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A Homotopy Analysis Approach to a 1D Fractional Order Problem with Lord-Shulman and Dual-Phase-Lag Model

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This paper is concerned with studying 1D fractional order generalized thermos-elastic half-space in the context of two distinct models: (LS) Lord-Shulman model and (DPL) Dual-phase-lag model. Exponentially varying heat source has been applied to the surface of the bounding plane x = 0. Proposed equations have been formulated by using Caputo-Fabrizio fractional order derivative. The homotopy perturbation technique is used to find its approximate solution. The effect of exponentially varying heat source and the fractional order parameter has been investigated on temperature, displacement, and stress under the (LS) Lord-Shulman and (DPL) Dual-phase-lag model. The procured results have been illustrated graphically using MATLAB, where the influence of the homotopy perturbation technique has been observed. The validity of this intended model is assessed by comparing it with previously published results. According to the authors, this explicit work will be more useful for studying the thermal behavior of various bodies in Geophysics, Mathematical Biology, and Acoustics, as well as for investigating real-life problems in the field of Engineering.

Keywords: Caputo-Fabrizio fractional order derivative, Homotopy Perturbation Technique, Nonlinear Problem, Dual-Phase-lag model, generalized thermos-elasticity, exponentially vary heating.

1. Introduction

In the current research paper two distinct models of thermos-elasticity viz., the Lord-Shulman (LS) Model and the Dual-Phase-Lag (DPL) Model, have been considered to study a 1D fractional order problem in half-space with the help of homotopy perturbation technique.

The LS model refers to a theory in the field of linear thermo-elasticity [3]. This was proposed by H. W. Lord and Y. Shulman in 1967. Lord and Shulman have tried to eliminate the paradox of infinite velocity of thermal differences inherent in CTE. This model is based on a modified Fourier's law but additionally a single relaxation time was also considered.

The DPL model was proposed by D. Y. Tzou [8]. It introduced a dual-phase-lagging effect to account for finite thermal wave speeds in heat conduction problems. This model considers two relaxation times associated with heat conduction process. Also this is used to describe the time delays between the temperature and the heat flux in a material and was established to modify the assumption made in Fourier's law and Canttaneo-Vernotte (CV) model [1,2] or thermal wave model or hyperbolic heat conduction model.

Dhaliwal R.S. et al., introduced the dynamic coupled thermo-elasticity and proposed various methods to solve system of PDEs [5]. Hetnarski R. B. et al., proposed the theory of generalized thermo-elasticity with improving the shortcomings of classic thermo-elasticity [12]. S. K. Roy Choudhuri, studied the problem of one-dimensional disturbances in an elastic half-space employed by dual-phase-lag effects and analyzed the underlying characteristic features [20]. S. Mukhopadhyay, et al., formulated fundamental solutions in an unbounded medium for a three- phase-lag thermoelastic problem [33]. K. R. Gaikwad, et al., determined the temperature and thermal deflection for a unsteady-state temperature field in response to internal heat generation in it. Moreover, a series solution in terms of Bessel's function is derived with the help of Hankel transform and generalized finite Fourier transform method [36]. H. M. Youssef, et al., obtained the exact analytical solution for the two-temperature dual-phase-lag model of bioheat transfer. Also the significant effects of dual-phase-lag time parameter, heat flux value, and two-temperature parameter have discussed [48]. K. R. Gaikwad, et al., analyzed the transient thermoelastic problem to obtain the analytical solution for the temperature field and thermal deflection for a thin circular plate using integral transform method [50].

In recent decades Fractional Calculus has been used as a powerful mathematical tool to model real world problems in the field of science, technology and engineering. Many definitions of fractional order derivative have derived till now [4, 7, 10, 13, 17, 37]. Lots of different thermoelastic problems have solved using fractional derivatives. H. H. Sherief, applied the theory of fractional order derivative to study a 1D problem of half space [41]. Lin W., discussed the IVP for a class of FDEs. Feed-back control of chaotic FDE is investigated theoretically and to verify analytical result the fractional Lorenz system is provided as a numerical example [19]. S. G. Khavale, et al., analyzed the temperature, displacement and stress field for a fractional order thermoelastic problem by employing axi-symmetric heat source and obtained the results using integral trans- form technique including Mittag-Leffler function [52]. Yang X. J., proposed a new fractional order derivative with non-singular kernel and derived an analytical solution of fractional order heat conduction equation using it [46]. S. G. Khavale and K. R. Gaikwad, analyze a magnto- thermo-viscoelastic problem in spherical region for an isotropic medium and derive the general solution in the context of fractional order derivative [49]. K. R. Gaikwad and V. G. Bhandwalkar, modified a problem of two-temperature for finite piezoelectric rod by employing three different heating applications viz., thermal shock, ramp-type and harmonically vary heating and derived the general solutions with the help of Caputo-Fabrizio fractional order derivative [51].

In the current paper we are using Caputo-Fabrizio fractional order derivative, which is a fractional derivative without singular kernel, firstly introduced by M. Caputo and M. Fabrizio [44]. This fractional derivative has various interesting properties whose significance can be seen for modeling in many branches of sciences. Many researchers studied this new derivative and used this as a tool in deriving solutions of complex thermoelastic problems.

Researchers introduced various methods for solving FDEs such as Lagrange characteristics method [18], Generalized transform method [22,27], Finite element method [26], Adomian decomposition method [28, 32], Variational iteration method [29], Fractional sub-equation method [35], Finite difference method [38], Homotopy analysis method [14]. In this paper we are using Homotopy perturbation technique [11].

A homotopy analysis method (HAM) was proposed by Liao, which transforms a non-linear prob- lem into an infinite number of linear problems without using perturbation techniques [9, 14]. Using these techniques we can solve linear and non-linear fractional PDEs. He improved and dis- cussed a powerful, easy-to-use analytic tool, namely the homotopy analysis method for nonlinear problems [6, 16]. A homotopy perturbation technique (HPM) is a special case of homotopy analysis method. Both methods HPM and HAM are based on Taylor series respect to the embedding parameter 'p'. If initial guess and auxiliary linear operator are good enough then both the methods give good approximate solutions. The only difference is that HPM had use of a good initial guess which is not necessary in HAM. Ji-Huan He, defined the Homotopy perturbation method (HPM) and discussed its validity and solved some examples by using the technique [11, 15].

