

On the Partial Square Sum Process in Stochastic Process

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In this article we introduced partial square sum processes. Let $\{X_n, n = 1, 2, \dots\}$ be a sequence of independent and non-negative random variables and let $G(x)$ be the distribution function of X . Then $\{X_n, n = 1, 2, \dots\}$ is called a partial square sum process, if the distribution function of X_{n+1} is $G(\beta_n^2 x)$ ($n = 1, 2, \dots$) where $\beta_n^2 > 0$ are constants and $\beta_n^2 = \beta_1^2 + \beta_2^2 + \dots + \beta_{n-1}^2$. We study some properties of partial square sum process.

Keywords: geometric process, partial product process, partial square sum process.

1. Introduction

A stochastic process $\{X_n, n = 1, 2, \dots\}$ is said to be a Geometric process, if there exists a real $\beta_0 > 0$ such that $\{\beta_0^{n-1} X_n, n = 1, 2, \dots\}$ forms a renewal process. The positive number β_0 is called the ratio of the G.P. A Geometric process is stochastically increasing if the ratio $0 < \beta_0 \leq 1$. It is stochastically decreasing if the ratio $\beta_0 > 1$. A Geometric process will become a renewal process if the ratio $\beta_0 = 1$. Therefore Geometric process is a simple monotone process and is a generalization of the renewal process. Assume that $\{X_n, n = 1, 2, \dots\}$ is a Geometric process with ratio β_0 . Let the Distribution function and density function of X_1 be G and g respectively and denote $E(X_1) = \lambda$ and $\text{Var}(X_1) = \sigma^2$. Then $E(X_n) = \frac{\lambda}{\beta_0^{2^{n-1}}}$ and $\text{Var}(X_n) = \frac{\sigma^2}{\beta_0^{2^n}}$. Thus β_0, λ and σ^2 are three important parameters of a Geometric process. Let $\{X_n, n = 1, 2, \dots\}$ be a sequence of non-negative independent random variables and let $G(x)$ be the distribution function of X_1 . Then $\{X_n, n = 1, 2, \dots\}$ is called a partial product process, if

the distribution function of X_{i+1} is $G(\beta_k x)$ ($k = 1, 2, \dots$) where $\beta_k > 0$ are constants and $\beta_k = \beta_0 \beta_1 \beta_2 \dots \beta_{k-1}$.

2. Partial square sum process

Let $\{X_n, n = 1, 2, \dots\}$ be a sequence of independent and non-negative random variables and let $G(x)$ be the distribution function of X . Then $\{X_n, n = 1, 2, \dots\}$ is called a partial square sum process, if the distribution function of X_{n+1} is $G(\beta_n^2 x)$ ($n = 1, 2, \dots$) where $\beta_n^2 > 0$ are constants and $\beta_n^2 = \beta_1^2 + \beta_2^2 + \dots + \beta_{n-1}^2$.

Lemma 2.1. $\beta_n^2 = \beta_0^2 + \beta_1^2 + \dots + \beta_{n-1}^2$ show that $\beta_n^2 = 2^{n-1} \beta_0^2$

Proof. When $n = 1$, $\beta_1^2 = \beta_0^2$. Thus, the result is true for $n = 1$.

Assume that the result is true for $n = k$.

$$\beta_k^2 = 2^{k-1} \beta_0^2 \dots (1)$$

Then we have to prove that the result is true for $n = k + 1$.

$$\begin{aligned} \beta_{k+1}^2 &= \beta_0^2 + \beta_1^2 + \dots + \beta_{k-1}^2 + \beta_k^2 \\ &= \beta_k^2 + \beta_k^2 \text{ [From (1)]} \\ &= 2 \beta_k^2 \\ &= 2 \times 2^{k-1} \beta_0^2 \\ \beta_{k+1}^2 &= 2^{k+1-1} \beta_0^2 = 2^k \beta_0^2 \end{aligned}$$

Thus, the result is true for $n = k + 1$ also.

$$\beta_n^2 = 2^{n-1} \beta_0^2; n = 1, 2, 3, \dots$$

Remark 2.2. By Lemma 2.1, the distribution function of X_{k+1} is $G(2^{n-1} \beta_0^2 x)$ for $n = 1, 2, \dots$ the term β_0^2 as the ratio of the partial square sum process.

