Oscillation Condition for a System of Non-Linear Differential Equations with Multiple Delays

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In this paper,we provide sufficient conditions for the oscillation of every solution of a non-linear differential equations with multiple delays of the form

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) f_i(x(\tau_i(t))) = 0$$

where the coefficients $p_i \in R$ and the delays $\tau_i \in R^+$ for i = 1,2,…n.
We provide new sufficient conditions for the oscillatory solution of this equation. Our results essentially improve the conditions in the literature.
Keywords: Non-Linear differential equation, Oscillation, Multiple delays.

1. Introduction

In this paper ,we consider the first -order non-linear differential equation with multiple delays of the form

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) f_i(x(\tau_i(t))) = 0 \quad (1.1)$$

where the coefficients $p_i \in R$ and the delays $\tau_i \in R^+$ for i = 1,2,…n are functions of positive real numbers such that

$$\tau_i(t) < t,\ t \geq t_0 \text{ and } \lim_{t \to \infty} \tau_i(t) = \infty \quad (1.2)$$

By a solution of (1.1) , we mean a function that is continuous for $t \geq T_0$, where

$T_0 = \min\{\inf\{\tau_i(t): t_0 \leq t\}, 1 \leq i \leq n\}\}$ and differentiable for $t \geq t_0$.A solution of (1.1) is called oscillatory if it has arbitrarily large zeros and otherwise ,it is called non- oscillatory. A solution is called eventually positive if there exists a t_1 such that $x(t) \geq 0$ and eventually negative if $x(t) < 0$ for $t \geq t_1$. .

In [3] , Julio Dix, Nurten Kilic and Özkan Öcalan used the following conditions and notations.

(H1) $\tau_i \in C(R,R)$, $\tau(t) \leq t$, $\lim_{t \to \infty} \tau_i(t) = \infty$ for i = 1,2,…n.

$$h_i(t) = \sup\{ \tau_i(s) : t_0 \leq s \leq t\},\ h(t) = \max\{ h_i(t): 1 \leq i \leq n\}. \quad (1.3)$$

(H2) $p_i \in C(t_0, \infty), R)$, $p_i(t) \geq 0$.

(H3) $f_i \in C(R,R)$, $x f_i(x) > 0$ for $x \neq 0$ and

$$M_i = \lim_{x \to 0} \sup \frac{x}{f_i(x)}, \quad 0 < M_i < \infty.$$

In (H1) , $\tau_i(t)$ are not necessarily monotonic, $\tau_i(t) \le h_i(t) \le h(t) \le t$ and h is non-decreasing.

In 2017,[1] G.E.Chatzarakis and Tongxing Li considered the differential equations generated by several deviating arguments

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) x(\tau_i(t))) = 0 \tag{1.4}$$

Throughout this paper, we use the following notations:

$$\alpha = \lim_{x \to \infty} \inf \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds \tag{1.5}$$

$$D(\alpha) = \begin{cases} 0 & \text{if } \omega > 1/e \\ \frac{1-\omega-\sqrt{1-2\omega-\omega^2}}{2} & \text{if } \omega \in [0,1/e] \quad \text{if } \omega \in [0,1/e] \end{cases} \tag{1.6}$$

$$\Delta = \lim_{t \to \infty} \sup \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds \tag{1.7}$$

where $\tau(t) = \max_{1 \le i \le m} \tau_i(t)$ and $\tau_i(t)$ in (1.2) are non-decreasing , i = 1,2,…n.

By Remark 2.7.3 in [7], it is clear that if $\tau_i(t)$, i = 1,2,…n, are non-decreasing and

$$\Delta > 1 , \tag{1.8}$$

then all solutions of (1.4) are oscillatory. This result is similar to Theorem 2.1.3[7] which is a special case of [5].

In 1978, Ladde [4] and in 1982, Ladas and Stavroulakis [6] proved that if

$$\alpha > 1/e, \tag{1.9}$$

then all solutions of (1.4) are oscillatory.

In 1984, Hunt and Yorke [2] proved that if $\tau_i(t)$ are non-decreasing ,

$t - \tau_i(t) \le \tau_0$, $1 \le i \le n$ and

$$\lim_{x \to \infty} \inf \sum_{i=1}^{n} p_i(t)(t - \tau_i(t)) > \frac{1}{e}, \tag{1.10}$$

then all solutions of (1.4) are oscillatory.

Assume that $\tau_i(t)$, i = 1,2,…n, are not necessarily monotone .

Set $h_i(t) = \sup_{t_0 \le s \le t} \tau_i(s)$ and $h(t) = \max_{1 \le i \le n} h_i(t)$, i = 1,2,…n. (1.11)

for $t \ge t_0$

Clearly, $h_i(t)$, $h(t)$ are nondecreasing and $\tau_i(t) \le h_i(t) < t$ for all $t \ge t_0$.

2. Preliminary results:

The following Lemmas and Theorems are used to prove the main result.

Lemma 2.1: ([6], Lemma 2.1.1).

If (H1) and $\liminf_{x\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds$ hold, then

$\liminf_{x\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds = \liminf_{x\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds$

where $\tau(t) = \max\{\tau_i(t) : 1 \le i \le n\}$.

Theorem 2.2: ([8,Theorem 2.1, 2.2])

Assume (H1) – (H3) , $0 < M_i < \infty$ and one of the following two conditions hold:

$$\liminf_{t\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds > \frac{M^*}{e}, \tag{2.1}$$

$$\limsup_{x\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{n} p_i(s)\,ds > M^*, \tag{2.2}$$

Then, every solution of (1.1) is oscillatory , where $\tau(t) = \max\{\tau_i(t) : 1 \le i \le n\}$ and $M^* = \max\{M_i : 1 \le i \le n\}$.