Further lots of thermoelastic problems have been solved by many of the researchers using homotopy perturbation technique. S. Momani, et al., proposed and discussed the homotopy perturbation method for solving nonlinear PDEs of non-integer order [21]. A. Golbabai, et al., applied Homotopy perturbation method to the numerical solution for solving eighth-order boundary value problems and compared its results with the results obtained by modified decomposition method [23]. M. Ghasemi, et al., applied HPM to solve nonlinear integro-differential equations and discussed the effectiveness of method [24]. Zaid M. Odibat, present modification of the homotopy perturbation method and conduct a comparative study between the new modification and the homotopy perturbation method [25]. Xicheng Li, et al., extended the HPM method to solve time-fractional diffusion equation with a moving boundary condition and obtained an accurate approximate solution [30]. J.H. He, introduced the concepts of the homotopy perturbation method and paid particular attention to give an intuitive grasp for the solution procedure [31].

P. K. Gupta, et al., applied a homotopy perturbation technique to solve the fractional order Fornberg-Whitham equation [34]. A. M. A. El-Sayed, et al., proposed a homotopy perturbation technique to solve a class of initial boundary value problems of partial differential equations of arbitrary fractional order in finite domain in terms of illustrating good approximate solution [39].

S. K. Rana, et al., studied wave motion in an infinite transversely isotropic, thermoelastic plate in the context of conventional coupled thermo-elasticity, Lord-Shulman and Green-Lindsay theories of thermo-elasticity using homotopy perturbation method [40]. Sudhakar Yadav, et al., studied a one-dimensional problem for a half-space in the context of the L-S theory of generalized thermo-elasticity with one relaxation time parameter with the help of Homotopy analysis method proposed by S. J. Liao [42]. A. Allahverdizadeh, et al., formulated nonlinear analysis of a thin rectangular functionally graded plate in terms of von-Karman's dynamic equations and the governing equations of motion reduced to a Duffing's equation and then solved by using homotopy perturbation method [43]. Sudhakar, Y. et al., analyze a thermoelastic problem in the context of exponentially varying heat source using homotopy analysis method [45]. Roul. P., found the numerical solution of singular boundary value problem by a domain decomposition homotopy perturbation approach [47]. Sayed Abo-Dahab, et al., investigate the wave propagation in transversely isotropic thermoelastic two-dimensional plates in the context of GL model and applied the technique of homotopy perturbation technique to get the approximate solution with boundary condition [53]. Adedapo Ismaila Alaje, et al., established the proposed technique by coupling a power series function of arbitrary order with the renowned homotopy perturbation method and proved the convergence of the method using the Banach fixed point theorem [54]. Althobaiti N. et al., analyzed solution of half-space in the context of two thermoelastic models employs by homotopy perturbation technique [55]. Payam Jalili, et al., present a general method of HPM to make this method more straightforward to use for future studies in solving heat transfer problems using Python [56].

Nomenclature:

λ, μ	Lame's Constants
ρ	Density
C_E	Specific heat per unit mass
T_0	Reference temperature
t	Time
k	Thermal Conductivity
α	Fractional differential operator
e_{ij}	Strain tensor components
α_t	Coefficient of linear thermal expansion
σ_{ij}	Stress vector components
τ	the thermal relaxation time
τ_θ	the phase-lag of temperature gradient
$p \in [0, 1]$	the embedding parameter
$\gamma = (3\lambda + 2\mu)\alpha_t$	

2. The Models Used in the Problem

The two models depend on thermal relaxation time are as follows [12, 13]:

Lord-Shulman (LS) Model:

$$\kappa T_{,ii} = \left(\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \gamma T_0 e) \qquad (1)$$

Dual-Phase-Lag (DPL) Model:

$$\kappa\left(1+\tau_\theta \frac{\partial}{\partial t}\right)T_{,ii} = \left(\frac{\partial}{\partial t}+\tau \frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \gamma T_0 e) \qquad (2)$$

3. Homotopy Perturbation Technique Fractional Order Problem - Formulation

Consider a general equation of type,

L(u) = 0

Assuming the homotopy convex as follows [15]:

$$H(u, p) = (1 - p)F(u) + pL(u) \qquad (4)$$

where F (u) is the functional operator for known solution u0. It is shown that

$$H(u, p) = 0 \qquad (5)$$

The embedding parameter p is monotonically increases in [0,1], hence it can be taken as an expanding parameter to obtain the solution,

$$U = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2u_2 + \ldots. \qquad (6)$$

When p → 1, the approximate solution is:

$$U = \lim_{p\to 1} u = \sum_{i=0}^{\infty} u_i \qquad (7)$$

Definition of Caputo-Fabrizio Fractional Order Derivative

The Caputo-Fabrizio fractional order derivative is defined in [35] as follows:

$${}_aD^{(\alpha)}{}_t\, f(t) = \frac{M(\alpha)}{1-\alpha}\int_a^t f'(\tau)\exp\left[\frac{\alpha(t-\tau)}{1-\alpha}\right]d\tau$$

where, $M(\alpha)$ is the normalization function such that,

$$M(0) = M(1) = 0, \quad 0 \le \alpha \le 1, \quad -\infty < a < t, \quad f \in H^1(a, b), \quad a < b$$

We suppose that the function M (α) = 1 and substituting a = 0 in the definition defined in above equation.

Fractional Order Problem - Formulation

Consider a generalized thermoelastic half space in isotropic homogeneous medium subjected to exponentially varying heat source on the boundary plane surface x = 0. The surface is assumed to be traction free.

The displacement is,

$$u_i = (u, 0, 0), \quad u_y = u_z = 0 \qquad (8)$$

The fundamental equations [7,20] will take the form:

$$\rho\frac{\partial^{\alpha+1}u}{\partial t^{\alpha+1}} = (\lambda+2\mu)\frac{\partial^2 u}{\partial x^2} - \gamma\frac{\partial T}{\partial x} \qquad (9)$$

$$k\left(1+\tau\theta\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2 T}{\partial x^2} = \left(\frac{\partial^\alpha}{\partial t^\alpha} + \tau\frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}}\right)(\rho C_E T + \gamma T_0 e) \qquad (10)$$

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma(T - T_0) \qquad (11)$$

The non-dimensional variables are defined as:

$c_1^2 = \frac{\lambda+2\mu}{\rho}$, $\frac{T}{T_0} = \theta, c_1^2 t = t'$, $\alpha_1 = \frac{\gamma T_0}{\lambda+2\mu}$, $\eta = \frac{\rho C_E}{k}$, $\eta x = x'$, $a_2 = \frac{\gamma}{\rho C_E}$,

$E = \frac{\partial u}{\partial x}$, $\sigma'_{xx} = \frac{\sigma_{xx}}{\lambda+2\mu}$, $a_1 a_2 = \epsilon$, $\theta - 1 = \theta$ (12)

