{D} \subseteq \boldsymbol{V} -$

$\boldsymbol{D_1}, \boldsymbol{V - D_1}$ contains a dominating set $\boldsymbol{S}$. Therefore $\boldsymbol{V - D_1}$ itself is a dominating set, which is a contradiction.

Corollary 3.1. In any graph G, any two $\boldsymbol{\gamma_{cned}^{D}}$ - sets intersect.

Theorem 3.3. For any graph $\boldsymbol{G}, \boldsymbol{\delta + 1 \leq \gamma_{cned}^{D}(G) \leq \gamma(G) + \delta}$.

Proof. Let $\boldsymbol{D}$ be a $\boldsymbol{\gamma}_{\text{cned}}^{\boldsymbol{D}}$ -set of $\boldsymbol{G}$. Since $\boldsymbol{V - D}$ is not a $\boldsymbol{D}$-eccentric dominating set, there exists a vertex $\boldsymbol{v \in D}$ which is not adjacent to any of the vertices in $\boldsymbol{V - D}$. Therefore $\boldsymbol{N[v] \subseteq D}$ which implies that $\boldsymbol{|N[v]| \leq |D|}$, that is $\boldsymbol{d(v) + 1 \leq |S|}$ and so $\boldsymbol{\delta + 1 \leq \gamma_{cned}^{D}(G)}$. Let $\boldsymbol{D_1}$ be a $\boldsymbol{\gamma}$ set of $\boldsymbol{G}$. Let $\boldsymbol{u \in V}$ such that $\boldsymbol{d(u) = \delta}$. Then atleast one vertex $\boldsymbol{u_1 \in N[u]}$ such that $\boldsymbol{u_1 \in D_1}$. Now $\boldsymbol{D_1 \cup (N[u] - \{u_1\})}$ is CNDED-set of $\boldsymbol{G}$, which implies that $\boldsymbol{\gamma}_{\text{cned}}^{\boldsymbol{D}}\boldsymbol{(G) \leq |D_1 \cup (N[u] - \{u_1\})| \leq |D_1| + |N[u] - \{u_1\}| = \gamma + \delta}$. Therefore $\boldsymbol{\delta + 1 \leq \gamma}_{\text{cned}}^{\boldsymbol{D}}\boldsymbol{(G) \leq \gamma + \delta}$.

Theorem 3.4. For any graph $\boldsymbol{G}$ with $\boldsymbol{p > 1}, \left\lceil \frac{\boldsymbol{p}}{\boldsymbol{\Delta+1}} \right\rceil < \boldsymbol{\gamma_{cned}^{D}(G) \leq 2q - p + 1}$. Also if $\boldsymbol{\gamma}_{\text{cned}}^{\boldsymbol{D}}\boldsymbol{(G) = 2q - p + 1}$, then $\boldsymbol{G}$ is a tree.

Proof. Since $\left\lceil \frac{\boldsymbol{p}}{\boldsymbol{\Delta+1}} \right\rceil \leq \boldsymbol{\gamma(G) < \gamma}_{\text{cned}}^{\boldsymbol{D}}\boldsymbol{(G)}$, the first inequality follows. For any graph $\mathbf{G}, \boldsymbol{\gamma}_{\mathbf{cned}}^{\mathbf{D}}\mathbf{(G) \leq p - 1} = 2(\mathrm{p} - 1) - \mathrm{p} + 1 \leq 2\mathrm{q} - \mathrm{p} + 1$.
Also if $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = 2\mathrm{q} - \mathrm{p} + 1$. Then $2\mathrm{q} - \mathrm{p} + 1 \leq \mathrm{p} - 1$ and so $\mathrm{q} \leq \mathrm{p} - 1$. Hence $\mathrm{q} = \mathrm{p} - 1$. Therefore G must be a tree.

Proposition 3.2. For any graph $\mathrm{G}, \gamma_{\mathrm{s}}(\mathrm{G}) < \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G})$.

Proof. Let D be a $\gamma_{\text{cned}}^{\mathrm{D}}$ -set of G. Then there exists a vertex $\mathrm{v} \in \mathrm{D}$ such that $\mathrm{N[v]} \subseteq \mathrm{D}$. Clearly $\mathrm{D} - \{\mathrm{v}\}$ is a split dominating set and so $|\mathrm{D} - \{\mathrm{v}\}| \geq \gamma_{\mathrm{s}}(\mathrm{G})$. Therefore $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) - 1 \geq \gamma_{\mathrm{s}}(\mathrm{G})$. Hence $\gamma_{\mathrm{s}}(\mathrm{G}) \leq \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) - 1 < \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G})$.

Theorem 3.5. For any graph $\mathrm{G}, \Gamma(\mathrm{G}) + \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) \leq \mathrm{p} + 1$.

Proof. Let D be Γ-set of G. Then there exists a vertex $\mathrm{v} \in \mathrm{D}$ such that $\mathrm{D} - \{\mathrm{v}\}$ is not a dominating set of G. But $\mathrm{V} - (\mathrm{D} - \{\mathrm{v}\})$ is a dominating set and so $\mathrm{V} - (\mathrm{D} - \{\mathrm{v}\})$ is a CNDED-set. Therefore $|(\mathrm{V} - \mathrm{D}) \cup \{\mathrm{v}\}| \geq \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G})$. Hence $\Gamma(\mathrm{G}) + \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) \leq \mathrm{p} + 1$.

