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Existence of Solutions for the Initial Value Problem of Variable Fractional Order Functional Differential Equations with Infinite Delay

R. Rama, P. Selvasankari

Quaid - E - Millath Govt College for Women, Chennai, India.

In this paper, we discuss the existence of solutions for the Initial Value Problem of Variable Fractional Order Functional Differential Equations with Infinite Delay and Neutral variable order fractional differential equations with Infinite Delay by using Banach's fixed point theorem and the non-linear alternative of Leray-Schauder type.

Keywords: Variable order fractional derivatives; functional differential equations; Existence of fixed points.

1. Introduction

This paper induces the existence of solutions for the Initial Value Problem (IVP) of variable fractional order functional differential equations with infinite delay of the form

$$D^{\alpha(t)}u(t) = f(t, u_t), \text{ for each } t \in A = (0,1], \; 0 < \alpha(t) < 1 \tag{1.1}$$

$$u(t) = \varphi(t), t \in (-\infty, 0] \tag{1.2}$$

where $D^{\alpha(t)}$ is the standard Riemann-Liouville's variable order fractional derivative,

$f: A \times B \to \mathbb{R}$ is a given function, $\varphi \in B, f(0, \varphi) = 0$ that is $\varphi(0) = 0$ and B is called a Phase space [4], which is defined in section 2.

Also, for any function $u(t)$ defined on $(-\infty, 1]$ and any $t \in A$, we denote the element of B as u_t, which is defined as

$$u_t(\tau) = u(t + \tau), \tau \in (-\infty, 0] \tag{1.3}$$

in which $u_t(\tau)$ represents the history of the state from the time $-\infty$ to the present time t.

In section 4, we discuss the existence of variable order fractional neutral functional differential equations with infinite delay of the form

$$D^{\alpha(t)}[u(t) - g(t, u_t)] = f(t, u_t), \text{ for each } t \in A = (0,1], \ 0 < \alpha(t) < 1 \tag{1.4}$$

$$u(t) = \varphi(t), t \in (-\infty, 0] \tag{1.5}$$

where f and φ are defined as in (1.1) and (1.2), and $g: A \times B \to \mathbb{R}$ is a given function,

$\varphi \in B, g(0, \varphi) = 0$ that is $\varphi(0) = 0$.

Here, the role of the Phase space B, plays an important role, which is a seminormed space satisfying some axioms, which was introduced by J. Hale, J. Kato [4]. For the detailed explanations on this topic we can refer the book [5]. Many authors discussed about the theory of fractional order functional differential equations [9-13]. We can refer [1,2,7,8,14] for the existence theory of fractional differential equations with infinite delay.

Our approach towards the existence of the solutions for (1.1), (1.2) and (1.4), (1.5) is based on the Banach's fixed point theorem and on the non-linear alternative of Leray-Schauder type [3].

2. Preliminaries

We introduce the preliminary definitions, facts and notations, which we are going to use in this paper.

We consider $\mathbb{R}^+ = \{u \in \mathbb{R}: u > 0\}, C^0(\mathbb{R}^+)$ the space of all continuous functions on $\mathbb{R}^+$. Also, consider the space $C^0(\mathbb{R}^+)$ of all real functions on $\mathbb{R}_0^+ = \{u \in \mathbb{R}: u \geq 0\}$, with the class of all $f \in C^0(\mathbb{R}^+)$ such that $\lim_{t \to 0^+} f(t) = f(0^+) \in \mathbb{R}$.

We denote the Banach space of all continuous functions from A into R as $C(A, \mathbb{R})$, with the suitable complete norm on $\mathbb{R}$ as $\|y\|_\infty = \sup\{|y(t)|: t \in A\}$

Definition 2.1: The variable order fractional integral equation of a function $f: \mathbb{R}^+ \to \mathbb{R}$, is defined by

$$I_0^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} f(s) ds, \ 0 < \alpha(t) \tag{2.1}$$

Note that, if $f \in C^0(\mathbb{R}^+)$ then $I^{\alpha(t)} f \in C^0(\mathbb{R}^+)$, and more over $I^{\alpha(t)} f(0) = 0$.

Definition 2.2: The variable order fractional derivative of a continuous function $f: \mathbb{R}^+ \to \mathbb{R}$, is defined by

$$\frac{d^{\alpha(t)} f(t)}{dt^{\alpha(t)}} = \frac{1}{\Gamma(1-\alpha(t))} \frac{d}{dt} \int_0^t (t-s)^{-\alpha(t)} f(s) ds = \frac{d}{dt} I_0^{1-\alpha(t)} f(t), \ 0 < \alpha(t) \tag{2.2}$$

In this paper, we consider the state space $(B, \|.\|_B)$, which is a semi-normed linear space of a functions that maps $(-\infty, 0] \to \mathbb{R}$, satisfying the following fundamental axioms [4].