Lemma 2.3. The partial square sum process $\{X_n, n = 1, 2, \dots\}$ is

- (i) Stochastically decreasing if $\beta_0^2 > 1$
 (ii) Stochastically increasing if $0 < \beta_0^2 < 1$.

Proof. Let $\beta_0^2 > 1$. Note that for any $\gamma \geq 0$

$$G(\gamma) \leq G(2^2 \beta_0 \gamma) \leq G((2^2)^2 \beta_0 \gamma) \leq \dots \leq G((2^2)^{n-1} \beta_0 \gamma)$$

This implies

$$P(X_1 \geq \gamma) \geq P(X_2 \geq \gamma) \geq P(X_3 \geq \gamma) \dots \geq P(X_n \geq \gamma)$$

This implies that $\{X_n, n = 1, 2, \dots\}$ is Stochastically decreasing if $\beta_0^2 > 1$. Similarly we can prove that it is Stochastically increasing if $0 < \beta_0^2 < 1$.

Lemma 2.4. Let $E(X_1) = \mu$, $\text{Var}(X_1) = \sigma^2$. Then for $n = 1, 2, \dots$

$$E(X_{n+1}) = \frac{\mu}{2^{n-1} \beta_0^2}, \text{Var}(X_{n+1}) = \frac{\sigma^2}{2^{2(n-1)} \beta_0^4}$$

Proof. By Lemma 2.3, for $n = 1, 2, 3, \dots$ the density function of X_{n+1} is $2^{n-1} \beta_0^2 g(2^{n-1} \beta_0^2 x)$ where g is the density function of X_1 .

$$\text{Now } E(X_{n+1}) = \int x [2^{n-1} \beta_0^2 g(2^{n-1} \beta_0^2 x)] dx$$

$$= 2^{n-1} \beta_0^2 \int x [g(2^{n-1} \beta_0^2 x)] dx$$

$$= 2^{n-1} \beta_0^2 \int \left(\frac{y}{2^{n-1} \beta_0^2} \right) g(y) \frac{dy}{2^{n-1} \beta_0^2} \quad \text{where } y = 2^{n-1} \beta_0^2 x$$

$$= \frac{1}{2^{n-1} \beta_0^2} \int y g(y) dy$$

$$= \frac{1}{2^{n-1} \beta_0^2} E(X_1)$$

$$\text{Therefore } E(X_{n+1}) = \frac{\mu}{2^{n-1} \beta_0^2}$$

$$\text{Now } E(X_{n+1}^2) = \int x^2 [2^{n-1} \beta_0^2 g(2^{n-1} \beta_0^2 x)] dx$$

$$= 2^{n-1} \beta_0^2 \int x^2 g(2^{n-1} \beta_0^2 x) dx$$

$$= 2^{n-1} \beta_0^2 \int \left(\frac{y}{2^{n-1} \beta_0^2} \right)^2 g(y) \frac{dy}{2^{n-1} \beta_0^2} \quad \text{where } y = 2^{n-1} \beta_0^2 x$$

$$= \frac{1}{(2^{n-1} \beta_0^2)^2} \int y^2 g(y) dy$$

$$= \frac{1}{2^{2(n-1)} \beta_0^4} E(X_1^2)$$

$$= \frac{E(X_1^2)}{2^{2(n-1)} \beta_0^4}$$

$$\text{Var}(X_{n+1}) = E(X_{n+1}^2) - [E(X_{n+1})]^2$$

$$= \frac{E(X_1^2)}{2^{2(n-1)} \beta_0^4} - \left[\frac{E(X_1)}{2^{n-1} \beta_0^2} \right]^2 = \frac{E(X_1^2)}{2^{2(n-1)} \beta_0^4} - \frac{[E(X_1)]^2}{2^{2(n-1)} \beta_0^4}$$

