# Inverse D- Eccentric Domination in Graphs

**Prasanna A[1], Mohamedazarudeen N[2,3], Muthupandiyan S[4]**

[1]*Assistant Professor, PG and Research Department of Mathematics, Jamal Mohamed College (Autonomous), (Affiliated to Bharathidasan University), India.*
[2]*Research Scholar, PG and Research Department of Mathematics, Jamal Mohamed College (Autonomous), (Affiliated to Bharathidasan University), India,*
[3]*Assistant Professor, PG and Research Department of Mathematics, C. Abdul Hakeem College, (Autonomous), (Affiliated to Thiruvalluvar University), India.*
[4]*Assistant Professor, Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, India.*
*Email: apj_jmc@yahoo.co.in*

A subset $D$ of the vertex set $V(G)$ of a graph $G(V,E)$ is said to be a dominating set if every vertex not in $D$ is adjacent to at least one vertex in $D$. A dominating set $D$ is said to be an $D$- eccentric dominating set if for every $v \in V - D$, there exists at least one $D$- eccentric vertex $u$ of $v$ in $D$. If $V - D$ contains a $D$- eccentric dominating set $D'$ of $G$, then $D'$ is called an inverse $D$- eccentric dominating set with respect to $D$. The minimum of the cardinality of the inverse $D$- eccentric dominating set of $G$ is called the inverse $D$- eccentric domination number ${\gamma^{-1}}^{D}_{\text{ed}}(G)$. In this paper, we obtain some bounds and some theorems stated and proved related to ${\gamma^{-1}}^{D}_{\text{ed}}(G)$.

**Keywords:** Dominating set, Eccentric dominating set, D- Eccentric dominating set, D- Eccentric domination number, inverse dominating set, inverse D-eccentric dominating set and its number.

## 1. Introduction

In 1962 O. Ore [1] proposed a new idea dominating set and domination number. In 1972, F. Harary [2] proposed the idea about Graph Theory. In 1991, Kulli VR, Sigarkanti SC [3]. Introduced the concept of Inverse domination in graphs. In 1998 T. W. Haynes et al., [4] Fundamentals of domination in graphs. In 2010, V.R. Kulli [5] introduced the theory of domination in graphs. In 2010 T. N. Janakiraman et al., [6] illustrated eccentric domination in graphs. In 2013 L. N. Varma et al., [7] determined D- Distance in graphs. In 2021, Prasanna A. and N. Mohamedazarudeen [8] initiated the concept of D- Eccentric domination in graph.

In 2022, Prasanna A. and N. Mohamedazarudeen [9] initiated the concept of Detour D-Eccentric domination in graph. In 2021, Jahir Hussain and Fathima Begam [10] intiated the concept inverse eccentric domination in graphs. Prasanna A. and N. Mohamedazarudeen [11] stated the concept of connected D- Eccentric domination in graphs in 2024.

In this article, the concept of inverse D- eccentric dominating set and its numbers are found. The bounds for inverse D-eccentric domination numbers are found. Theorems related to inverse D-eccentric dominating set and its number are stated and proved.

## 2. Preliminaries

Definition 2.1[1]: For any graph G $(V, E)$,V is the vertex set of G or V(G) and E is the edge set of G or E(G). The cardinality $|V|$ of the vertex set V is called order of G and the cardinality $|E|$ of the edge set E is called size of G. Let u, v ∈ V be two vertices of G. The standard or usual distance d (u, v) between u and v is the length of the shortest u – v path in G.

Definition 2.2[7]: If u ,v are any two vertices of a connected graph G, then the D-length of a u – v path s is defined as $l^D(s) = d(u, v) + \deg(u) + \deg(v) + \sum \deg(w)$ where sum runs over all intermediate vertices w of s. The D- distance $d^D(u, v) = \min\{ l^D(s) \}$, where the minimum is taken over all u – v paths in G.

Definition 2.3[7]: For a vertex v, each vertex at a D-distance $e^D(v)$ from v is a D- eccentric vertex of v. D-eccentric set of a vertex v is defined as $E^D(v) = \{ u \in V / d^D(v) = e^D(v) \}$ or any vertex u for which $d^D(u, v) = e^D(v)$ is called D-eccentric vertex of v and a vertex u is said to be D- eccentric vertex of G if it is the D- eccentric vertex of some vertex.