[A]: If $u: (-\infty, 1] \to \mathbb{R}$, and $u_0 \in B$, then for every $0 \leq t \leq 1$ the following conditions hold:

(i) $u_t \in B$,

(ii) $\|u_t\|_B \leq K(t)\sup\{|u(s)|: 0 \leq s \leq t\} + M(t)\|u_0\|_B$,

(iii) $|u(t)| \leq H\|u_t\|_B$.

where $H \geq 0$ is a constant, $K: [0,1] \to [0, +\infty)$ is a continuous function,

$M: [0, +\infty) \to [0, +\infty)$ is locally bounded and K, M are independent of $u(t)$.

[A1]: For the function u in [A], u_t is a B valued continuous function on $[0,1]$.

[A2]: The space B is complete.

3. Variable fractional order functional differential equations with infinite delay

To get the solution of the IVP (1.1) and (1.2), we consider the space

$\psi = \{u: (-\infty, 1] \to \mathbb{R} \ /u_t \in B \text{for} -\infty < t \leq 0 \text{ and } u_t \text{ is continuous for } 0 < t \leq 1] \}$
(3.1)

Definition 3.1: A function $u \in \psi$ is said to be the solution of (1.1) and (1.2), if u satisfies the equation $D^{\alpha(t)}u(t) = f(t, u_t)$, for each $t \in A = (0,1]$, $0 < \alpha(t) < 1$ and $u(t) = \varphi(t)$,

$t \in (-\infty, 0]$. i.e., if u satisfies (1.1) and (1.2).

We state the following Lemma to derive the existence results of (1.1) and (1.2).

Lemma 3.2: [13] Let $0 < \alpha(t) < 1$ and $f: (0,1] \to \mathbb{R}$ be continuous and

$\lim_{t \to 0^+} f(t) = f(0^+) \in \mathbb{R}$. Then, $u(t)$ is the solution of the fractional order integral equation

$I_0^{\alpha(t)} f(t) = \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} f(s) ds, \ \alpha(t) > 0$

if and only if $u(t)$ is the solution of the initial value problem for the fractional differential equation

$$\left. \begin{aligned} D^{\alpha(t)}u(t) &= f(t), 0 < t \leq 1 \\ u(0) &= 0 \end{aligned} \right] \tag{3.2}$$

Now, we give our first existence result of the IVP (1.1) and (1.2) based on the Banach's contraction principle.

Theorem 3.3: Let $f: A \times B \to \mathbb{R}$, and assume the hypothesis

[H] there exists some $l > 0$ such that

$$|f(t, u) - f(t, v)| \leq l\|u - v\|_B \text{ for } t \in A \text{ and for every } u, v \in B. \tag{3.3}$$

If $\frac{K_1 t^{\alpha(t)} l}{\Gamma(\alpha(t)+1)} < 1$, where $K_1 = \sup\{|K(t)|: t \in [0,1]\}$,

(3.4)

then there exists a unique solution for the initial value problem (1.1) and (1.2) in the interval $(-\infty, 1]$.

Proof: First we transform the equations (1.1) and (1.2) into a fixed point equations.

Consider the operator $\boldsymbol{N}: \boldsymbol{\psi} \to \boldsymbol{\psi}$, defined as

$$(\boldsymbol{Nu})(\boldsymbol{t}) = \begin{cases} \boldsymbol{\varphi}(\boldsymbol{t}) \ \boldsymbol{for} \ \boldsymbol{t} \in (-\infty, \mathbf{0}] \\ \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))} \int_{\mathbf{0}}^{\boldsymbol{t}} (\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\mathbf{1}} \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{u}_{\boldsymbol{s}}) \boldsymbol{ds} \ \boldsymbol{for} \ \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}] \end{cases}$$

(3.5)

Let $\boldsymbol{x}: (-\infty, \mathbf{1}] \to \mathbb{R}$ be the function defined by

$$\boldsymbol{x}(\boldsymbol{t}) = \begin{cases} \mathbf{0}, \boldsymbol{if} \ \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}] \\ \boldsymbol{\varphi}(\boldsymbol{t}) \ \boldsymbol{if} \ \boldsymbol{t} \in (-\infty, \mathbf{0}] \end{cases}$$

(3.6)

then $\boldsymbol{x}_{\mathbf{0}} = \boldsymbol{\varphi}$. For each $\boldsymbol{z} \in \boldsymbol{C}([\mathbf{0}, \mathbf{1}], \mathbb{R})$ with $\boldsymbol{z}(\mathbf{0}) = \mathbf{0}$, the function $\bar{\boldsymbol{z}}$ is defined as

$$\bar{\boldsymbol{z}}\,(\boldsymbol{t}) = \begin{cases} \boldsymbol{z}(\boldsymbol{t}), \boldsymbol{if} \ \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}] \\ \mathbf{0}, \ \boldsymbol{if} \ \boldsymbol{t} \in (-\infty, \mathbf{0}] \end{cases}$$