From (H3) , we can choose M_i for $1 \le i \le n$ such that

$f_i(x(\tau_i(t)) \ge \frac{1}{M_i} x(\tau_i(t))$ (See eq.(8) in [3]).

Thus (1.1) can be rewritten as

$$\dot{x}(t) + \sum_{i=1}^{n} \frac{p_i(t)}{M_i} x(\tau_i(t)) \le 0 \tag{2.3}$$

Lemma 2.3: ([1,lemma 2]

Assume that x is an eventually positive solution of (1.4), h(t) is defined by (1.11) and α by (1.5) with $0 < \alpha \le 1/e$.Then

$$\liminf_{t\to\infty} \frac{x(t)}{x(h(t))} \ge D(\alpha). \tag{2.4}$$

Lemma 2.4:([1,lemma 3]

Assume that x is an eventually positive solution of (2.3), h(t) is defined by (1.11) and α by (1.5) with $0 < \alpha \le 1/e$.Then

$$\liminf_{t\to\infty} \frac{x(h(t))}{x(t)} \ge \lambda_0, \tag{2.5}$$

where λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$.

3. Main Result:

Theorem 3.1:

Assume that h(t) is defined by (1.11) and for some j ϵ N

$$\lim_{t\to\infty}\sup \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds > 1, \quad (3.1)$$

where

$$S_j(t,\varepsilon)=\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\left[1+\int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_{j-1}(\psi,\varepsilon)\,d\psi\right)du\right)ds\right] \quad (3.2)$$

with $S_0 = \left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)(\lambda_0 - \varepsilon)$ and λ_0 is the smaller root of the transcendental equation

$\lambda = e^{\alpha\lambda}$.

Then all solutions of (2.3) are Oscillatory.

Proof:

Assume for the sake of contradiction , that there exists a non-oscillatory solution x(t) of (2.3). Since -x(t) is also a solution of (2.3) ,we can confine our discussion only to the case where the solution x(t) is eventually positive. Then there exists a $t_1 > t_0$,

such that x(t) > 0 and $x(\tau_i(t)) > 0$, $1 \le i \le n$,for all $t \ge t_1$.

Thus from (2.3) , we have

$$\dot{x}(t) = -\sum_{i=1}^{n}\frac{p_i(t)}{M_i}x(\tau_i(t)) \le 0 \qquad \text{for all } t \ge t_1,$$

which means that x(t) is an eventually nonincreasing function of positive integers. Considering $\tau_i(t) \le h(t)$, (2.3) implies that

$$\dot{x}(t) + \sum_{i=1}^{n}\frac{p_i(t)}{M_i}x(h(t)) \le \dot{x}(t) + \sum_{i=1}^{n}\frac{p_i(t)}{M_i}x(\tau_i(t)) = 0 \quad \text{for all } t \ge t_1,$$

or

$$\dot{x}(t) + \sum_{i=1}^{n}\frac{p_i(t)}{M_i}x(h(t)) \le 0 \text{ for all } t \ge t_1. \quad (3.3)$$

Observe that (2.5) implies that ,for each ε >0,there exists a t_ε such that

$$\frac{x(h(t))}{x(t)} > \lambda_0 - \varepsilon \text{ for all } t \ge t_\varepsilon \ge t_1. \quad (3.4)$$

Combining inequalities (3.3) and (3.4) ,we obtain

$$\dot{x}(t) + \sum_{i=1}^{n}\frac{p_i(t)}{M_i}(\lambda_0 - \varepsilon)\,x(t) \le 0 \text{ for all } t \ge t_\varepsilon \quad (3.5)$$

or

$$\dot{x}(t) + S_0\,x(t) \le 0 \text{ for all } t \ge t_\varepsilon \quad (3.6)$$

where $S_0 = \sum_{i=1}^{n}\frac{p_i(t)}{M_i}(\lambda_0 - \varepsilon)$

Applying Grönwall inequality in (3.5) ,we conclude that

$$x(s) \geq x(t)\exp\left(\int_s^t S_0(\vartheta,\varepsilon)\, d\vartheta\right),\ t \geq s \geq t_\varepsilon \tag{3.7}$$

Now we divide (2.3) by x(t) > 0 and integrate on [s,t],so

$$-\int_s^t \frac{\dot{x}(u)}{x(u)}\, du = \int_s^t \sum_{i=1}^{n} \frac{p_i(u)}{M_i}\ \frac{x(\tau_i(u))}{x(u)}\, du$$

$$\geq \int_s^t \sum_{i=1}^{n} \frac{p_i(u)}{M_i}\ \frac{x(\tau(u))}{x(u)}\, du$$

or

$$\ln\frac{x(s)}{x(t)} \geq \int_s^t \sum_{i=1}^{n} \frac{p_i(u)}{M_i}\ \frac{x(\tau(u))}{x(u)}\, du,\quad t \geq s \geq t_\varepsilon . \tag{3.8}$$

since $\tau_i(u) < \tau(u) < u$, $u = t, s = \tau_i(u)$ in (3.7)

$$x(\tau(u)) \geq x(u)\exp\left(\int_s^t S_0(\vartheta,\varepsilon)\, d\vartheta\right),\ t \geq s \geq t_\varepsilon \tag{3.9}$$

Combining (3.8) and (3.9) , we obtain for sufficiently large t,

$$\ln\frac{x(s)}{x(t)} \geq \int_s^t \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_s^t S_0(\vartheta,\varepsilon)\, d\vartheta\right) du,$$

or

$$x(s) \geq x(t)\exp\left(\int_s^t \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_s^t S_0(\vartheta,\varepsilon)\, d\vartheta\right) du\right) \tag{3.10}$$

Hence,

$$x(\tau(s)) \geq x(t)\exp\left(\int_{\tau(s)}^t \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_s^t S_0(\vartheta,\varepsilon)\, d\vartheta\right) du\right) \tag{3.11}$$