Definition 2.4[8]: The D-eccentricity of a vertex v is defined by $e^D(v) = \max \{d^D(u,v) / u \in V\}$

Definition 2.5[8]: The D- radius, defined and denoted by $r^D(G) = \min \{ e^D(v): v \in V \}$.The D-diameter, defined and denoted by $d^D(G) = \max \{ e^D(v): v \in V \}$.

Definition 2.6[8]: The vertex v in G is a D- central vertex if $r^D(G) = e^D(v)$ and the D-center $C^D(G)$ is the set of all central vertices. The D-central sub graph $< C^D(G) >$ is induced by the center.

Definition 2.7[8]: The D- peripheral of G, $P^D(G) = d^D(G) = e^D(G)$. V is a D-peripheral vertex if $e^D(v) = d^D(G)$. The D-periphery $P^D(G)$ is the set of all peripheral vertices. The D-peripheral sub graph $< P^D(G) >$ is induced by the periphery.

Definition 2.8[5]: A set $D \subseteq V(G)$of vertices in a graph G = (V, E) is called a dominating set of G if every vertex $v \in V - D$ is adjacent to some vertex in D.

Definition 2.9[6] Let $D \subseteq V(G)$ of a graph G is dominating set. Then if $V - D$ contains a dominating set $D'$ with respect to D then $D'$ is called an inverse dominating set of G.

Definition 2.10[6] A set $S \subseteq V(G)$ is known as an eccentric point set of G if for every $v \in V - S$ there exist at least one eccentric vertex u in S such that $u \in E(v)$.

Definition 2.11[8] A set $S \subseteq V(G)$ is known as an D- eccentric point set of G if for every $v \in$

$V - S$ there exist at least one D-eccentric vertex u in S such that $u \in E(v)$.

Definition 2.12[6] A set $D \subseteq V(G)$ is an eccentric dominating set if D is a dominating set of G and also for every v in $V - D$ there exist at least one eccentric point of v in D.

Definition 2.13[8] In a dominating set $D \subseteq V$ of a graph G (V, E) if there exists at least one D-eccentric vertex u of v in D for every $v \in V - D$ then it is called a D- eccentric dominating set.

Definition 2.14[2]: The neighbourhood N(u) of a vertex u is the set of all vertices adjacent to u in G. N [v] = N(v) ∪ {v} is called the closed neighbourhood of v.

Definition 2.15[10]: A subgraph that has the same vertex set as G is called linear factor the degree of all vertices is one.

Definition 2.16 [10]: A spider is a tree on $2n + 1$ vertices obtained by subdividing each edge of a star $K_{1,n}$ where $n \geq 3$.

In this paper, only non trivial simple connected undirected graphs are considered and for all the other undefined terms one can refer [1, 2].

## 3. Inverse D- Eccentric Vertex Set In Graphs

Definition 3.1. Let $S \subseteq V(G)$ be a D- eccentric vertex set of a graph $G(V, E)$, then if $V - S$ contains a D- eccentric vertex set $S'$ of G, then $S'$ is called an inverse D- eccentric vertex set with respect to S. An inverse D- eccentric vertex set $S'$ of G is called minimal inverse D- eccentric vertex set, if no proper subset $S''$ of $S'$ is an inverse D- eccentric vertex set of G. The minimum cardinality of a minimal inverse D- eccentric vertex set of $D'$ is called the inverse D- eccentric number and is denoted by $e^{-D}(G)$ and simply denoted by $e^{-D}$. The maximum cardinality of a minimal inverse D- eccentric vertex set is called the upper inverse D- eccentric number and is denoted by $E^{-D}(G)$ and simply denoted by $E^{-D}$.

Note 3.1: The minimum inverse D- eccentric vertex set is denoted by $e^{-D}$ - set.

Example 3.1: Consider the graph given in the figure. 1.

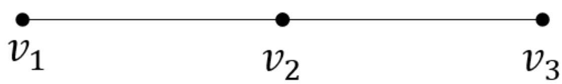


Figure.1 Inverse D- eccentric vertex set

From the figure. 1 given in example 3.1, the following points are observed.

1. The D- eccentric vertex set is $S_1 = \{v_1\}$ and $e^D(G) = 1$.
2. An inverse D- eccentric vertex set is $S_2 \subseteq V - S_1 = \{v_3\} = 1$ and $e^{-D}(G) = 1$.

Remark 3.1: The set $S_2 \subseteq V - S_1 = \{v_3\} = 1$ is also an upper inverse D- eccentric vertex set and hence, $E^{-D}(G) = 1$

Results 3.1:

(i) For any connected graph G, $e^{-1}(G) \leq e^{-D}_{ed}(G) \leq E^{-D}_{ed}(G)$.