(3.7)

If $\boldsymbol{u}(\boldsymbol{t})$ satisfies the integral equation

$\boldsymbol{u}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))} \int_{\mathbf{0}}^{\boldsymbol{t}} (\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\mathbf{1}} \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{u}_{\boldsymbol{s}}) \boldsymbol{ds}$, we can decompose $\boldsymbol{u}(\boldsymbol{t})$ as follows

$$\boldsymbol{u}(\boldsymbol{t}) = \bar{\boldsymbol{z}}\,(\boldsymbol{t}) + \boldsymbol{x}(\boldsymbol{t}) = \boldsymbol{z}(\boldsymbol{t}) + \mathbf{0} = \boldsymbol{z}(\boldsymbol{t}) \ \boldsymbol{for} \ \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}] \quad (3.8)$$

which implies $\boldsymbol{u}_{\boldsymbol{t}} = \bar{\boldsymbol{z}}_{\boldsymbol{t}} + \boldsymbol{x}_{\boldsymbol{t}} \ \boldsymbol{for} \ \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}]$

(3.9)

Therefore, by (3.8) $\boldsymbol{z}(\boldsymbol{t})$ satisfies the equation

$$\boldsymbol{z}(\boldsymbol{t}) = \frac{\mathbf{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))} \int_{\mathbf{0}}^{\boldsymbol{t}} (\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\mathbf{1}} \boldsymbol{f}(\boldsymbol{s}, \boldsymbol{z}_{\boldsymbol{s}}) \boldsymbol{ds}.$$

Set $\boldsymbol{C}_{\mathbf{0}} = \{\boldsymbol{z} \in \boldsymbol{C}([\mathbf{0}, \mathbf{1}], \mathbb{R}) : \ \boldsymbol{z}_{\mathbf{0}} = \mathbf{0}\}$.

(3.10)

Let $\|\boldsymbol{z}\|_{\mathbf{1}} = \boldsymbol{sup}\{|\boldsymbol{z}(\boldsymbol{t})| : \boldsymbol{t} \in (\mathbf{0}, \mathbf{1}]\}, \boldsymbol{z} \in \boldsymbol{C}_{\mathbf{0}}$.

(3.11)

$\boldsymbol{C}_{\mathbf{0}}$ is the Banach space with norm $\|.\|_{\mathbf{1}}$

Let the operator $\boldsymbol{F}: \boldsymbol{C}_{\mathbf{0}} \to \boldsymbol{C}_{\mathbf{0}}$ be defined as

$$(Fz)(t) = \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s, \bar{z}_s + x_s)ds,\ t \in (0,1] \tag{3.12}$$

The operator N has a fixed point is equivalent to the fixed point in F. Hence; it is enough to prove that F has a fixed point. We will show that $F: C_0 \to C_0$ is a contraction mapping. Then, for each $t \in (0,1]$ and $z_1, z_2 \in C_0$

$$|Fz_1(t) - Fz_2(t)| \leq \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} |f(s, \bar{z}_{1s} + x_s) - f(s, \bar{z}_{2s} + x_s)| ds$$

$$\leq \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} l \|\bar{z}_{1s} - \bar{z}_{2s}\|_B ds$$

$$\leq \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} l \sup_{s\in[0,t]} \|z_1(s) - z_2(s)\| ds$$

$$\leq \frac{K_1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} l \|z_1 - z_2\|_1 ds = \frac{lK_1 t^{\alpha(t)} \|z_1 - z_2\|_1}{\Gamma(\alpha(t)+1)}$$

Therefore,

$$\|Fz_1 - Fz_2\|_1 \leq \frac{lK_1 t^{\alpha(t)} \|z_1 - z_2\|_1}{\Gamma(\alpha(t)+1)},$$

and hence F is a contraction mapping. Therefore, by Banach's contraction principle, F has a unique fixed point.

Next, we discuss an existence result for (1.1) and (1.2) based on the non-linear alternative of Leray-Schauder type. For this, we state the generalization of Gronwall's lemma for singular kernel [6], which is an essential tool to prove the main result of this section.

Lemma 3.4: Let $k: (0,1] \to [0, +\infty)$ be a real function and $r(t)$ is a non negative and locally integrable function on $[0,1]$, and there are constants $a > 0, 0 < \alpha(t) < 1$ such that

$$k(t) \leq r(t) + a\int_0^t \frac{k(s)}{(t-s)^{\alpha(t)}} ds, \tag{3.13}$$

then, there exists a constant $K = K(\alpha(t))$ such that

$$k(t) \leq r(t) + Ka\int_0^t \frac{r(s)}{(t-s)^{\alpha(t)}} ds \tag{3.14}$$

for every $t \in (0,1]$.