Integrating (2.3) from τ(t) to t ,we have

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) x(\tau_i(s))\, ds \leq 0$$

$$x(t) - x(\tau(t)) + \int_{\tau(t)}^t \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) x(\tau(s))\, ds \leq 0 \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^t \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^t \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(s)}^u S_0(\vartheta,\varepsilon)\, d\vartheta\right) du\right) ds \leq 0.$$

Multiplying the last inequality by $\sum_{i=1}^{n} \frac{p_i(t)}{M_i}$, we get

$$\left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) x(t) - \left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) x(\tau(t)$$

$$+\left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) x(t)\int_{\tau(t)}^t \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^t \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(s)}^u S_0(\vartheta,\varepsilon)\, d\vartheta\right) du\right) ds \leq 0$$

. (3.13)

Furthermore,

as $\dot{x}(t) \leq - \sum_{i=1}^{n} \frac{p_i(t)}{M_i} x(\tau_i(t))$

$$\dot{x}(t) \leq - \sum_{i=1}^{n} \frac{p_i(t)}{M_i} x(\tau_i(t)) \leq - \sum_{i=1}^{n} \frac{p_i(t)}{M_i} x(\tau(t)), \quad (3.14)$$

Combining inequalities (3.13) and (3.14) , we have

$$\left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right)x(t) + \dot{x}(t) + \left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) x(t) \int_{\tau(t)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{t} \sum_{i=1}^{n} \frac{p_i(u)}{M_i} \exp\left(\int_{\tau(s)}^{u} S_0(\vartheta, \varepsilon) d\vartheta \right) du \right) ds \leq 0. \quad (3.15)$$

That is,

$$\dot{x}(t)+\left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right)x(t)+ \left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) x(t) \int_{\tau(t)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(s)}^{u} S_0(\vartheta, \varepsilon) d\vartheta \right) du \right) ds \leq 0 .$$

Hence,

$$\dot{x}(t)+\left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) \left[1 + \int_{\tau(t)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(s)}^{u} S_0(\vartheta, \varepsilon) d\vartheta \right) du \right) ds\right]x(t) \leq 0$$

$$\dot{x}(t) + S_1(t, \varepsilon) x(t) \leq 0 \quad (3.16)$$

where

$$S_1(t, \varepsilon) = \left(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\right) \left[1 + \int_{\tau(t)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(s)}^{u} S_0(\vartheta, \varepsilon) d\vartheta \right) du\ ds\right]$$

Integrating (3.16) on [s ,t] leads to

$$x(s) \geq x(t) \exp\left(\int_{s}^{t} S_1(\psi, \varepsilon) d\psi\right) \quad (3.17)$$

We notice from (3.6) to (3.11) that x satisfies the inequality

$$x(\tau(u)) \geq x(u) \exp\left(\int_{s}^{t} S_1(\psi, \varepsilon) d\psi\right) \quad (3.18)$$

Combining now (3.8) and (3.18) ,we obtain

$$x(s) \geq x(t) \exp \left(\int_{s}^{t} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u} S_1(\psi, \varepsilon) d\psi\right) du\right),$$

from which we take

$$x(\tau(s)) \geq x(t) \exp \left(\int_{\tau(s)}^{t} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u} S_1(\psi, \varepsilon) d\psi\right) du\right). \quad (3.19)$$

By (3.12) and (3.19) , we have

$x(t) - x(\tau(t)) + x(t)\int_{\tau(t)}^{t}\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\exp\left(\int_{\tau(s)}^{t}\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds \le 0$,

Multiplying the last equation by $\sum_{i=1}^{n}\frac{p_i(t)}{M_i}$,we find

$$x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right) - x(\tau(t))\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right) + x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds \le 0.$$

As $\tau_i(t) < \tau(t)$,

Furthurmore,

$$x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right) - x(\tau_i(t))\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right) + x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds \le 0,$$

$$x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right) + \dot{x}(t) + x(t)\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds \le 0,$$

$$\dot{x}(t)+\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\left[1 + \int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds\right]x(t) \le 0.$$

Therefore , for sufficiently large t,

$$\dot{x}(t) + S_2(t,\varepsilon)x(t) \le 0, \tag{3.20}$$

where ,

$$S_2(t,\varepsilon) = \left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\left[1 + \int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_1(\psi,\varepsilon)\,d\psi\right)du\right)ds\right]$$

Repeating the above procedure ,it follows by induction that for sufficiently large t

$$\dot{x}(t) + S_j(t,\varepsilon)x(t) \le 0, \quad j \in N$$

where

$$S_j(t,\varepsilon)=\left(\sum_{i=1}^{n}\frac{p_i(t)}{M_i}\right)\left[1 + \int_{\tau(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_{j-1}(\psi,\varepsilon)\,d\psi\right)du\right)ds\right].$$

Moreover ,since $\tau(s) \le h(s) \le h(t)$, we have

$$x(\tau(s)) \ge x(h(t))\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right). \tag{3.21}$$

Integrating (2.3) from h(t) to t and using (3.21), we obtain

$0 \geq x(t) - x(h(t)) + \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) x(\tau_i(s))ds$

$\geq x(t) - x(h(t)) + \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)x(\tau(s))ds$

$\geq x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds.$

That is,

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds < 0 \quad (3.22)$$

The strict inequality is valid if we omit x(t) > 0 on the left-hand side. Therefore,

$$x(h(t)) \left[\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds - 1\right] < 0,$$

or

$$\int_{h(t)}^{t}\left[\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds - 1\right] < 0.$$

Taking the limit as $t \to \infty$, we have

$$\limsup_{t\to\infty} \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds < 1.$$

Since ε may be taken arbitrarily small,this inequality contradicts (3.1) .This completes the proof.