$$= \frac{\sigma^2}{2^{2(n-1)} \beta_0^4}$$

3. Properties of Partial Square Sum Process

In this section we study some properties of partial square sum process. Let G and g be the distribution function and density function of X_1 respectively. and denote $E(X_1) = \mu$, and $\text{Var}(X_1) = \sigma^2$. Then by Lemma 2.4, for $i = 1, 2, \dots$ We have

$$E(X_{i+1}) = \frac{\mu}{2^{i-1}\beta_0^2}$$

$$\text{and Var}(X_{i+1}) = \frac{\sigma^2}{2^{2(i-1)}\beta_0^4}$$

Thus β_0^2, μ and σ^2 are important three parameters of partial square sum process.

Define $u_0 = 0$ and $u_n = \sum_{i=1}^n X_i$

Let $G_n = \sigma(X_1, X_2, \dots, X_n)$ be the σ -algebra generated by $\{X_i, i = 1, 2, \dots, n\}$.

Theorem 3.1. If $\beta_0^2 > 1$, the $\{U_n, n = 1, 2, \dots\}$ is a nonnegative submartingale with respect to $G_n = \sigma(X_1, X_2, \dots, X_n)$.

Proof. Obviously $\{U_n, n = 1, 2, \dots\}$ is a sequence of increasing nonnegative random variables with

$$E[U_{n+1}|G_n] = U_n + E[X_{n+1}] \geq U_n \dots (1)$$

$$\begin{aligned} \text{Also } \sup_{n \geq 0} E[|U_n|] &= \lim_{n \rightarrow \infty} E[U_n] \\ &= \lim_{n \rightarrow \infty} E\left[\sum_{i=1}^n X_i\right] \\ &= \lim_{n \rightarrow \infty} \sum_{i=1}^n E[X_i] \\ &= \lim_{n \rightarrow \infty} \left[\mu + \sum_{i=2}^n \frac{\mu}{2^{i-1}\beta_0^2}\right] \\ &= \mu \left[1 + \sum_{i=2}^n \frac{1}{2^{i-1}\beta_0^2}\right] < \infty \dots (2) \end{aligned}$$

Where eqn (2) is due to the fact that the series $\sum_{i=2}^n \frac{1}{2^{i-1}\beta_0^2}$ is convergent by comparing it with geometric series if $\frac{1}{\beta_0^2} < 1$. Thus from eqn (1) and (2), by definition $\{U_n, n = 1, 2, \dots\}$ is a nonnegative submartingale with respect to $\{G_n, n = 1, 2, \dots\}$ if $\beta_0^2 > 1$.

Theorem 3.2. If $\beta_0^2 > 1$, there exist a random variable U such that the sequence $\{U_n\}$ converges almost sure to U .

Proof. $\{U_n, n = 1, 2, \dots\}$ is a nonnegative sub martingale with respect to $\{G_n, n = 1, 2, \dots\}$

$$\sup_{1 \leq n \leq \infty} E[|U_n|] \leq \infty$$

Then with probability 1, and random variable $U = \lim_{n \rightarrow \infty} U_n$ and $E[|U|] < \infty$

\therefore the sequence $\{U_n\}$ converges almost sure to U .

Theorem 3.3. If $\beta_0^2 > 1$, $\{U_n, n = 1, 2, \dots\}$ has a unique decomposition such that

$$U_n = M_n - B_n \dots (3)$$

Where $\{M_n, n = 1, 2, \dots\}$ is a martingale. $\{B_n, n = 1, 2, \dots\}$ is a decreasing with $B_1 = 0$ and $B_n \in G_{n-1}$.