Theorem 3.5: Assume that the following hypothesis holds.

H1: f is a continuous function,

H2: there exists $p, q \in C(A, \mathbb{R}^+)$ such that

$$|f(t,u)| \leq p(t) + q(t)\|u\|_B \tag{3.15}$$

for each $\boldsymbol{u} \in \boldsymbol{B}$ and $\boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1}]$ and $\left\|\boldsymbol{I}^{\boldsymbol{\alpha}(\boldsymbol{t})}\right\|_{\infty} < \infty$.

Then the initial value problem (1.1) and (1.2) has atleast one solution on $(-\infty, \boldsymbol{1}]$.

Proof: Let the operator $\boldsymbol{F}: \boldsymbol{C_0} \to \boldsymbol{C_0}$ be defined as in (3.12). In this theorem, in step 1-3, we prove that $\boldsymbol{F}$ is continuous and completely continuous, and in step 4 we prove that there exists a fixed point for $\boldsymbol{F}$ as a consequence of the non-linear alternative of Leray-Schauder type [3].

Step 1: Here we prove that $\boldsymbol{F}$ is continuous

Let $\{\boldsymbol{z_n}\}$ be the sequence such that in $\boldsymbol{C_0}$. Then,

$$|(\boldsymbol{F}\boldsymbol{z_n})(\boldsymbol{t}) - (\boldsymbol{F}\boldsymbol{z})(\boldsymbol{t})\ | \leq \frac{\boldsymbol{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))}\int_{\boldsymbol{0}}^{\boldsymbol{t}}(\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\boldsymbol{1}}|\boldsymbol{f}(\boldsymbol{s}, \bar{\boldsymbol{z}}_{\boldsymbol{ns}} + \boldsymbol{x_s}) - \boldsymbol{f}(\boldsymbol{s}, \bar{\boldsymbol{z}}_{\boldsymbol{s}} + \boldsymbol{x_s})|\boldsymbol{ds}$$

since $\boldsymbol{f}$ is continuous (by H1), we have

$$\|\boldsymbol{F}(\boldsymbol{z_n}) - \boldsymbol{F}(\boldsymbol{z})\|_{\boldsymbol{1}} \leq \frac{\boldsymbol{t}^{\boldsymbol{\alpha}(\boldsymbol{t})}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t})+\boldsymbol{1})}\|\boldsymbol{f}(\boldsymbol{s}, \bar{\boldsymbol{z}}_{\boldsymbol{ns}} + \boldsymbol{x_s}) - \boldsymbol{f}(\boldsymbol{s}, \bar{\boldsymbol{z}}_{\boldsymbol{s}} + \boldsymbol{x_s})\|_{\infty} \to \boldsymbol{0}\ \boldsymbol{as}\ \boldsymbol{n} \to \infty$$

Therefore, $\boldsymbol{F}$ is continuous.

Step 2: In this, we prove that $\boldsymbol{F}$ maps bounded sets into bounded sets in $\boldsymbol{C_0}$.

For this, it is enough to prove that for any $\boldsymbol{\delta} > \boldsymbol{0}$, there exists a positive constant $\boldsymbol{m} > \boldsymbol{0}$ such that for each $\boldsymbol{z} \in \boldsymbol{B_\delta} = \{\boldsymbol{z} \in \boldsymbol{C_0} :\ \|\boldsymbol{z}\|_{\boldsymbol{\delta}} \leq \boldsymbol{\delta}\}\ \boldsymbol{and}\ \|\boldsymbol{F}(\boldsymbol{z})\|_{\infty} \leq \boldsymbol{m}$. Let $\boldsymbol{z} \in \boldsymbol{B_\delta}$. Since, $\boldsymbol{f}$ is a continuous function, for every $\boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1}]$,

$$|(\boldsymbol{F}\boldsymbol{z})(\boldsymbol{t})| \leq \frac{\boldsymbol{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))}\int_{\boldsymbol{0}}^{\boldsymbol{t}}(\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\boldsymbol{1}}|\boldsymbol{f}(\boldsymbol{s}, \bar{\boldsymbol{z}}_{\boldsymbol{s}} + \boldsymbol{x_s})|\boldsymbol{ds}$$

$$\leq \frac{\boldsymbol{1}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t}))}\int_{\boldsymbol{0}}^{\boldsymbol{t}}(\boldsymbol{t}-\boldsymbol{s})^{\boldsymbol{\alpha}(\boldsymbol{t})-\boldsymbol{1}}[\boldsymbol{p}(\boldsymbol{s}) + \boldsymbol{q}(\boldsymbol{s})\|\bar{\boldsymbol{z}}_{\boldsymbol{s}} + \boldsymbol{x_s}\|_{\boldsymbol{B}}]\boldsymbol{ds}$$