Using variables from equation (12) in equations (9)-(11), we get,

$$\frac{\partial^{\alpha+1} u}{\partial t^{\alpha+1}} = \frac{\partial^2 u}{\partial x^2} - a_1 \frac{\partial \theta}{\partial x} \quad (13)$$

$$\left(1 + \tau_\theta \frac{\partial^\alpha}{\partial t^\alpha}\right) \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^\alpha \theta}{\partial t^\alpha} + \tau \frac{\partial^{\alpha+1} \theta}{\partial t^{\alpha+1}} + a_2 \frac{\partial^\alpha}{\partial t^\alpha}\left[\frac{\partial u}{\partial x} + \tau \frac{\partial^2 u}{\partial t \partial x}\right] \quad (14)$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} - a_1 \theta \quad (15)$$

4. Solution using Homotropy perturbation technique

The boundary conditions are assumed as:

$\theta(t, 0) = e^{-t}$, $\sigma_{xx}(t, 0) = 0$ (16)

From equations (13) and (15),

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} = \frac{\partial^{\alpha+1} \sigma_{xx}}{\partial t^{\alpha+1}} + a_1 \frac{\partial^{\alpha+1} \theta}{\partial t^{\alpha+1}} \quad (17)$$

From equations (14) and (15),

$$\frac{\partial^2 \theta}{\partial x^2} = (1+\epsilon) \frac{\partial^\propto \theta}{\partial t^\propto} + \tau(1+\epsilon) \frac{\partial^{\propto+1} \theta}{\partial t^{\propto+1}} + a_2 \left[\frac{\partial^\propto \sigma_{xx}}{\partial t^\propto} + \tau \frac{\partial^{\propto+1} \sigma_{xx}}{\partial t^{\propto+1}}\right] - \tau_\theta \frac{\partial^3 \theta}{\partial t \partial x} \quad (18)$$

Using homotopy perturbation technique [18], from equations (17) and (18) we have,

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + p\left[-\frac{\partial^{\alpha+1} \sigma_{xx}}{\partial t^{\alpha+1}} - a_1 \frac{\partial^{\alpha+1} \theta}{\partial t^{\alpha+1}}\right] = 0 \quad (19)$$

$$\frac{\partial^2 \theta}{\partial x^2} + p\left[-(1+\epsilon) \frac{\partial^\propto \theta}{\partial t^\propto} - \tau(1+\epsilon) \frac{\partial^{\propto+1} \theta}{\partial t^{\propto+1}} - a_2 \left[\frac{\partial^\alpha \sigma_{xx}}{\partial t^\alpha} + \tau \frac{\partial^{\alpha+1} \sigma_{xx}}{\partial t^{\alpha+1}}\right] + \tau_\theta \frac{\partial^3 \theta}{\partial t \partial x^2} = 0 \quad (20)$$

we can expand the solution as:

$$\sigma_{xx}(x,t) = {\sigma_{xx}}^0 + p^1 {\sigma_{xx}}^1 + p^2 {\sigma_{xx}}^2 + p^3 {\sigma_{xx}}^3 + \cdots \quad (21)$$

$$\theta(x,t) = \theta^0 + p^1 \theta^1 + p^2 \theta^2 + p^3 \theta^3 + \cdots \quad (22)$$

θ(x, t) from equations (21) and (22) in equations (19) and (20), we get the linear equations in series form. First few of them are as:

$$p^0: \frac{\partial^2 {\sigma_{xx}}^0}{\partial x^2} = 0 \quad (23)$$

$$p^0: \frac{\partial^2 \theta^0}{\partial x^2} = 0$$

$$p^1: \frac{\partial^2 {\sigma_{xx}}^1}{\partial x^2} = \frac{\partial^{\alpha+1} {\sigma_{xx}}^0}{\partial t^{\alpha+1}} + a_1 \ \frac{\partial^{\alpha+1}\theta}{\partial t^{\alpha+1}}$$

$$p^1: \frac{\partial^2 \theta^1}{\partial x^2} = (1+\ \epsilon)\frac{\partial^{\propto}\theta^0}{\partial t^{\propto}} + \tau(1+\ \epsilon)\frac{\partial^{\propto+1}\theta^0}{\partial t^{\propto+1}} - \tau_\theta \frac{\partial^3\theta^0}{\partial t \partial x^2} + a_2\left[\frac{\partial^{\propto} {\sigma_{xx}}^0}{\partial t^{\propto}} + \tau \frac{\partial^{\propto+1} {\sigma_{xx}}^0}{\partial t^{\propto+1}}\right] \quad (26)$$

$$p^2: \frac{\partial^2 {\sigma_{xx}}^2}{\partial x^2} = \frac{\partial^{\alpha+1} {\sigma_{xx}}^1}{\partial t^{\alpha+1}} + a_1 \ \frac{\partial^{\alpha+1}\theta^1}{\partial t^{\alpha+1}} \quad (27)$$

$$p^2: \frac{\partial^2 \theta^2}{\partial x^2} = (1+\ \epsilon)\frac{\partial^{\propto}\theta^1}{\partial t^{\propto}} + \tau(1+\ \epsilon)\frac{\partial^{\propto+1}\theta^1}{\partial t^{\propto+1}} - \tau_\theta \frac{\partial^3\theta^1}{\partial t \partial x^2} + a_2\left[\frac{\partial^{\propto} {\sigma_{xx}}^1}{\partial t^{\propto}} + \tau \frac{\partial^{\propto+1} {\sigma_{xx}}^1}{\partial t^{\propto+1}}\right] \quad (28)$$

$$p^3: \frac{\partial^2 {\sigma_{xx}}^3}{\partial x^2} = \frac{\partial^{\alpha+1} {\sigma_{xx}}^2}{\partial t^{\alpha+1}} + a_1 \ \frac{\partial^{\alpha+1}\theta^2}{\partial t^{\alpha+1}} \quad (29)$$