Theorem 3.2:

Assume that α defined by (1.5) with $0 < \alpha \leq 1/e$ and h(t) by (1.3).If for some $j \in N$,

$$\limsup_{t\to\infty} \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right) \exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds > 1 - D(\alpha). \quad (3.23)$$

where S_j is defined by (3.2),then all solutions of (2.3) are oscillatory.

Proof:

Let x be an eventually positive solution of (2.3) .Then , by Theorem 3.2, inequality (3.22) is satisfied.

That is,

$$x(t) - x(h(t)) + x(h(t)) \int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right) \exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds \leq 0$$

$$\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u} S_j(\psi,\varepsilon)\, d\psi\right)du\right)ds \leq 1 - \frac{x(t)}{x(h(t))},$$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds$$

$$\leq 1-\lim_{t\to\infty}\frac{x(t)}{x(h(t))}. \qquad (3.24)$$

By Lemma 2.3,

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \leq 1-\lim_{t\to\infty}D(\alpha).$$

Since ε may be taken arbitrarily small, this inequality contradicts (3.23).

The proof of the theorem is complete.

Theorem 3.3:

Assume that α is defined by (1.5) with $0 < \alpha \leq 1/e$ and h(t) by (1.11).If for some j ϵ N

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds > \frac{1}{D(\alpha)}-1, \quad (3.25)$$

where S_j is defined by (3.2), then all solutions of (2.3) are oscillatory.

Proof:

Assume for the sake of contradiction,that there exists a non-oscillatory solution x of (2.3) and that x is eventually positive.Then , as in the proof of Theorem 3.1,(3.21) is satisfied, which yields

$$x(\tau(s)) \geq x(h(t))\exp\left(\int_{\tau(s)}^{h(t)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right).$$

Integrating (2.3) from h(t) to t,we have

$$x(t)-x(h(t))+\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)x(\tau_i(s))\,ds \leq 0$$

or

$$x(t)-x(h(t))+\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)x(\tau(s))\,ds \leq 0.$$

Thus by (3.21),the last inequality gives

$$x(t)-x(h(t))+\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)x(t)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \leq 0.$$

or

$$x(t)-x(h(t))+x(t)\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \leq 0.$$

Thus, for all sufficiently large t , it holds

$$\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \le \frac{x(h(t))}{x(t)} - 1.$$

Letting $t \to \infty$,we take

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \le \lim_{t\to\infty}\frac{x(h(t))}{x(t)} - 1,$$

which ,in view of (2.4) gives

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds \le \frac{1}{D(\alpha)} - 1.$$

Since ε is arbitrary small, this inequality contradicts (3.25).

The proof of the theorem is complete.

Theorem 3.4:

Assume that α is defined by (1.5) with $0 < \alpha \le 1/e$ and h(t) by (1.11).If for some $j \in N$

$$\limsup_{t\to\infty}\int_{h(t)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(s)}{M_i}\right)\exp\left(\int_{\tau(s)}^{t}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right)ds > \frac{1+\ln\lambda_0}{\lambda_0} - D(\alpha), \quad (3.26)$$

where S_j is defined by (3.2) and λ_0 is the smaller root of the transcendental equation $\lambda = e^{\alpha\lambda}$,

then all solutions of (2.3) are oscillatory.

Proof:

Assume ,for the sake of contradiction, that there exists a non-oscillatory solution x of (2.3) and that x is eventually positive. Then ,as in Theorem 3.1 ,(3.21) is satisfied. By (2.5), for each $\varepsilon > 0$,there exists a t_ε ,such that

$$\lambda_0 - \varepsilon < \frac{x(h(t))}{x(t)} \text{ for all } t \ge t_\varepsilon, \quad (3.27)$$

Since $\frac{x(h(t))}{x(s)}$ is non increasing in s ,we have

$$1 = \frac{x(h(t))}{x(h(t))} \le \frac{x(h(t))}{x(s)} \le \frac{x(h(t))}{x(t)},\ t_\varepsilon \le h(t) \le s \le t,$$

In particular for $\varepsilon \in (0, \lambda_0 - 1)$, by continuity there exists a $t^* \in (h(t),t]$ such that

$$1 < \lambda_0 - \varepsilon = \frac{x(h(t))}{x(t^*)}. \quad (3.28)$$

By (3.2), we have

$$x(\tau(s)) \ge x(h(s))\exp\left(\int_{\tau(s)}^{h(s)}\left(\sum_{i=1}^{n}\frac{p_i(u)}{M_i}\right)\exp\left(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\,d\psi\right)du\right). \quad (3.29)$$

Integrating (2.3) from t^* to t,we have

$x(t) - x(t^*) + \int_{t*}^{t}(\sum_{i=1}^{n} \frac{p_i(s)}{M_i})\, x(\tau(s))\, ds \le 0,$

By using (3.29) along with h(s) ≤ h(t) in combination with the non increasingness of x ,we have

$x(t) - x(t^*) + x(h(t)) \int_{t*}^{t}(\sum_{i=1}^{n} \frac{p_i(s)}{M_i})\ \exp\, (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\, \exp(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi)\, du))\, ds \le$ 0,

or

$\int_{t*}^{t}((\sum_{i=1}^{n} \frac{p_i(s)}{M_i})\ \exp\, (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\, \exp(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi)\, du))\, ds) \le \frac{x(t*)}{x(h(t))} - \frac{x(t)}{x(h(t))}.$

In view of (3.28) and Lemma 2.3 for the ε considered, there exists a $t_\varepsilon^{'} \ge t_\varepsilon$, such that

$\int_{t*}^{t}(\sum_{i=1}^{n} \frac{p_i(s)}{M_i})\ \exp\, (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\exp(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi)\, du))\, ds < \frac{1}{\lambda_0 - \varepsilon} - D(\alpha) + \varepsilon$ (3.30)

for $t \ge t_\varepsilon^{'}$.