$$\leq \frac{\boldsymbol{t}^{\boldsymbol{\alpha}(\boldsymbol{t})}\|\boldsymbol{p}\|_{\infty}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t})+\boldsymbol{1})} + \frac{\boldsymbol{t}^{\boldsymbol{\alpha}(\boldsymbol{t})}\|\boldsymbol{q}\|_{\infty}}{\boldsymbol{\Gamma}(\boldsymbol{\alpha}(\boldsymbol{t})+\boldsymbol{1})}\boldsymbol{\delta}^{*} = \boldsymbol{m}\ (\boldsymbol{say})$$

where

$$\|\bar{\boldsymbol{z}}_{\boldsymbol{s}} + \boldsymbol{x_s}\|_{\boldsymbol{B}} \leq \|\bar{\boldsymbol{z}}_{\boldsymbol{s}}\|_{\boldsymbol{B}} + \|\boldsymbol{x_s}\|_{\boldsymbol{B}} \leq \boldsymbol{K_1}\boldsymbol{\delta} + \boldsymbol{M_1}\|\boldsymbol{\varphi}\|_{\boldsymbol{B}} = \boldsymbol{\delta}^{*}\ (\boldsymbol{say})$$

and

$$\boldsymbol{M_1} = \boldsymbol{sup}\{|\boldsymbol{M}(\boldsymbol{t})| :\ \boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1}]\}.$$

Hence, $\|\boldsymbol{F}(\boldsymbol{z})\|_{\infty} \leq \boldsymbol{m}$.

Therefore, $\boldsymbol{F}$ maps bounded sets into bounded sets in $\boldsymbol{C_0}$.

Step 3: Here, we prove that $\boldsymbol{F}$ maps bounded sets into equicontinuous sets in $\boldsymbol{C_0}$.

To prove this, consider $\boldsymbol{t_1}, \boldsymbol{t_2} \in (\boldsymbol{0}, \boldsymbol{1}],\ \boldsymbol{t_1} < \boldsymbol{t_2}$.

Let $\boldsymbol{B_\delta} = \{\boldsymbol{z} \in \boldsymbol{C_0} :\ \|\boldsymbol{z}\|_{\boldsymbol{\delta}} \leq \boldsymbol{\delta}\}$ defined in step be a bounded set.

Let $\boldsymbol{z} \in \boldsymbol{B_\delta}$, for every $\boldsymbol{t} \in (\boldsymbol{0}, \boldsymbol{1}]$, we have

$$|(\boldsymbol{F}\boldsymbol{z})(\boldsymbol{t_2}) - (\boldsymbol{F}\boldsymbol{z})(\boldsymbol{t_1})|$$

$$\leq \left|\frac{1}{\Gamma(\alpha(t))}\int_0^{t_2}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds \ -\frac{1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_1-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds\right|$$

$$\leq \left|\frac{-1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_1-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds - \frac{1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_0^{t_2}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds\right|$$

$$\leq \left|\frac{-1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_1-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_0^{t_1}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_{t_1}^{0}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_0^{t_2}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds\right|$$

$$\leq \left|\frac{1}{\Gamma(\alpha(t))}\int_0^{t_1}\left[(t_2-s)^{\alpha(t)-1}-(t_1-s)^{\alpha(t)-1}\right]f(s,\bar{z}_s+x_s)ds + \frac{1}{\Gamma(\alpha(t))}\int_{t_2}^{t_2}(t_2-s)^{\alpha(t)-1}f(s,\bar{z}_s+x_s)ds\right|$$

$$\leq \frac{\|p\|_\infty+\|q\|_\infty\delta^*}{\Gamma(\alpha(t)+1)}\left[(t_2-t_1)^{\alpha(t)}+{t_1}^{\alpha(t)}-{t_2}^{\alpha(t)}\right]+\frac{\|p\|_\infty+\|q\|_\infty\delta^*}{\Gamma(\alpha(t)+1)}\left[(t_2-t_1)^{\alpha(t)}\right]$$

$$\leq \frac{2(\|p\|_\infty+\|q\|_\infty\delta^*)}{\Gamma(\alpha(t)+1)}(t_2-t_1)^{\alpha(t)}$$

which $\to 0$ as $t_1 \to t_2$.

Therefore, F maps bounded sets into equicontinuous sets in C_0

Hence, from step 1-3 and by Arzela-Ascoli's theorem, we can conclude that F is continuous and completely continuous.

Step 4: Here we will show that there exists an open set $V \subseteq C_0$ with $z \neq \lambda F(z)$ for $0 < \lambda < 1$ and $z \in \partial V$.