Using the boundary conditions from equation (16) the solutions of system of linear equations can be obtained as:

$$\theta^0(x,t) = \theta(t,0) = \ e^{-t}, \qquad {\sigma_{xx}}^0(x,t) \ = \ \sigma_{xx}(t,0) = 0 \quad (31)$$

$${\sigma_{xx}}^1(x,t) = \ a_1\left(\frac{x^2}{2!}\right)(-1)^{\propto+1}e^{-t} \quad (32)$$

$$\theta^1(x,t) = \ (-1)^{\propto}(1+\tau)(1+\ \epsilon)\left(\frac{x^2}{2!}\right)e^{-t} \quad (33)$$

$${\sigma_{xx}}^2(x,t) = \ a_1(-1)^{2\propto}[1-(1+\tau)(1+\ \epsilon)]\left(\frac{x^4}{4!}\right)e^{-t} \quad (34)$$

$$\theta^2(x,t) = \ (-1)^{2\propto}[(1-\ \tau)[\ (1+\tau)(1+\ \epsilon)^2 - \epsilon]\ \left(\frac{x^4}{4!}\right) - \ \tau_\theta(1+\tau)(1+\ \epsilon)\left(\frac{x^2}{2!}\right)]e^{-t} \quad (35)$$

$${\sigma_{xx}}^3(x,t) = (-1)^{3\propto}a_1\,[\tau_\theta(1+\tau)(1+\ \epsilon)\left(\frac{x^4}{4!}\right) - [1-(1+\tau)(1+\ \epsilon) + (1-\tau^2)(1+\epsilon)^2 - \epsilon(1-\tau)]\ \left(\frac{x^6}{6!}\right)e^{-t} \quad (36)$$

$$\theta^3(x,t) = \ (-1)^{3\propto}(1-\tau)[(\epsilon - 2\epsilon(1+\epsilon) + (1-\tau^2)\ (1+\epsilon)^3)\left(\frac{x^6}{6!}\right) - \ \tau_\theta\ \epsilon\left(\frac{x^4}{4!}\right)]\ e^{-t} \quad (37)$$

In the series form the solution can be expressed as:

$$\sigma_{xx}(x,t) = \sum_{r=0}^{3} {\sigma_{xx}}^r(x,t) = \left[l_1\frac{x^2}{2!} + \ l_2\frac{x^4}{4!} + l_3\frac{x^6}{6!}\right]e^{-t} \quad (38)$$

$$\theta(x,t) = \sum_{r=0}^{3}\theta^r(x,t) = \left[1 + m_1\frac{x^2}{2!} + \ m_2\frac{x^4}{4!} + m_3\frac{x^6}{6!}\right]e^{-t} \quad (39)$$

From equation (15) we have,

$$\frac{\partial u}{\partial x} = [a_1\ + (a_1\,m_1 + l_1)\frac{x^2}{2!} + \ (a_1\,m_2 + l_2)\frac{x^4}{4!} + (a_1\,m_3 + l_3)\frac{x^6}{6!}]\ e^{-t} \quad (40)$$

Integrating equation (40) w.r.t. ′x′, we get,

$$u(x,t) = \ [a_1\ \frac{x^1}{1!} + (a_1\,m_1 + l_1)\frac{x^3}{3!} + \ (a_1\,m_2 + l_2)\frac{x^5}{5!} + (a_1\,m_3 + l_3)\frac{x^7}{7!}]\ e^{-t} \quad (41)$$

Where,

$l_1 = (-1)^{\alpha+1} a_1,$
$l_2 = (-1)^{2\alpha} a_1[1 - (1+\tau)(1+\epsilon) + \tau_\theta(-1)^\alpha(1+\tau)(1+\epsilon)],$
$l_3 = (-1)^{3\alpha+1} a_1[1 - (1+\tau)(1+\epsilon) + (1-\tau^2)(1+\epsilon)^2 - \epsilon(1-\tau)],$
$m_1 = (-1)^\alpha(1+\tau)(1+\epsilon)(1-(-1)^\alpha\tau_\theta),$

$m_2 = (-1)^{2\alpha}(1-\tau)[(1+\tau)(1+\epsilon)^2 - \epsilon - (-1)^\alpha\tau_\theta\epsilon],$

$m_3 = (-1)^{3\alpha}(1-\tau)[\epsilon - 2\epsilon(1+\epsilon) + (1-\tau^2)(1+\epsilon)^3]$

5. Graphical Representation and Discussion

The numerical values of displacement, temperature, and stress for copper material are demonstrated in this problem. MATLAB software is utilized to calculate the values of field variables by utilizing the solutions of equations (38), (39), and (41). The results are plotted for temperature, displacement, and stress against space variable x at t = 0.2 and four distinct values of the fractional order parameter α = 0, 0.25, 0.50, 0.75, 1.

Numerical Values of Constants:

Physical Constants	Value
Lame's constant (λ)	$7.76 \times 10^{10}\ Nm^{-2}$
Lame's constant (μ)	$3.86 \times 10^{10}\ kgm^{-1}s^{-2}$
Density (ρ)	$8954\ kgm^{-3}$
Reference temperature (T_0)	$293^o\ k$
Coefficient of linear thermal expansion (α_t)	$1.78 \times 10^{-5}\ k^{-1}$
Thermal conductivity (k)	$8886.73\ sm^{-3}$
Specific heat per unit mass (C_E)	383.1

These physical constants for copper material are chosen for numerical calculation [44].

Figure 1-5 indicates the stress variation under the LS and DPL model at t = 0.2 for distinct values of α = 0, 0.25, 0.50, 0.75, 1 respectively, when the boundary of the half-space is subjected to a heat source varying with time t at zero stress.

When α = 0, 0.25, 0.5, in Figure 1-3 the stress function gradually decreases with increase in value of x and after travelling some distance x it reaches to zero for both LS and DPL model. The effect of relaxation time appears in LS and DPL model as follows:

In Figure 1, is evident in the increase at distinct values of relaxation time τ = 0.8, 1.0, and τθ = 0 for LS model but decrease at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0 for DPL model. In Figure 2, is evident in the decrease at distinct values of relaxation time τ = 0.8, 1.0, and τθ = 0 for LS model but increase at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0 for DPL model. In Figure 3, it is evident that the relaxation time decreases at different values for both LS model and DPL models.

We observed in Figure 1 that (σxx)LS > (σxx)DP L, for all intervals of x. Hence, the rate of convergence of the series solution for the LS model is higher than for the DPL model. In Figure 2 and 3, the effect of (σxx)LS < (σxx)DP L is shown for all intervals of x. Thus, series solutions