Dividing (2.3) by x(t) and integrating from h(t) to $t*$, we find

$\int_{h(t)}^{t*}(\sum_{i=1}^{n} \frac{p_i(t)}{M_i})\ \frac{x(\tau(s))}{x(s)} \le - \int_{h(t)}^{t*} \frac{\dot{x}(s)}{x(s)}\, ds,$

And using (3.29) , we get

$\int_{h(t)}^{t*}(\sum_{i=1}^{n} \frac{p_i(t)}{M_i})\ \frac{x(h(s))}{x(s)}\ \mathbf{exp}\ \left(\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\ \mathbf{exp}\left(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, \mathbf{d\psi}\right) \mathbf{du}\right) \mathbf{ds} \le - \int_{h(t)}^{t*} \frac{\dot{x}(s)}{x(s)}$ ds (3.31)

By (3.26), for $s \ge h(t) \ge t_\varepsilon^{'}$,we have $\frac{x(h(s))}{x(s)} > \lambda_0 - \varepsilon$, so from (3.31) we get

$(\lambda_0 - \varepsilon) \int_{h(t)}^{t*}(\sum_{i=1}^{n} \frac{p_i(t)}{M_i})\ \mathbf{exp}\ \left(\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\mathbf{exp}\left(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, \mathbf{d\psi}\right) \mathbf{du}\right) \mathbf{ds} < - \int_{h(t)}^{t*} \frac{\dot{x}(s)}{x(s)}$ ds .

Hence for sufficiently large t, we have

$$\int_{h(t)}^{t*}(\sum_{i=1}^{n} \frac{p_i(t)}{M_i})\, \mathbf{exp}\ \left(\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\ \mathbf{exp}\left(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, \mathbf{d\psi}\right) \mathbf{du}\right) \mathbf{ds}$$

$$< - \frac{1}{(\lambda_0 - \varepsilon)} \int_{h(t)}^{t*} \frac{\dot{x}(s)}{x(s)}\, ds$$

$$= \frac{1}{(\lambda_0 - \varepsilon)} \ln \frac{x(h(t))}{x(t*)}$$

$$= \frac{\ln(\lambda_0 - \varepsilon)}{(\lambda_0 - \varepsilon)} ,$$

(3.32)

Adding (3.30) and (3.32), and then taking the limit as t ⟶ ∞, we have

$$\limsup_{t \to \infty} \int_{h(t)}^{t}(\sum_{i=1}^{n} \frac{p_i(s)}{M_i})\ \mathbf{exp}\ (\int_{\tau(s)}^{t}(\sum_{i=1}^{n} \frac{p_i(u)}{M_i})\ \mathbf{exp}(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\ \mathbf{d\psi})\mathbf{du})\mathbf{ds}$$

$$\leq \frac{1+\ln(\lambda_0 - \varepsilon)}{(\lambda_0 - \varepsilon)} - D(\alpha) + \varepsilon.$$

Since ε may be taken arbitrarily small, this inequality contradicts (3.26).

The proof of the theorem is complete.

Theorem 3.5:

Assume that h(t) is defined by (1.3) and for $j \in N$

$$\liminf_{t \to \infty} \int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \exp\Big(\int_{\tau(s)}^{h(s)} \Big(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\Big) \exp\Big(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi\Big) du\Big) ds > \frac{1}{e},$$
(3.33)

where $\mathbf{S_j}$ is defined by (3.2) .Then all solutions of (2.3) are oscillatory.

Proof:

Assume for the sake of contradiction, that there exists a non-oscillatory solution x(t) of (2.3).

Since -x(t) is also a solution of (2.3),we can confine our discussion only to the case where the solution x(t) is eventually positive. Then there exists a $t_1 > t_0$ such that $x(t) > 0$ and $x(\tau_i(t)) > 0$, $1 \leq i \leq m$ for all $t \geq t_1$. Thus , from (2.3) we have

$$\dot{x}(t) = -\Big(\sum_{i=1}^{n} \frac{p_i(t)}{M_i}\Big) x(\tau_i(t)) \leq 0 \quad \text{for all } t \geq t_1,$$

which means that x(t) is an eventually non-increasing function of positive numbers. Moreover, as in previous theorem, (3.29) is satisfied.

Dividing (2.3) by x(t) and integrating from h(t) to t, for some $t_2 \geq t_1$, we get

$$\ln\Big(\frac{x(t)}{x(h(t))}\Big) + \int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \frac{x(\tau_i(s))}{x(s)}\, ds \leq 0$$

$$\ln\Big(\frac{x(h(t))}{x(t)}\Big) \geq \int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \frac{x(\tau(s))}{x(s)}\, ds \tag{3.34}$$

Combining inequalities (3.29) and (3.34),we obtain

$$\ln\Big(\frac{x(h(t))}{x(t)}\Big) \geq \int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \frac{x(h(s))}{x(s)} \exp\Big(\int_{\tau(s)}^{h(s)} \Big(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\Big) \exp\Big(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi\Big) du\Big) ds.$$

Taking into account that x is non-increasing and h(s) < s, the last inequality becomes

$$\ln\Big(\frac{x(h(t))}{x(t)}\Big) \geq \int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \exp\Big(\int_{\tau(s)}^{h(s)} \Big(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\Big) \exp\Big(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi\Big) du\Big) ds.$$
(3.35)

From (3.33) ,it follows that there exists a constant c > 0 such that for sufficiently large t

$$\int_{h(t)}^{t} \Big(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\Big) \exp\Big(\int_{\tau(s)}^{h(s)} \Big(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\Big) \exp\Big(\int_{\tau(u)}^{u} S_j(\psi, \varepsilon)\, d\psi\Big) du\Big) ds \geq c > \frac{1}{e},$$

Choose $\mathbf{c}'$ such that $c > \mathbf{c}' > \frac{1}{e}$. For every $\varepsilon > 0$ such that $c - \varepsilon > \mathbf{c}'$, we have

$\int_{h(t)}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\ \mathbf{exp}\ (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\ \mathbf{exp}(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\ d\psi)du)ds \ \geq\ c\ -\ \varepsilon\ >\ c'\ >\ \frac{1}{e}$,
(3.36)

Combining inequalities (3.35) and (3.36), we obtain

$\ln(\frac{x(h(t))}{x(t)}) \geq c'$, $t \geq t_3$.