(ii) Every inverse D- eccentric vertex set is a vertex set but the converse is not true.

(iii) If $r^{D}(G) = d^{D}(G)$, then $e^{-1}(G) = e^{-D}_{ed}(G)$.

(iv) Every superset of an inverse D- eccentric vertex set is also an inverse D- eccentric vertex set.

(v) The subset of an inverse D- eccentric vertex set need not be an inverse D- eccentric vertex set.

(vi) In a tree every inverse D- eccentric vertex set contains at least one pendent vertex.

(vii) Star graph has no an inverse D- eccentric vertex set.

Observation 3.1:

(i) $e^{-D}(K_n) = 1, n \geq 2.$

(ii) $e^{-D}\left(K_{n,m}\right) = 2, n, m \geq 2.$

(iii) $e^{-D}\left(K_{1,n}\right) = 2, n \geq 2.$

(iv) $e^{-D}(W_n) = 2, n \geq 2.$

(v) $e^{-D}(P_n) = 1, n = 2, 3.$

## 4. Inverse $\boldsymbol{D}$- Eccentric Dominating Set in Graphs

Definition 4.1: Let $\boldsymbol{D} \subseteq \boldsymbol{V}$(G) be a $\boldsymbol{D}$- eccentric dominating set of a graph $\boldsymbol{G}(\boldsymbol{V}, \boldsymbol{E})$, then if $\boldsymbol{V} - \boldsymbol{D}$ contains a $\boldsymbol{D}$- eccentric dominating set $\boldsymbol{D'}$ of $\boldsymbol{G}$, then $\boldsymbol{D'}$ is called an inverse $\boldsymbol{D}$- eccentric dominating set with respect to $\boldsymbol{D}$. An inverse D- eccentric dominating set $\boldsymbol{D'}$ of G is called minimal inverse D- eccentric dominating set, if no proper subset $\boldsymbol{D''}$ of $\boldsymbol{D'}$ is an inverse D- eccentric dominating set of G. The minimum cardinality of a minimal inverse D -eccentric dominating set of $\boldsymbol{D'}$ is called the inverse D- eccentric domination number and is denoted by$\boldsymbol{\gamma^{-D}_{ed}}(\boldsymbol{G})$ and simply denoted by $\boldsymbol{\gamma^{-D}_{ed}}$. The maximum cardinality of a minimal inverse D- eccentric dominating set is called the upper inverse D- eccentric domination number and is denoted by $\boldsymbol{\Gamma^{-D}_{ed}}(\boldsymbol{G})$ and simply denoted by $\boldsymbol{\Gamma^{-D}_{ed}}$

Note 4.1: The minimum inverse $\boldsymbol{D}$- eccentric dominating set is denoted by $\boldsymbol{\gamma^{-D}_{ed}}$ -Set.

Example 4.1 Consider the graph given in the following figure $\mathbf{2}$.

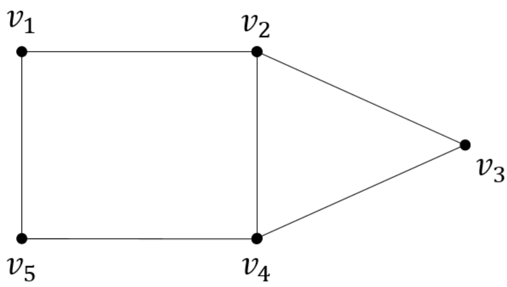


Figure.**2** Inverse $\boldsymbol{D}$-eccentric dominating set.

From the graph given in figure **2**, the following points are observed.

The $\boldsymbol{D}$- eccentricity and $\boldsymbol{D}$- eccentric set of a vertices $\boldsymbol{v_1, v_2, v_3, v_4}$ and $\boldsymbol{v_5}$ are $\boldsymbol{e^D(v_1) = e^D(v_2) = e^D(v_3) = (v_4) = e^D(v_5) = 9}$ and $\boldsymbol{E^D(v_1) = \{v_3\}}$, $\boldsymbol{E^D(v_2) = \{v_5\}}$, $\boldsymbol{E^D(v_3) = \{v_1, v_5\}}$, $\boldsymbol{E^D(v_4) = \{v_1\}}$ and $\boldsymbol{E^D(v_5) = \{v_2, v_3\}}$ respectively. Hence,

(i) $\boldsymbol{D}$- Eccentric dominating set is $\boldsymbol{D_1 = \{v_1, v_2\}}$ and $\boldsymbol{\gamma_{ed}^{D}(G) = 2}$.