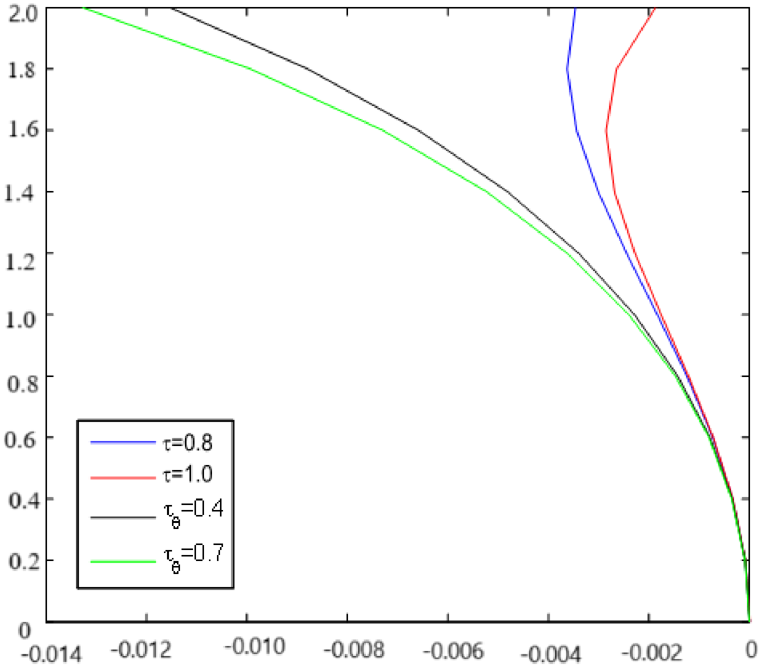

Fig. 1. Stress variation under LS and DPL model at t=0.2 when α = 0

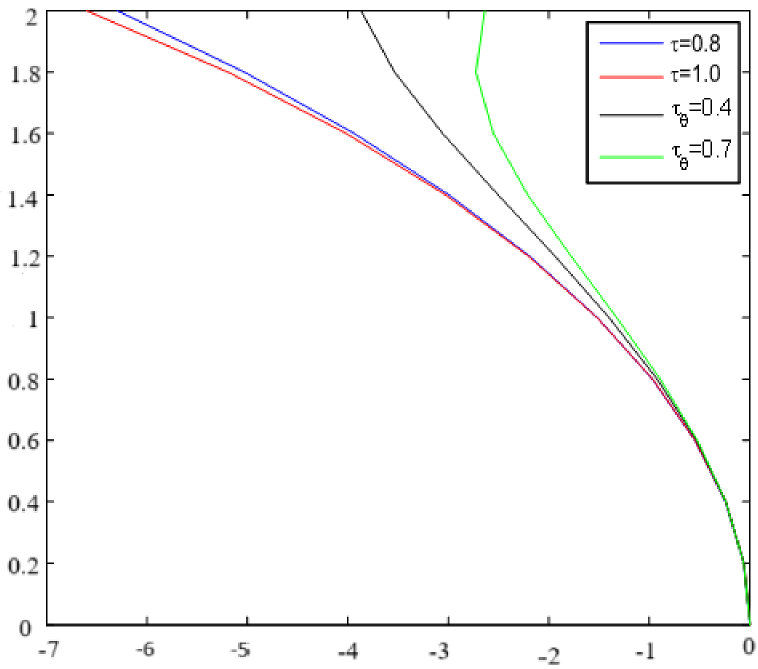

Fig. 2. Stress variation under LS and DPL model at t=0.2 when α = 0.25

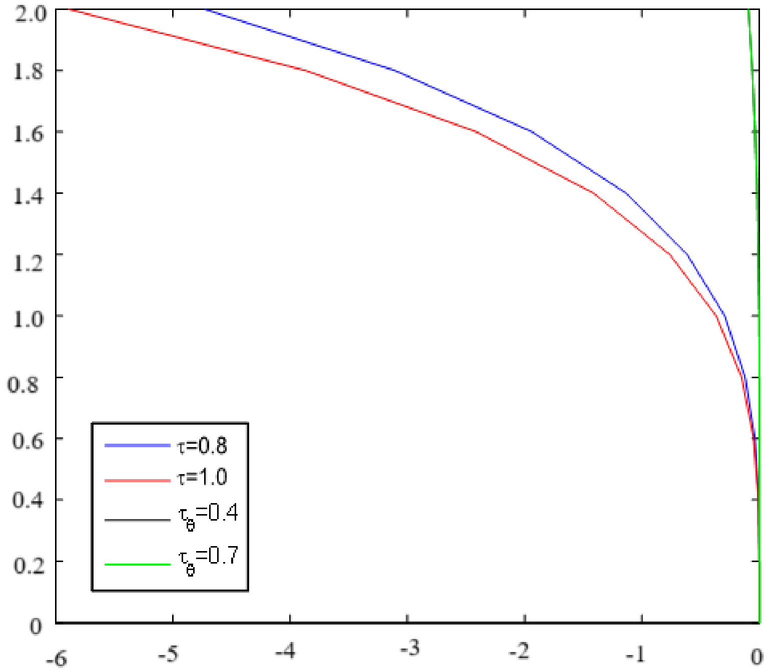

Fig. 3. Stress variation under LS and DPL model at t=0.2 when α = 0.5

for the DPL model have a higher convergence rate than those for the LS model.

When α = 0.75, 1, in Figure 4 and 5, the stress function increases with an increase in the value of x for both models. The effect of relaxation time appears in the LS and DPL models as follows:

In Figure 4, is evident in the decrease at distinct values of relaxation time τ = 0.8, 1.0, and τθ = 0 for LS model but increase at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0 for DPL model. In Figure 5, is evident in the decrease at distinct values of relaxation time for both LS model and DPL model.

Figure 4 displays, the impact of the relation (σxx)LS < (σxx)DP L for all intervals of x. Therefore, the convergence rate of series solutions for the DPL model is higher than that for the LS model. But in Figure 5, it is observed that the values of (sigmaxx)LS and (sigmaxx)DPL fluctuate for different values of relaxation time.

In Figure 6-10, we can see that there are temperature variations under both LS and DPL models for different values of α = 0, 0.25, 0.5, 0.75, 1.

For α = 0, 0.25 in Figure 6 and 7, respectively, the temperature θ gradually increases from a non-zero constant value with an increase in the value of space variable x under both LS and DPL models.

For α = 0, in Figure 6, the effect of relaxation time appears in both the LS and DPL models. It is evident in the increase at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0 for DPL model, while for LS model it first decreases then coincides and then increase at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0.