Thus

$(\frac{x(h(t))}{x(t)}) \geq e^{c'} \geq ec' > 1$,

which gives ,for some $t \geq t_4 \geq t_3$,

$x(h(t)) \geq (ec')\ x(t)$.

Repeating the above procedure ,it follows by induction that for any positive integer k,

$\frac{x(h(t))}{x(t)} \geq (ec')^k$ for sufficiently large t.

Since $ec' > 1$, there is a $k \in N$ satisfying $k > 2\ (\frac{(\ln(2)-\ln(c'))}{1+\ln(c')})$ such that for t sufficiently large

$$\frac{x(h(t))}{x(t)} \geq (ec')^k > (\frac{2}{c'})^2 \tag{3.37}$$

Next we split the integral in (3.36) into two integrals ,each integral being no less than $\frac{c'}{2}$.

$\int_{h(t)}^{t_n}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\mathbf{exp}\ (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\ \mathbf{exp}(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\ d\psi)du)ds \qquad \geq \qquad \frac{c'}{2}$,
(3.38)

$\int_{t_n}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\mathbf{exp}\ (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\ \mathbf{exp}(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\ d\psi)du)ds \qquad \geq \qquad \frac{c'}{2}$.
(3.39)

Integrating (2.3) from t_n to t, we deduce that

$x(t) - x(t_n) + \int_{t_n}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})x(\tau_i(s))ds = 0$

or

$x(t) - x(t_n) + \int_{t_n}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\ x(\tau(s))ds \leq 0$.

which in view of (3.29),gives

$x(t) - x(t_n) + x(h(t)) \int_{t_n}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\ \mathbf{exp}\ (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\mathbf{exp}(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\ d\psi)\ du)\ ds$
≤ 0.

The strict inequality is valid if we omit $x(t) > 0$ on the left-hand side.

$-x(t_n) + x(h(t)) \int_{t_n}^{t}(\sum_{i=1}^{n}\frac{p_i(s)}{M_i})\ \mathbf{exp}\ (\int_{\tau(s)}^{h(s)}(\sum_{i=1}^{n}\frac{p_i(u)}{M_i})\ \mathbf{exp}(\int_{\tau(u)}^{u}S_j(\psi,\varepsilon)\ d\psi)\ du)\ ds < 0$.

Using the inequality in (3.39), we conclude that

$$x(\mathbf{t_n}) > \frac{c'}{2} x(h(t)). \quad (3.40)$$

Similarly, integrating (2.3) between h(t) to $\mathbf{t_n}$ with the later application of (3.29) leads to

$$x(\mathbf{t_n}) - \mathbf{x}(\mathbf{h}(\mathbf{t})) + \mathbf{x}(\mathbf{h}(\mathbf{t_n})) \int_{h(t)}^{t_n} \left(\sum_{i=1}^{n} \frac{p_i(s)}{M_i}\right) \mathbf{exp}\left(\int_{\tau(s)}^{h(s)} \left(\sum_{i=1}^{n} \frac{p_i(u)}{M_i}\right) \mathbf{exp}\left(\int_{\tau(u)}^{u} \mathbf{S_j}(\psi, \varepsilon)\, d\psi\right) du\right) ds \leq \mathbf{0}.$$

The strict inequality is valid if we omit $\mathbf{x}(\mathbf{t_n}) > 0$ on the left-hand side.

$$x(h(t)) > \frac{c'}{2} x(h(\mathbf{t_n})). \quad (3.41)$$

Combining inequalities (3.40) and (3.41) ,we obtain

$$x(h(\mathbf{t_n})) < \frac{2}{c'} x(h(t)) < \left(\frac{2}{c'}\right)^2 x(\mathbf{t_n}),$$

which contradicts (3.37) .

Thus the proof of the theorem is completed.

Example 1:

Consider the non-linear delay differential equation

$$\dot{\mathbf{x}}(t) + \frac{71}{50} x(\boldsymbol{\tau_1}(\mathbf{t})) \left(|\mathbf{x}(\boldsymbol{\tau_1}(\mathbf{t}))| + \frac{1}{5}\right) + \frac{63}{50} x(\boldsymbol{\tau_2}(\mathbf{t})) \left(|\mathbf{x}(\boldsymbol{\tau_2}(\mathbf{t}))| + \frac{1}{4}\right) + \frac{56}{50} x(\boldsymbol{\tau_3}(\mathbf{t})) \left(|\mathbf{x}(\boldsymbol{\tau_3}(\mathbf{t}))| + \frac{1}{2}\right) = 0,\ t \geq 0,$$

with (3.42)

$$\boldsymbol{\tau_1}(\mathbf{t}) = \begin{cases} -t + 12k - 2 & \text{if } t \in [6k, 6k+1] \\ 4t - 18k - 7 & \text{if } t \in [6k+1, 6k+2] \\ -t + 12k + 3 & \text{if } t \in [6k+2, 6k+3] \\ t - 3 & \text{if } t \in [6k+3, 6k+4] \\ -2t + 18k + 9 & \text{if } t \in [6k+4, 6k+5] \\ 5t - 24k - 26 & \text{if } t \in [6k+2, 6k+3] \end{cases}$$

and $\boldsymbol{\tau_2}(\mathbf{t}) = \boldsymbol{\tau_1}(\mathbf{t}) - 0.1$, $\boldsymbol{\tau_3}(\mathbf{t}) = \boldsymbol{\tau_2}(\mathbf{t}) - 0.2$,

where k ϵ $\mathbf{N_0}$ and $\mathbf{N_0}$ is the set of non negative integers.