(ii) Inverse $\boldsymbol{D}$- Eccentric dominating set is $\boldsymbol{D_2 = \{v_3, v_4, v_5\}}$ and $\boldsymbol{\gamma_{ed}^{-D}(G) = 3}$.

(iii) Upper inverse $\boldsymbol{D}$- Eccentric dominating set is $\boldsymbol{D = \{v_3, v_4, v_5\}}$ and $\boldsymbol{\Gamma_{ed}^{-D}(G) = 3}$

Remark 4.1: Let D be a minimum inverse dominating set of a graph G and S be a minimum D-eccentric vertex set of G. Then clearly D ∪ S is a inverse D-eccentric dominating set of G.

Results 4.1:

(i) For any connected graph G, $\boldsymbol{\gamma^{-1}}$(G) ≤ $\boldsymbol{\gamma^{-D}_{ed}}$(G) ≤ $\boldsymbol{\Gamma_{ed}^{-D}(G)}$.

(ii) Every inverse D- eccentric dominating set is a dominating set but the converse is not true.

(iii) If r $^D$(G) = d $^D$(G), then $\boldsymbol{\gamma^{-1}}$(G) =$\boldsymbol{\gamma_{ed}^{-D}}$(G).

(iv) Every superset of an inverse D- eccentric dominating set is also an inverse D-eccentric dominating set.

(v) The subset of a inverse D- eccentric dominating set need not be an inverse D- eccentric dominating set.

(vi) In a tree every inverse D- eccentric dominating set contains at least one pendent vertex.

(vii) Path and Star graph has no an inverse $\boldsymbol{D}$-eccentric dominating set.

Theorem 4.1 $\boldsymbol{\gamma_{ed}^{-D}(K_n) = 1, n \geq 2}$.

Proof. Let $\boldsymbol{v_1, v_2, \ldots, v_n}$ be the vertices of $\boldsymbol{K_n}$ where $\boldsymbol{n \geq 2}$. We know that $\boldsymbol{\gamma_{ed}^{D}(K_n) = 1}$.

Let $\boldsymbol{D = \{v_1\}}$ be the minimum $\boldsymbol{D}$- eccentric dominating set of $\boldsymbol{K_n}$. Any vertex $\boldsymbol{v_i}$ in $\boldsymbol{V - D}$ is a minimum inverse $\boldsymbol{D}$- eccentric dominating set. That is $\boldsymbol{D' = \{v_i \in V - \{v_1\}\}}$ and anyone $\boldsymbol{v_i}$

in $\boldsymbol{D'}$ is a minimum inverse $\boldsymbol{D}$- eccentric dominating set with respect to $\boldsymbol{D}$. Then $\boldsymbol{\gamma_{ed}^{-D}(K_n) = 1, n \geq 2}$.

Theorem 4.2 $\boldsymbol{\gamma_{ed}^{-D}\left(K_{n,m}\right) = 2,\ \ n, m \geq 2}$.

Proof. Let $\mathbf{G = K_{m,n}}$. Then $\mathbf{V(G)} = V_1 \cup V_2 \cdot |v_1| = m$ and $|V_2| = n$ where $m \geq 2$ and $n \geq 2$. We know that $\gamma_{ed}^{D}\left(K_{m,n}\right) = 2$.

Let $D = \{u_1, v_1\}$ where $u_1 \in V_1$ and $v_1 \in V_2$ is a minimum D- eccentric dominating set. Then any vertex $u_i \in V_1 - \{u_1\}$ dominates all the vertices of $v_2$ and it is a D-eccentric vertex of all the vertices in $v_1 - \{u_i\}$. Similarly for $v_2 - \{v_1\}$.

Let $D' = \{u_i, v_i\}$ where $u_i \in V_1 - \{u_1\}$ and $v_i \in V_2 - \{v_1\}$. Then any two vertices $\{u_i, v_i\} \subseteq D'$ where $u_i \in V_1 - \{u_1\}$ and $v_i \in V_2 - \{v_1\}$ is a minimum inverse D- eccentric dominating set of G with respect to D. Therefore, $\gamma_{ed}^{-D}\left(K_{m,n}\right) = 2$ for all $m \geq 2$ and $n \geq 2$.