For α = 0.25, the effect of relaxation time is evident in Figure 7 as indicated by the decrease at different values of relaxation time. τ = 0.8, 1.0, and τθ = 0 for the LS model, but they almost

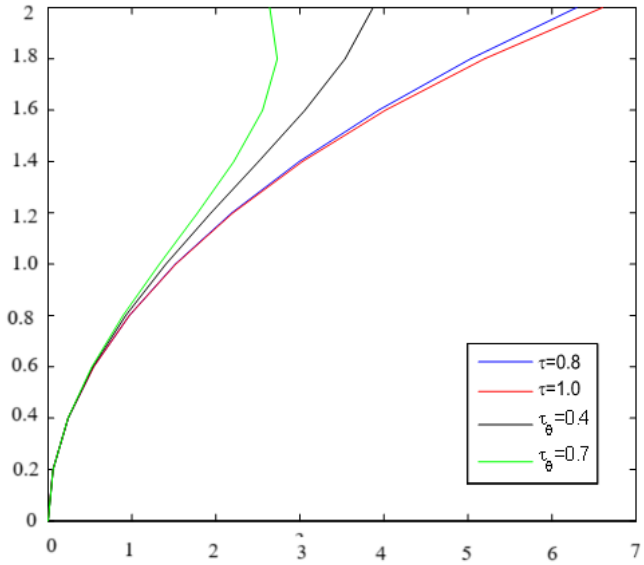

Fig. 4. Stress variation under LS and DPL model at t=0.2 when α = 0.75

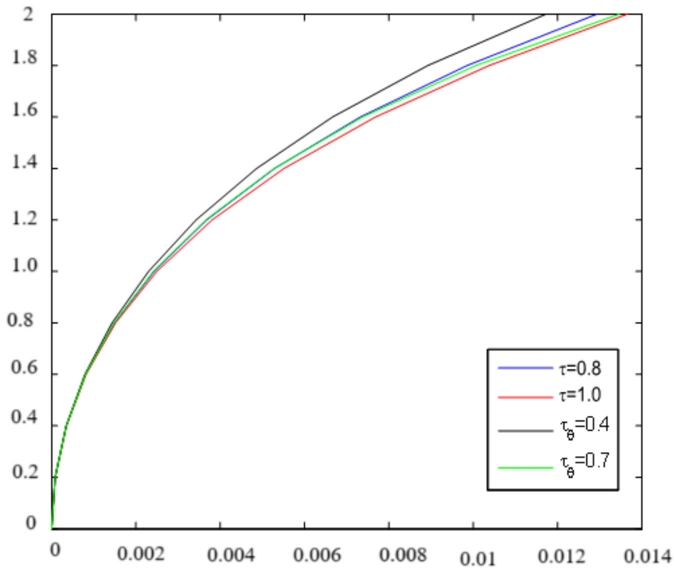

Fig. 5. Stress variation under LS and DPL model at t=0.2 when α = 1

coincide at distinct relaxation time τθ = 0.4, 0.7, and τ = 0 for the DPL model. Figure 8-10 demonstrates that the temperature rapidly decreases as x increases for α = 0.5, 0.75, 1.

Here, we note that θLS < θDP L for distinct values of α = 0, 0.25, 0.5, 0.75, 1 in all intervals of x.

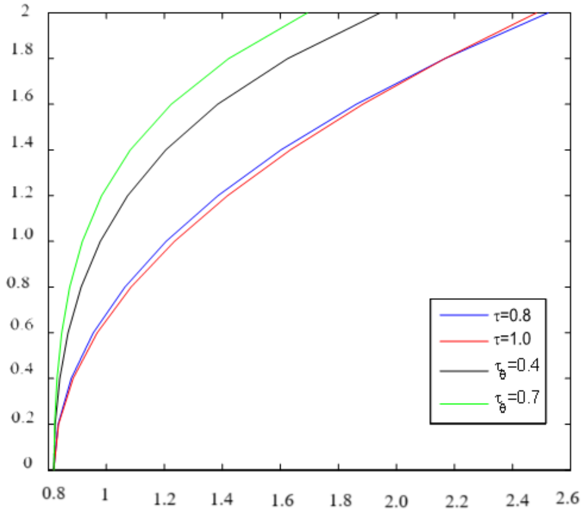

Fig. 6. Temperature variation under LS and DPL model at t=0.2 when α = 0

Figure 11-15 shows variations in displacement under the LS and DPL models for distinct values of α = 0, 0.25, 0.5, 0.75, and1 respectively.

For all distinct values of α = 0, 0.25, 0.5, 0.75, 1, the displacement increases with increasing the space variable x in both the LS and DPL models.

For α = 0, 0.25, uLS < uDP L for all intervals of x. While for α = 0.5, 0.75, 1, we have uLS > uDP L for all intervals of x.

For α = 0, the effect of relaxation time appears in both the LS and DPL models and is evident in the decrease at distinct values of relaxation time τ = 0.8, 1.0, and τθ = 0 in the LS model, while is in the increase at distinct values of relaxation time τθ = 0.4, 0.7, and τ = 0 for DPL model as shown in Figure 11. For α = 0.25, 0.5, the effect of relaxation time appears and decreases at distinct relaxation time values for both the LS and DPL models, as shown in Figures 12 and 13, respectively. In Figures 14 and 15, for α = 0.75, 1,the effect of relaxation time appears and is in the increase at distinct relaxation time values for both the LS and DPL models.

6. Conclusion

In the present work, the homotopy perturbation technique has been utilized to analyze the displacement, temperature and stress for a half space under an exponentially varying heat source using a generalized thermo-elasticity theory based on fractional order heat conduction with Caputo- Fabrizio time-fractional derivative.