By (1.11),we see that

$$\mathbf{h_1}(\mathbf{t}) = \begin{cases} 6k - 2, & \text{if } t \in [6k, 6k + 1.25], \\ 4t - 18k - 7, & \text{if } t \in [6k + 1.25, 6k+2], \\ 6k + 1, & \text{if } t \in [6k+2, 6k + 5.4], \\ 5t - 24k - 26 & \text{if } t \in [6k + 5.4, 6k + 6], \end{cases}$$

and $\mathbf{h_2}(\mathbf{t}) = \mathbf{h_1}(\mathbf{t}) - \mathbf{0.1}$, $\mathbf{h_3}(\mathbf{t}) = \mathbf{h_2}(\mathbf{t}) - \mathbf{0.1}$, and consequently,

$h(t) = \max_{1 \le i \le 3} h_i(t) = h_1(t)$ and $\tau(t) = \max_{1 \le i \le 3} \tau_i(t) = \tau_1(t)$.

Also we have

$$\alpha = \liminf_{x\to\infty} \int_{\tau(t)}^{t} \sum_{i=1}^{3} p_i(s)\, ds = \liminf_{x\to\infty} \int_{6k+1}^{6k+2} 3.8\, ds = 3.8$$

and therefore ,the smaller root of $e^{3.8\lambda} = \lambda$ is $\lambda_0 = -0.357$.

By (H3) ,

$$M_i = \limsup_{x\to 0} \frac{x}{f_i(x)}, \quad 0 < M_i < \infty.$$

$$M_1 = \limsup_{x\to 0} \frac{x}{f_1(x)} = \limsup_{x\to 0} \frac{x}{x\,(\,|x(\tau_1(t))| + \frac{1}{5})} = 5,$$

$$M_2 = \limsup_{x\to 0} \frac{x}{f_2(x)} = \limsup_{x\to 0} \frac{x}{x\,(\,|x(\tau_1(t))| + \frac{1}{4})} = 4,$$

$$M_3 = \limsup_{x\to 0} \frac{x}{f_2(x)} = \limsup_{x\to 0} \frac{x}{x\,(\,|x(\tau_1(t))| + \frac{1}{2})} = 2.$$

Thus $\sum_{i=1}^{3} \frac{p_i(t)}{M_i} = \frac{p_1(t)}{M_1} + \frac{p_2(t)}{M_2} + \frac{p_3(t)}{M_3} = \frac{1.42}{5} + \frac{1.26}{4} + \frac{1.12}{2} = 1.159.$

Let us prove that the solutions of (3.42) is oscillatory by showing (3.1) of Theorem 3.1 holds.

The function $F_j = \int_{h(t)}^{t} (\sum_{i=1}^{3} \frac{p_i(s)}{M_i}) \exp\,(\int_{\tau(s)}^{h(t)} (\sum_{i=1}^{3} \frac{p_i(u)}{M_i}) \exp(\int_{\tau(u)}^{u} S_j(\,\psi, \varepsilon)\, d\psi) du) ds$

attains its maximum at t = 6k + 5.4, k ϵ N_0 , for every j ≥ 1.Specifically,

$$F_1(\,t = 6k + 5.4) = \int_{6k+1}^{6k+5.4} (\sum_{i=1}^{3} \frac{p_i(s)}{M_i})\, \exp\,(\int_{\tau(s)}^{6k+1} (\sum_{i=1}^{3} \frac{p_i(u)}{M_i}) \exp(\int_{\tau(u)}^{u} S_1(\,\psi, \varepsilon)\, d\psi) du) ds$$

with

$$S_1(\,t, \varepsilon\,) = (\sum_{i=1}^{3} \frac{p_i(t)}{M_i})[\,1 + \int_{\tau(\varepsilon)}^{\varepsilon} (\sum_{i=1}^{3} \frac{p_i(s)}{M_i}) \exp\,(\int_{\tau(v)}^{\varepsilon} (\sum_{i=1}^{3} \frac{p_i(w)}{M_i}) \exp\,(\int_{\tau(w)}^{w} \lambda_0\,(\sum_{i=1}^{n} \frac{p_i(z)}{M_i})\; dz\,)\, dw\,) dv]\;.$$

We obtain

$$S_1(\,t, \varepsilon\,) = 1.159\,[\,1 + \int_{6k+1}^{6k+5.4} 1.159 \exp\,(\int_{6k+1}^{6k+5.4} 1.159\;\; \exp\,(\int_{6k+1}^{6k+5.4} (-0.357)(1.159)\;\, dz\,)\, dw\,) dv]$$

$S_1 = 11.36.$

Thus ,

$$F_1(\,t = 6k + 5.4) = \int_{6k+1}^{6k+5.4} 1.159 \exp\,(\int_{6k+1}^{6k+1} 1.159\; \exp(\int_{\tau(u)}^{u} (11.36) d\psi) du) ds$$

$\mathbf{F_1}(t = 6k + 5.4) \approx 5.09$ and so

$\lim_{t\to\infty} \sup \mathbf{F_1(t)} \approx 5.09 > 1$, thus satisfying condition (3.1) of Theorem 3.1 is for j = 1, and therefore all solutions of (3.42) are oscillatory.