Theorem 4.3

(i) $\gamma_{ed}^{-D}(W_4) = 1$

(ii) $\gamma_{ed}^{-D}(W_5) = 2$

(iv) $\gamma_{ed}^{-D}(W_6) = 3$

(iv) $\gamma_{ed}^{-D}(W_7) = 2$

Proof.
(i) Let $G = W_4 = K_4$. Hence by theorem 4.1, $\gamma_{ed}^{-D}(W_4) = 1$.

(ii) Let $G = W_5$. We know that $\gamma_{ed}^{D}(W_5) = 2$.

Let $D = \{u_1, u_2\}$ is a minimum D- eccentric dominating set. Then $D' = V - D = \{u_3, u_4, v\}$ where v is the D- central vertex of $W_4$. Consider $D'' = \{u_3, u_4\} \subseteq D'$ which is a minimum inverse D-eccentric dominating set of G with respect to D. Therefore, $\gamma_{ed}^{-D}(W_5) = 2$.
(iii) Let $G = W_6$. We know that $\gamma_{ed}^{D}(W_6) = 3$

Let $D = \{u_1, u_2, v\}$ where $u_1$ and $u_2$ are adjacent non-D- central vertices and v is a D-central vertex. Consider $D' = V - D = \{u_3, u_4, u_5\}$ which is a minimum inverse D- eccentric dominating set. Therefore, $\gamma_{ed}^{-D}(W_6) = 3$

(iv) Let $G = W_7$. We know that $\gamma_{ed}^{D}(W_7) = 2$.

Let $D = \{u_1, u_4\}$ is a minimum D- eccentric dominating set. Then $D' = V - D = \{u_2, u_3, u_5, u_6, v\}$ where v is the D- central vertex. Consider $D'' = \{u_2, u_5\}$ where $u_2$ dominates $u_1, u_3, v$ and $u_5$ dominates $u_6, u_4, v$ and also where $u_2$ is an eccentric point of $u_6, u_4, v$ and $u_5$ is an D- eccentric point of $u_1, u_3, v$. Therefore, $D''$ is the minimum inverse D- eccentric dominating set of G with respect to D. Hence $\gamma_{ed}^{-D}(W_7) = 2$.

Results 4.2: For a wheel graph $W_n$,

(i) $\gamma_{ed}^{-1^{D}}(W_n) = \left\lceil \frac{n}{3} \right\rceil n \geq 8$ and $n \neq 3m + 1$ where $m \geq 3$.

(ii) $\gamma_{ed}^{-1^D}(W_n) = \frac{(n-1)}{3} n \geq 8$ and $n = 3m+1$ where $m \geq 3$.

Theorem4.4. For a spider graph $T, \gamma_{ed}^{-1^D}(T) = n - \Delta(T) - 1$ when $n = 2k+1 \geq 9$ is the number of vertices of T.

Proof.
Le T be a spider graph with $n = 2k+1 \geq 9$. Then $r^D(T) = 2$ and $\mathrm{diam}^D(T) = 4$.

We know that $\gamma_{ed}^D(T) = n - \Delta(T) - 1 = k = |N(u)|$ where u is the D- central vertex of T.

Let D be a minimum D- eccentric dominating set containing $k-2$ vertices of $N(u)$ and 2 pendent vertices of T which are not adjacent to the vertex which we have selected from $N(u)$ to form D. Then $D' = V - D$ contains remaining $k-2$ pendent vertices, and 2 vertices from $N(u)$ and the D-central vertex u.

Let $D''$ be the subset of $D'$ containing $k-2$ pendent vertices and 2 vertices of $N(u)$. Then $D''$ is the minimum inverse D- eccentric dominating set of T with respect to D. Therefore, $\gamma_{ed}^{-D}(T) = |D''| = k - 2 + 2 = k = n - \Delta(T) - 1$.

Results 4.3:

(i) Let π denote the family of minimum dominating sets of G. If for every minimum dominating set $D \in \pi$, $V - D$ is independent, then $\gamma_{ed}^D(G) + \gamma_{ed}^{-D}(G) = p$.

(ii) Let T be a tree such that every nonend vertex is adjacent to at least one end vertex. Then $\gamma_{ed}^D(G) + \gamma_{ed}^{-D}(G) = p$

## 5. Conclusion

Here, the study of inverse D- eccentric domination in graphs are discussed. Theorems related to inverse D- eccentric dominating set and its number are stated and proved. Also, some bounds for inverse D- eccentric domination number of a graph are studied. Exact value of a inverse D-eccentric domination number for some standard graphs are found.

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