Theorem 3.6. For any graph $\mathrm{G}, \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = 2$ if and only if $\gamma(\mathrm{G}) = 1$ and $\delta = 1$.

Proof. Suppose $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = 2$. Then $\gamma(\mathrm{G}) < \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = 2$, implies $\gamma(\mathrm{G}) = 1$. Therefore, there exits a vertex u in G such that $\mathrm{d(u)} = \mathrm{p} - 1$. If $\delta > 1$, since $\delta + 1 \leq \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G})$, which implies that $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) > 2$, which is a contradiction to $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = 2$. Converse is obivious.

Theorem 3.7. Let G be a graph with $\delta > 1$ and $\mathrm{p} > 3$. Then $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) = \mathrm{p} - 1$ if and only if $\delta = \mathrm{p} - 2$.

Proof. Suppose that $\gamma_{\text{cnde}}^{\mathrm{D}}(\mathrm{G}) = \mathrm{p} - 1$. On the contrary, let us assume that $\delta \leq \mathrm{p} - 3$. Then there exists a vertex $\mathrm{v} \in \mathrm{V(G)}$ such that v is not adjacent to atleast two vertices say u, w in $\mathrm{V(G)}$. Then $\mathrm{V} - \{\mathrm{u}, \mathrm{w}\}$ is a CNDED-set. So $|\mathrm{V} - \{\mathrm{u}, \mathrm{w}\}| \geq \gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G})$. Hence $\gamma_{\text{cned}}^{\mathrm{D}}(\mathrm{G}) \leq \mathrm{p} - 2$, which is a contradiction to the hypothesis. Conversely if $\delta = \mathrm{p} - 2$, there exists a vertex v in G with $d(v) = p - 2$ and N[v] is a CNDED-set. Therefore $|N[v]| \geq \gamma_{cned}^{D}$ (G), so $\delta + 1 \geq \gamma_{cned}^{D}(G)$. Hence, , $\gamma_{cned}^{D}(G) = \delta + 1$. Therefore, $\gamma_{cned}^{D}(G) = p - 1$.

Observations 3.1:

(i) Let G be graph with $\delta = 1$. Then there exists a γ_{cned}^{D} - set which contains atleast one pendant vertex.

(ii) Let T be a tree. Then there exists a γ_{cned}^{D}- set which contain all the supports and exactly one pendant vertex. Hence $s + 1 \leq \gamma_{cned}^{D}(G)$, where s is the number of supports in T.

Theorem 3.8. Let G be a graph with diam (G) ≥ 3. Then $\gamma_{cned}(G) \leq p - \delta$.

Proof. Let $v \in V$ with $d^{D}(v) = \delta$. Since $diam^{D}$ (G) ≥ 3, there exists a vertex $u \in V - N[v]$ but u is not adjacent to any vertex in N[v]. Now, $V - N(v)$ is a D-eccentric dominating set but N(v) is not a D-eccentric dominating set. Therefore $|V - N(v)| \geq \gamma_{cned}^{D}(G)$ and $\gamma_{cned}^{D}(G) \leq p - \delta$.

Proposition 3.3. Let G be a bipartite graph with its complement $\overline{G}$ connected. Then $\gamma_{cned}^{D}(\overline{G}) = p - \Delta(G)$ or $p - \Delta(G) + 1$.

Proof. Let (X, Y) be a partition of G. In $\overline{G}$, $\langle X \rangle$ and $\langle Y \rangle$ are complete. Let u be a vertex in $\overline{G}$ such that $\delta(\overline{G}) = d(u)$. Without loss of generality we may assume that u is in X . If $N[u] \cap Y \neq \phi$, then $\gamma_{cned}^{D}(\overline{G}) = |N[u]| = \delta(\overline{G}) + 1$, since $\delta(\overline{G}) = p - 1 - \Delta(G), \gamma_{cned}(\overline{G}) = p - \Delta(G)$. If $N[u] \cap Y = \phi$, then N[u] together with a vertex from Y is a γ_{cned}^{D}-set of $\overline{G}$. In this case $\gamma_{cned}^{D}(\overline{G}) = \delta(\overline{G}) + 2, \gamma_{cned}^{D}(\overline{G}) = p - \Delta(G) + 1$.

Theorem 3.9. For any graph G, if $\gamma(G) = \frac{p}{2}$, then $\gamma_{cned}^{D}(G) = \frac{p}{2} + 1$. Proof. Suppose $\gamma(G) = \frac{p}{2}$. Let D be a γ-set of G, which implies $V - D$ is a dominating set with $|V - D| = \frac{p^2}{2}$. For any vertex $x \in V - D$, $(V - D) - \{x\}$ is not a dominating set. But $D \cup \{x\}$ is a dominating set and so $D \cup \{x\}$ is a CNDED-set. Therefore $|D \cup \{x\}| \geq \gamma_{cned}^{D}$ (G). So $\gamma(G) + 1 \geq \gamma_{cned}^{D}$ (G). Hence,, γ_{cned}^{D} (G) $= \gamma(G) + 1$.

4. Conclusion

Here, the study of complementary nil D-eccentric domination in graphs are discussed. Theorems related to complementary nil D-eccentric dominating set and its number are stated and proved. Also, some bounds for complementary nil Deccentric domination number of a graph are studied.

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