Let $z \in C_0$ and $z = \lambda F(z)$ for $0 < \lambda < 1$.

Then for every $0 < t \leq 1$, we have

$$z(t) = \lambda\left[\frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s, \bar{z}_s + x_s) ds\right].$$

By the hypothesis H2

$$|z(t)| \le \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1}[p(s) + q(s)\|\bar{z}_s + x_s\|_B] ds$$

$$\le \frac{t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} q(s)\|\bar{z}_s + x_s\|_B ds \tag{3.16}$$

But,

$$\|\bar{z}_s + x_s\|_B \le \|\bar{z}_s\|_B + \|x_s\|_B$$

$$\le K(t) sup\{|z(s): s \in [0,t]|\} + M(t)\|z_0\|_B + K(t) sup\{|x(s): s \in [0,t]|\} + M(t)\|x_0\|_B$$

$$\le K_1 sup\{|z(s): s \in [0,t]|\} + M_1\|\varphi\|_B = r(t) \ \ (say) \tag{3.17}$$

Therefore, (3.16) becomes

$$|z(t)| \le \frac{t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} q(s) r(s) ds \tag{3.18}$$

By (3.18) and by the choice of $r(t)$ we have,

$$r(t) \le M_1\|\varphi\|_B + \frac{K_1 t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{K_1\|q\|_\infty}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} r(s) ds.$$

By lemma 3.4, there exists $K = K(\alpha(t))$ such that

$$|r(t)| \le M_1\|\varphi\|_B + \frac{K_1 t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{KK_1\|q\|_\infty}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1}\, R ds$$

where, $P = M_1\|\varphi\|_B + \frac{K_1 t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)}$.

Hence, $\|r\|_\infty \le P + \frac{PKK_1\|q\|_\infty}{\Gamma(\alpha(t)+1)} = D \ (say)$

Then (3.18) becomes $\|z\|_\infty \le D\left\|I^{\alpha(t)} q\right\|_\infty + \frac{t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} = D^* \ (say)$

Set, $V = \{z \in C_0 : \|z\| < D^* + 1\}$.

Also, $F: \bar{V} \to C_0$ is continuous and completely continuous. By the choice of V, there is no $z \in \partial V$ such that $z = \lambda F(z)$ for $0 < \lambda < 1$.

Therefore, by non-linear alternative of Leray-Schauder type [3], we can conclude that F has a fixed point $z \in V$.

4. Variable order neutral fractional functional differential equations

In this section, we discuss the existence results for the initial value problems (1.4) and (1.5).

Definition 4.1: A function $\boldsymbol{u} \in \boldsymbol{\psi}$ is said to be solution of (1.4) and (1.5) if it satisfies $D^{\alpha(t)}[u(t) - g(t, u_t)] = f(t, u_t),\ for\ each\ t \in A = (0,1],\ 0 < \alpha(t) < 1$ and $u(t) = \varphi(t)$,

$t \in (-\infty, 0]$. $i.e.$, u satisfies (1.4) and (1.5).

In this section also, we provide the first existence result for the IVP (1.4) and (1.5) by means of Banach's contraction principle.

Theorem 4.2: Assume that the hypothesis [H] in theorem 3.3 holds and the assumption

[A] there exists a positive constant $c_1 > 0$, such that

$$|g(t,u) - g(t,v)| \leq c_1 \|u - v\|_B\ , for\ every\ u, v \in B \tag{4.1}$$

If $K_1 \left[c_1 + \frac{t^{\alpha(t)} l}{\Gamma(\alpha(t)+1)}\right] < 1$, then there exists a unique solution for (1.4) and (1.5) on $(-\infty, 1]$.

Proof: Consider the operator $N_1: \psi \to \psi$, defined as

$$(N_1 u)(t) = \begin{cases} \varphi(t)\ for\ t \in (-\infty, 0] \\ g(t, u_t) + \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} f(s, u_s) ds\ for\ t \in (0,1] \end{cases} \tag{4.2}$$

In view of theorem 3.5, we consider the operator $F_1: C_0 \to C_0$ be defined as

$$(F_1 z)(t) = \begin{cases} 0\ for\ t \in (-\infty, 0] \\ g(t, \overline{z_t} + x_t) + \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} f(s, \overline{z_s} + x_s) ds\ for\ t \in (0,1] \end{cases} \tag{4.3}$$

To prove that the operator F_1 is a contraction, let $z_1, z_2 \in \psi$.