Proof. Let $M_1 = U_1$ and $B_1 = 0$ for $n \geq 2$

We define $M_n = M_{n-1} + (U_n - E[U_n | G_{n-1}]) \dots (4)$

$B_n = B_{n-1} + (U_{n-1} - E[U_n | G_{n-1}]) \dots (5)$

From eqn. (4) and (5) we have

$$M_n - B_n = \sum_{i=2}^n (U_i - U_{i-1}) + U_1 - B_1 = U_n$$

and (3) follows. It checks $\{M_n, n = 1, 2, \dots\}$ and $\{B_n, n = 1, 2, \dots\}$ satisfy the terms. Next to prove that a decomposition is unique.

Suppose $U_n = M_n^* - B_n^*$ is another decomposition. Then $M_n = M_n^* = B_n - B_n^*$

Since $M_2 = M_2^* = B_2 - B_2^* \in G_1$

implies that $M_2 = M_2^*$. Then by induction

Hence $B_n = B_n^*$.

Definition 3.4. Given a partial square sum process $\{X_j, j = 1, 2, 3, \dots\}$ and if $w(t) = \sup_{j \in \mathbb{Z}} \{j: V_j \leq 1\}$, where $V_j = \sum_{i=1}^j X_i$ the age at t is defined by $A(t) = t - U_{w(t)}$ the residual life at t is defined by

$$B(t) = U_{w(t)+1} - t$$

and the total life at t is defined by

$$\begin{aligned} X_{w(t)+1} &= U_{w(t)+1} - U_{w(t)} \\ &= A(t) + B(t) \end{aligned}$$

Let G_j be the distribution function of U_j and $G_j(x) = 1 - U_j(x)$.

Theorem 3.5. If G is the distribution function of X_1 and $\bar{G}(x) = 1 - G(x)$.

Then

1. $P(A(t) > x) = \begin{cases} \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^{t-x} \bar{G}(2^{n-1} \beta_0^2(t-u)) dG_n(t), & 0 < x < t \\ 0, & x \geq t \end{cases}$
2. $P(B(t) > x) = \begin{cases} \bar{G}(t+x) + \sum_{n=1}^{\infty} \int_0^t \bar{G}(2^{n-1} \beta_0^2(x+t-y)) dG_n(y), & x > 0 \\ 1, & x \leq 0 \end{cases}$

$$3. P(X_{W(t)+1} > x) = \begin{cases} \bar{G}(t \vee x) + \sum_{n=1}^{\infty} \int_0^t \bar{G}(2^{n-1} \beta_0^2(x \vee t - y) dG_n(y), & x > 0 \\ 1, & x \leq 0 \end{cases}$$

$$4. P(U_{W(t)} \leq x) = \begin{cases} \bar{G}(t) + \sum_{n=1}^{\infty} \int_0^{t-x} \bar{G}(2^{n-1} \beta_0^2(t - y) dG_n(y), & 0 \leq x \leq t \\ 1, & x > t \end{cases}$$

Proof. 1. Assume $0 < x < t$

$$\begin{aligned} P(A(t) > x) &= P(U_{W(t)} < t - x) \\ &= \sum_{j=0}^{\infty} P(U_{W(t)} < t - x, W(t) = j) \\ &= \sum_{j=0}^{\infty} P(U_j < t - x, U_{W+1} > t) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} P(U_j > t \mid U_j = x) dG_j(x) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} P(X_{j+1} > t - x) dG_j(x) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} \bar{G}(2^{j-1} \beta_0^2(t - u) dG_j(u) \end{aligned}$$

For $x \geq t$, $P(A(t) > x) = 0$ is trivial.

Part (i) proof is completed.