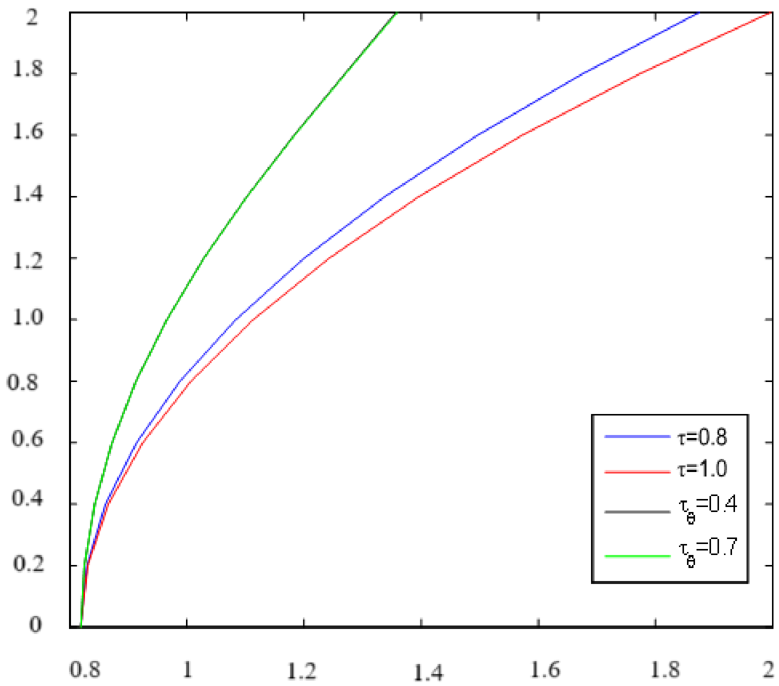

Fig. 7. Temperature variation under LS and DPL model at t=0.2 when α = 0.25

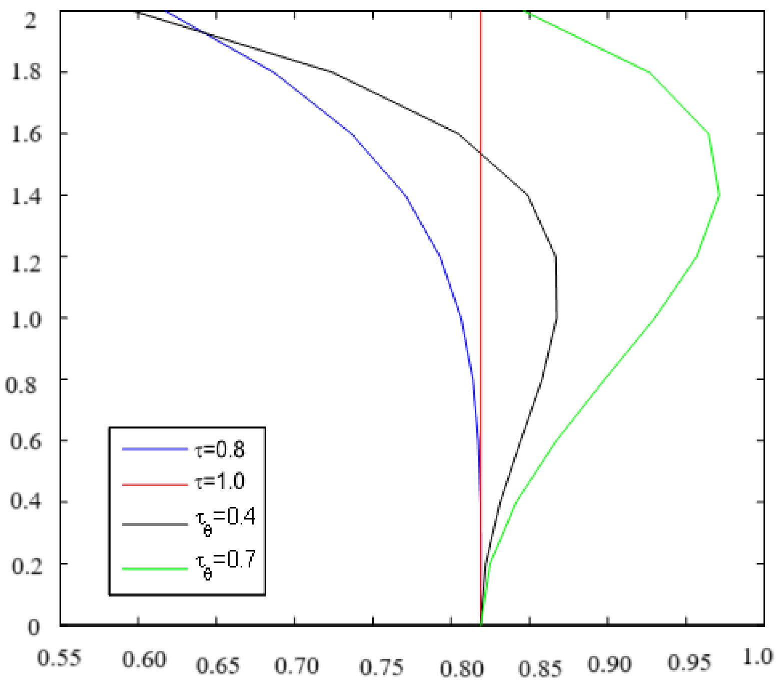

Fig. 8. Temperature variation under LS and DPL model at t=0.2 when α = 0.5

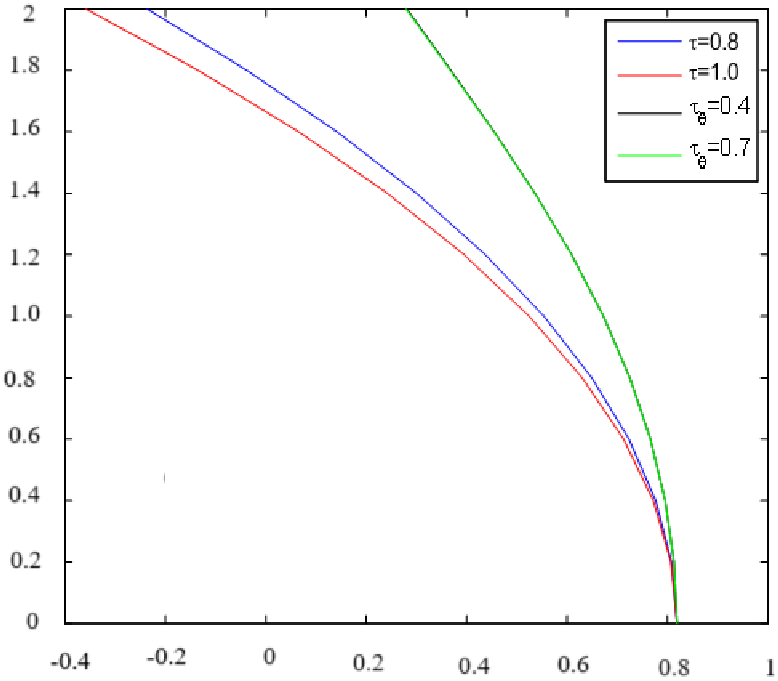

Fig. 9. Temperature variation under LS and DPL model at t=0.2 when α = 0.75

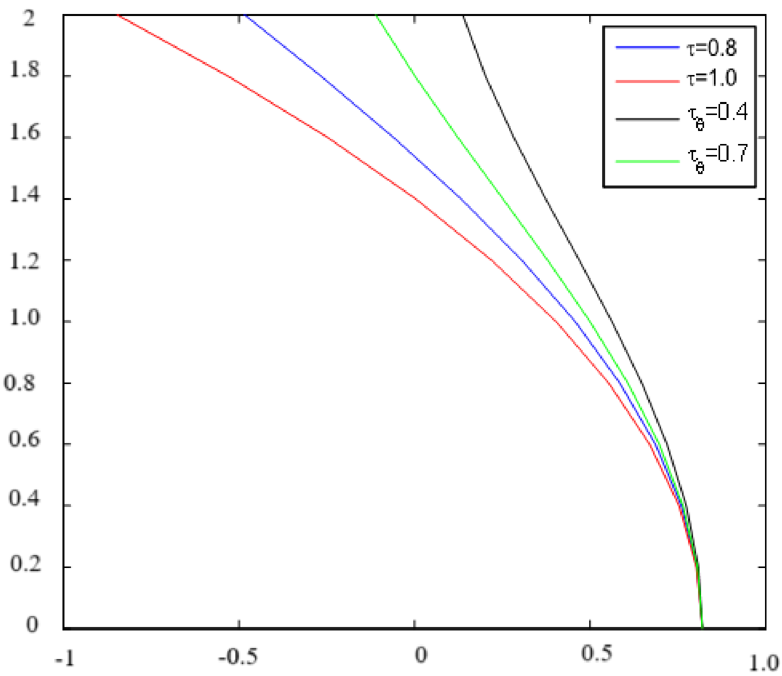

Fig. 10. Temperature variation under LS and DPL model at t=0.2 when α = 1