Example 2:

We consider the following first order non-linear delay differential equation

$\dot{x}(t) + \frac{93}{100}\, x(\tau_1(t)) \left(|x(\tau_1(t))| + \frac{1}{e} \right) + \frac{99}{100}\, x(\tau_2(t)) \left(|x(\tau_2(t))| + \frac{2}{e} \right) = 0\ ,\ t \geq 0,$ (3.43)

where

With

$$\tau_1(t) = \begin{cases} t - 1 & \text{if } t \in [3k, 3k+1] \\ -3t + 12k + 3 & \text{if } t \in [3k+1, 3k+2] \\ 5t - 12k - 13 & \text{if } t \in [3k+2, 3k+3] \end{cases}$$

, $k \in N_0$ and $\tau_2(t) = \tau_1(t) - 2$,

Also,

$$h_1(t) = \begin{cases} t - 1 & \text{if } t \in [3k, 3k+1] \\ -3k & \text{if } t \in [3k+1, 3k+2.6] \\ 5t - 12k - 13 & \text{if } t \in [3k+2.6, 3k+3] \end{cases}$$

and $h_2(t) = h_1(t) - 2$,

$h(t) = \max_{1 \leq i \leq 2} h_i(t) = h_1(t)$ and $\tau(t) = \max_{1 \leq i \leq 2} \tau_i(t) = \tau_1(t)$.

Also we have

$\alpha = \lim_{x\to\infty} \inf \int_{\tau(t)}^{t} \left(\sum_{i=1}^{2} p_i(s)\right) ds$.

$\sum_{i=1}^{2} p_i(s) = 0.93 + 0.99 = 1.92$.

Therefore , $\alpha = \lim_{x\to\infty} \inf \int_{\tau(t)}^{t} \left(\sum_{i=1}^{2} p_i(s)\right) ds = \lim_{x\to\infty} \inf \int_{t-1}^{t} \sum_{i=1}^{2} p_i(s)\, ds = 1.92.$

and the smaller root of $e^{1.92\lambda} = \lambda$ is $\lambda_0 = -1.086$.

By (H3) ,

$M_i = \lim_{x\to 0} \sup \frac{x}{f_i(x)}$, $0 < M_i < \infty$.

$M_1 = \lim_{x\to 0} \sup \frac{x}{f_1(x)} = \lim_{x\to 0} \sup \frac{x}{x\left(|x(\tau_1(t))| + \frac{1}{e}\right)} = 2.71,$

$$M_2 = \limsup_{x\to 0} \frac{x}{f_2(x)} = \limsup_{x\to 0} \frac{x}{x\,(\,|x(\tau_2(t))| + \frac{2}{e})} = 1.35.$$

Also $\sum_{i=1}^{2} \frac{p_i(t)}{M_i} = \frac{P_1(t)}{M_1} + \frac{P_2(t)}{M_2} = 1.076.$

Let us prove that the solutions of (3.43) is oscillatory by showing (3.23) of Theorem 3.2 holds.

To show that ,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\left(\int_{\tau(s)}^{h(t)} \sum_{i=1}^{2} \frac{p_i(u)}{M_i} \exp\left(\int_{\tau(u)}^{u} S_1(\psi,\varepsilon)\, d\psi\right) du\right) ds > 1 - D(\alpha)$$

where

$$S_1(\psi,\varepsilon) = \sum_{i=1}^{2} \frac{p_i(t)}{M_i} \left[1 + \int_{\tau(t)}^{t} \sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\left(\int_{\tau(s)}^{t} \sum_{i=1}^{2} \frac{p_i(u)}{M_i} \exp\left(\int_{\tau(u)}^{u} S_0(\psi,\varepsilon)\, d\psi\right) du\right) ds\right.$$

and

$$S_0 = \sum_{i=1}^{2} \frac{p_i(t)}{M_i} (\lambda_0 - \varepsilon) .$$

Hence ,

$$S_0 = \left(\frac{p_1(t)}{M_1} + \frac{p_2(t)}{M_2}\right)(\lambda_0 - \varepsilon) = (1.076)(-1.086) = -1.168 ,$$

$$S_1 = 1.076 \left[1 + \int_{t-1}^{t} 1.076 \exp\left(\int_{(s-1)}^{t} 1.076 \exp\left(\int_{u-1}^{u} -1.168\, d\psi\right) du\right) ds\right.$$

$$= 37.54.$$

Therefore,

$$\limsup_{t\to\infty} \int_{h(t)}^{t} \sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\left(\int_{\tau(s)}^{h(t)} \sum_{i=1}^{2} \frac{p_i(u)}{M_i} \exp\left(\int_{\tau(u)}^{u} S_1(\psi,\varepsilon)\, d\psi\right) du\right) ds$$

$$= \limsup_{t\to\infty} \int_{t-1}^{t} \sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\left(\int_{s-1}^{t-1} \sum_{i=1}^{2} \frac{p_i(u)}{M_i} \exp\left(\int_{u-1}^{u} S_1(\psi,\varepsilon)\, d\psi\right) du\right) ds$$

$$= \limsup_{t\to\infty} \int_{t-1}^{t} 1.076 \exp\left(\int_{s-1}^{t-1} 1.076 \exp\left(\int_{u-1}^{u} 37.54\, d\psi\right) du\right) ds$$

$$= 1.076 .$$

Also ,

as $\alpha = 1.92 > \frac{1}{e}$, by (1.6),

$D(\alpha) = 0.$

$1 - D(\alpha) = 1 - 0 = 1.$

Thus proving

$$\lim_{t \to \infty} \sup \int_{h(t)}^{t} \sum_{i=1}^{2} \frac{p_i(s)}{M_i} \exp\left(\int_{\tau(s)}^{h(t)} \sum_{i=1}^{2} \frac{p_i(u)}{M_i} \exp\left(\int_{\tau(u)}^{u} S_1(\psi, \varepsilon)\, d\psi\right) du\right) ds = 1.076$$

$$> 1 - D(\alpha)$$

$$= 1$$

Thus proving the inequality (3.22). Hence the solutions of (3.43) are oscillatory.

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