As discussed in theorem 3.3, we have

$$|F_1 z_1(t) - F_1 z_2(t)|$$

$$\leq |g(s, \bar{z}_{1t} + x_t) - g(s, \bar{z}_{2t} + x_t)| + \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} |f(s, \bar{z}_{1s} + x_s) - f(s, \bar{z}_{2s} + x_s)| ds,$$

$$\leq c_1 \|\bar{z}_{1t} - \bar{z}_{2t}\|_B + \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} l \|\bar{z}_{1s} - \bar{z}_{2s}\|_B ds,$$

$$\leq c_1 K_1 sup\{|z_1(s) - z_2(s)| : s \in [0,t]\} + \frac{1}{\Gamma(\alpha(t))} \int_0^t (t-s)^{\alpha(t)-1} K_1 l sup\{|z_1(s) - z_2(s)| : s \in [0,t]\} ds.$$

$$\leq \boldsymbol{K_1}\left[\boldsymbol{c_1}+\frac{\boldsymbol{l t^{\alpha(t)}}}{\boldsymbol{\Gamma(\alpha(t)+1)}}\right]\|\bar{\boldsymbol{z}}_{\mathbf{1}}-\bar{\boldsymbol{z}}_{\mathbf{2}}\|.$$

which show that $\boldsymbol{F_1}$ is a contraction.

Therefore, by Banach's fixed point principle $\boldsymbol{F_1}$ has a fixed point.

Next, we provide the existence result for the initial value problem (1.4) and (1.5) based on the non-linear alternative of Leray-Schauder.

Theorem 4.3: Assume that the conditions H1, H2 of theorem 3.5 holds along with

H3: the function $\boldsymbol{g}$ is continuous and completely continuous, and for any bounded set B in $\boldsymbol{\psi}$, the set $\{\boldsymbol{t} \to \boldsymbol{g}(\boldsymbol{t}, \boldsymbol{u_t}): \boldsymbol{u} \in \boldsymbol{B}\}$ is equicontinuous in $\boldsymbol{C}((\mathbf{0}, \mathbf{1}], \mathbb{R})$, and there exist constants

$\mathbf{0} \leq \mathbf{K_1 c_2} < \mathbf{1}, \mathbf{c_3} \geq \mathbf{0}$ such that

$|\mathbf{g}(\mathbf{t}, \mathbf{y})| \leq \mathbf{c_2}\|\mathbf{y}\|_{\mathbf{B}} + \mathbf{c_3}, \mathbf{t} \in (\mathbf{0}, \mathbf{1}]$ **and** $\mathbf{y} \in \mathbf{B}$.

Then, the initial value problem (1.4), (1.5) has atleast one solution in $(-\infty, 1]$.

Proof: Let $F_1: C_0 \to C_0$ is the function defined as in theorem 4.2. First we prove that the operator F_1 is continuous and equicontinuous. By H3, it is enough to prove that operator $F_2: C_0 \to C_0$ defined by

$$(F_2 z)(t) = g(t, \overline{z_t} + x_t) + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s, \overline{z_s} + x_s)ds \text{ for } t \in (0,1]$$

is continuous and completely continuous, which was already proved in theorem 3.5.

Hence, we show that there exists an open set $V \subseteq C_0$ with $z \neq \lambda F_1(z)$ for $0 < \lambda < 1$ and $z \in \partial V$.

Let $z \in C_0$ and $z = \lambda F_2(z)$ for $0 < \lambda < 1$. Then, for every $0 < t \leq 1$, we have

$$z(t) = \lambda\left[g(t, \overline{z_t} + x_t) + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s, \overline{z_s} + x_s)ds\right] \text{ for } t \in (0,1]$$

and

$$|z(t)| \leq c_2\|\overline{z_t} + x_t\|_B + d_2 + \frac{t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{1}{\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} f(s, \overline{z_s} + x_s)ds,$$

Thus,

$$r(t) \leq \frac{1}{1-K_1c_2}\left[2K_1c_3 + \frac{K_1 t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + \frac{K_1\|q\|_\infty}{(1-K_1c_2)\Gamma(\alpha(t))}\int_0^t (t-s)^{\alpha(t)-1} r(s)ds\right]$$

and so

$$\|r\|_\infty \leq P_1 + \frac{P_1 K_1 t^{\alpha(t)}\|q^*\|_\infty}{(1-K_1c_2)\Gamma(\alpha(t))} = D_1 \text{ (say)}$$

where $\|q^*\|_\infty = \frac{\|q\|_\infty}{(1-K_1c_2)}$ and $P_1 = \frac{1}{1-K_1c_2}\left[2K_1c_3 + \frac{K_1 t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)}\right]$.

Then,

$\|z\|_\infty \le c_2\|\varphi\|_B + 2c_3 + D_1c_2 + \frac{t^{\alpha(t)}\|p\|_\infty}{\Gamma(\alpha(t)+1)} + L\left\|I^{\alpha(t)}q\right\|_\infty = D_1{}^*$ (say)

Set, $V_1 = \{y \in C_0: \|y\| < D_1{}^* + 1\}$

By the choice of V_1, there is no $z \in \partial V_1$ such that $z = \lambda F_2(z)$ for $0 < \lambda < 1$.