2. Assume $x > 0$

$$\begin{aligned} P(B(t) > x) &= P(U_{W(t)} > t + x) \\ &= \sum_{j=0}^{\infty} P(U_{W(t)+1} > t + x, W(t) = j) \\ &= \sum_{j=0}^{\infty} P(U_{j+1} > t + x, U_j \leq t) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^t P(U_{j+1} > t + x \mid U_j = y) dG_j(y) \\ &= \bar{G}(t + x) + \sum_{j=1}^{\infty} \int_0^t P(X_{j+1} > t + x - y) dG_j(y) \\ &= \bar{G}(t + x) + \sum_{j=1}^{\infty} \int_0^t \bar{G}(2^{j-1} \beta_0^2(x + t - y) dG_j(y) \end{aligned}$$

For $y \leq 0$, $P(B(t) > x) = 1$ is trivial.

Part (2) proof completed.

3. Assume $x > 0$

$$\begin{aligned} P(X_{W(t)+1} > x) &= \sum_{j=0}^{\infty} P(X_{W(t)+1} > x, W(t) = j) \\ &= \sum_{j=1}^{\infty} \int_0^t P(X_{j+1} > x, U_j \leq t \leq U_{j+1} | U_j = y) dG_j(y) \\ &= \sum_{j=1}^{\infty} \int_0^t P(X_{j+1} > \max(x, t-y)) dG_j(y) \dots (6) \\ &= \bar{G}(t \vee x) + \sum_{j=1}^{\infty} \int_0^t \bar{G}(2^{j-1} \beta_0^2(x \vee (t-y))) dG_j(y) \end{aligned}$$

For $x \leq 0$, $P(X_{W(t)+1} > x) = 1$ is trivial.

Part (3) proof completed.

4. Assume $0 < x \leq t$

$$\begin{aligned} P(U_{W(t)+1} \leq x) &= P(t - U_{W(t)} \leq x) \\ &= P(U_{W(t)} \geq t - x) \\ &= \sum_{j=0}^{\infty} P(U_{W(t)} \geq t - x, W(t) = j) \\ &= \sum_{j=0}^{\infty} P(U_{j+1} > t - x, U_{j+1} > t) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^t P(U_{j+1} > t | U_j = x) dG_j(x) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} P(X_{j+1} > t - x) dG_j(x) \\ &= \bar{G}(t) + \sum_{j=1}^{\infty} \int_0^{t-x} \bar{G}(2^{j-1} \beta_0^2(t-u)) dG_j(u) \end{aligned}$$

For $x \geq t$, $P(U_{W(t)} \leq x) = 1$ is trivial.

Part (4) proof completed.

Limit Theorems for partial square sum process

Theorem 3.6. Weld's Equation for partial square sum process

Suppose $\{X_n, n = 1, 2, 3, \dots\}$ forms a partial square sum process with ratio β_0^2 with $E(X_1) = \mu < \infty$ then for $t > 0$, we have

$$E(U_{W(t)+1}) = \mu E \left[1 + \sum_{n=2}^{W(t)+1} \frac{1}{2^{n-2} \beta_0^2} \right]$$

Proof. Let I_A be the indicator function of event A . Then $I_{\{U_{n-1} \leq t\}} = I_{\{W_{n+1} \geq n\}}$ and X_n are independent. Consequently, for $t > 0$ we get

$$\begin{aligned} E(U_{W(t)+1}) &= E \left[\sum_{n=1}^{W(t)+1} X_n \right] \\ &= \sum_{n=1}^{\infty} E[X_n I_{\{W_{n+1} \geq n\}}] \\ &= \sum_{n=1}^{\infty} E(X_n) P(W_{(t)+1} \geq n) \\ &= \sum_{j=1}^{\infty} \left(\sum_{n=1}^j E(X_n) \right) P(W_{(t)+1} = j) \\ &= E \left[E(X_1) + \sum_{n=2}^{W(t)+1} E(X_n) \right] \\ &= \mu E \left[1 + \sum_{n=2}^{W(t)+1} \frac{1}{2^{n-2} \beta_0^2} \right] \end{aligned}$$

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