Therefore, by non-linear alternative of Leray-Schauder type [3], we can conclude that F_2 has a fixed point $z \in V_1$.

Hence, F_1 has a fixed point which is the solution of (1.4) and (1.5).

5. Example

Consider the variable order fractional functional differential equation

$$D^{\alpha(t)}u(t) = \frac{ke^{-\delta t+t}\|u_t\|}{(e^t+e^{-t})(1+\|u_t\|)}, t \in (0,1] \text{ and } 0 < \alpha(t) < 1 \quad (5.1)$$

$$u(t) = \varphi(t), t \in (-\infty, 0]. \quad (5.2)$$

where $\alpha(t) = \begin{cases} 0.75, t \in (0,1] \\ 0.5, \ t \in (-\infty, 0] \end{cases}$ (5.3)

$k\Gamma(\alpha(t)) = k_0 \int_0^1 s^{\alpha(t)-1} e^{-s} ds, k_0 > 1$ is a fixed constant.

Let δ be a positive and real constant and

$B_\delta = \left\{u \in C((-\infty, 0], \mathbb{R}): \lim_{\vartheta \to -\infty} e^{\delta\vartheta}u(\vartheta), \text{exists in } \mathbb{R}\right\}.$

The norm of $B_\delta$is given by

$\|u\|_\delta = \sup_{-\infty<\vartheta\le 0} e^{\delta\vartheta}|u(\vartheta)|.$

Let $u: (-\infty, 1] \to \mathbb{R}$ such that $u_0 \in B_\delta$. Then,

$\lim_{\vartheta\to-\infty} e^{\delta\vartheta}u_t(\vartheta) = \lim_{\vartheta\to-\infty} e^{\delta\vartheta}u(t+\vartheta) = \lim_{\vartheta\to-\infty} e^{\delta(\vartheta-t)}u(\vartheta) = e^{-\delta t} \lim_{\vartheta\to-\infty} e^{\delta\vartheta}u_0(\vartheta) < \infty.$

Hence, $u_t \in B_\delta$.

Finally, we prove that $\|u_t\|_\delta \le K(t)\sup\{|u(s)|: 0 \le s \le t\} + M(t)\|u_0\|_\delta$,

where $K = M = 1$ and $H = 1$. We have $|u_t(\vartheta)| = |u(t+\vartheta)|$.

If $t + \vartheta \le 0$, we get

$|u_t(\vartheta)| \le \sup\{|u(s)|: -\infty < s \le 0\}.$

If $t + \vartheta > 0$, we get

$|u_t(\vartheta)| \leq \sup\{|u(s)|: 0 < s \leq t\}.$

Thus, for all $t + \vartheta \in (0,1]$, we get

$|u_t(\vartheta)| \leq \sup\{|u(s)|: -\infty < s \leq 0\} + \sup\{|u(s)|: 0 < s \leq t\}.$

Then, $\|u_t\|_\delta \leq \|u_0\|_\delta + \sup\{|u(s)|: 0 < s \leq t\}.$

It is obvious that $(B_\delta, \|u_t\|_\delta)$ is a Banach space. Hence, we can conclude that B_δ is a Phase space.

Set $f(t,x) = \frac{e^{-\delta t+t}x}{k(e^t+e^{-t})(1+x)}$, $(t,x) \in (0,1] \times B_\delta$.

Let $x_1, x_2 \in B_\delta$. Then, we get

$$|f(t,x_1) - f(t,x_2)| \leq \frac{e^{-\delta t+t}}{k(e^t + e^{-t})}\left|\frac{x_1}{1+x_1} - \frac{x_2}{1+x_2}\right|$$

$$= \frac{e^{-\delta t+t}}{k(e^t + e^{-t})}\left|\frac{x_1 - x_2}{(1+x_1)(1+x_2)}\right|$$

$$\leq \frac{e^{-\delta t}e^t}{k(e^t + e^{-t})}\frac{|x_1 - x_2|}{(1+x_1)(1+x_2)}$$

$$\leq \frac{e^t\|x_1 - x_2\|_{B_\delta}}{k(e^t + e^{-t})}$$

$$\leq \frac{\|x_1 - x_2\|_{B_\delta}}{k(e^t + e^{-t})}$$

$$\leq \frac{\|x_1 - x_2\|_{B_\delta}}{k}$$

Thus, the condition [H] of theorem 3.3 holds. Since, $K = 1$

$$\frac{t^{\alpha(t)}}{k\Gamma(\alpha(t)+1)} < 1.$$

Hence, by Theorem 3.3 there exists a unique solution for the equations (5.1), (5.2) on $(0,1]$.

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