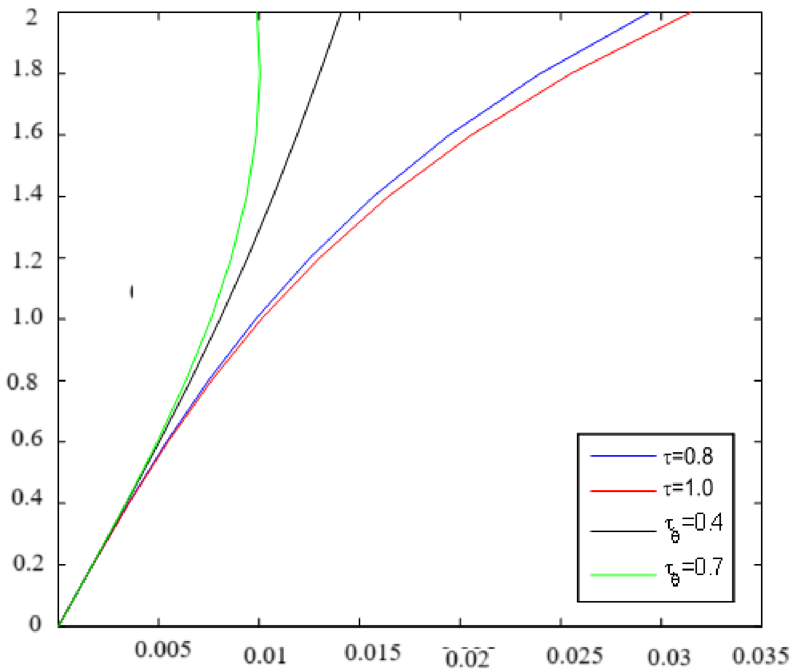

Fig. 11. Displacement variation under LS and DPL model at t=0.2 when α = 0

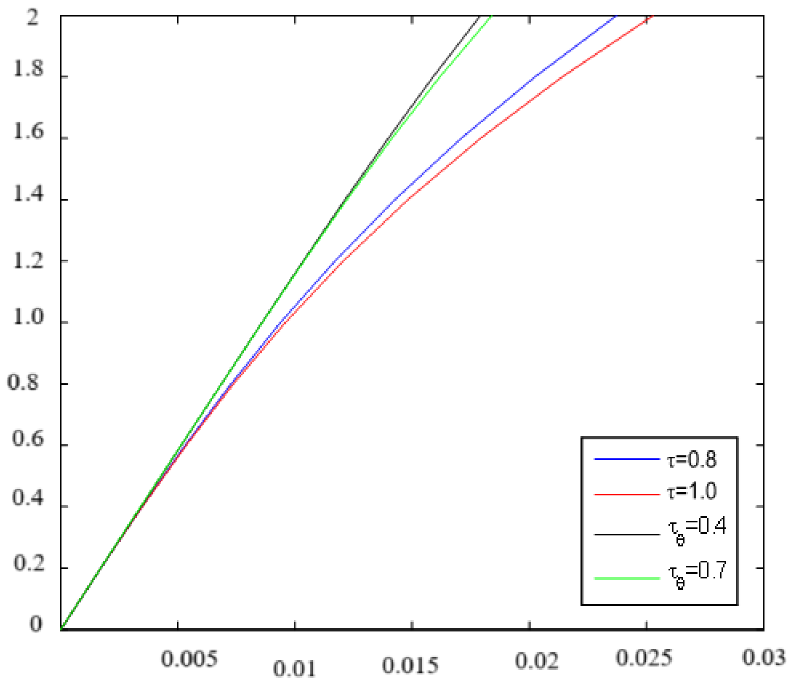

Fig. 12. Displacement variation under LS and DPL model at t=0.2 when α = 0.25

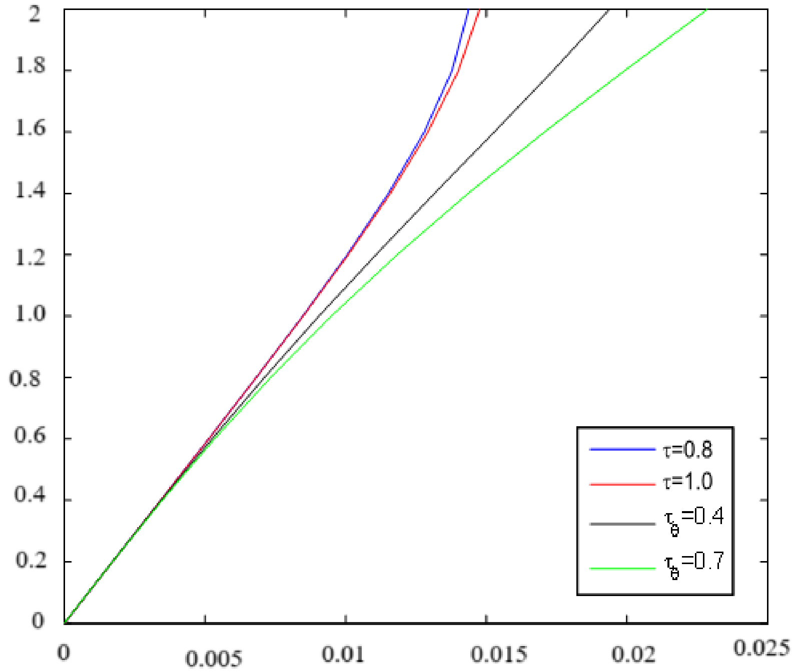

Fig. 13. Displacement variation under LS and DPL model at t=0.2 when α = 0.5

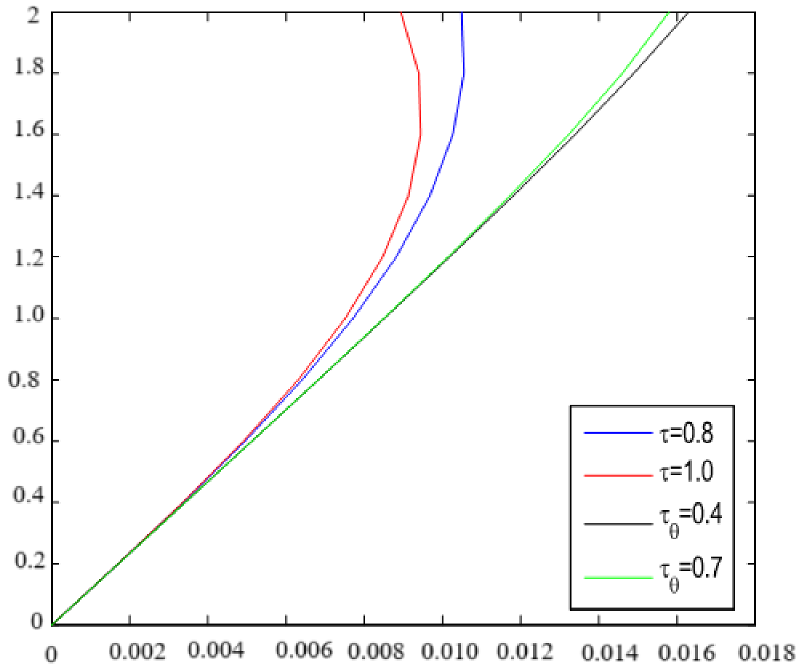

Fig. 14. Displacement variation under LS and DPL model at t=0.2 when α = 0.75

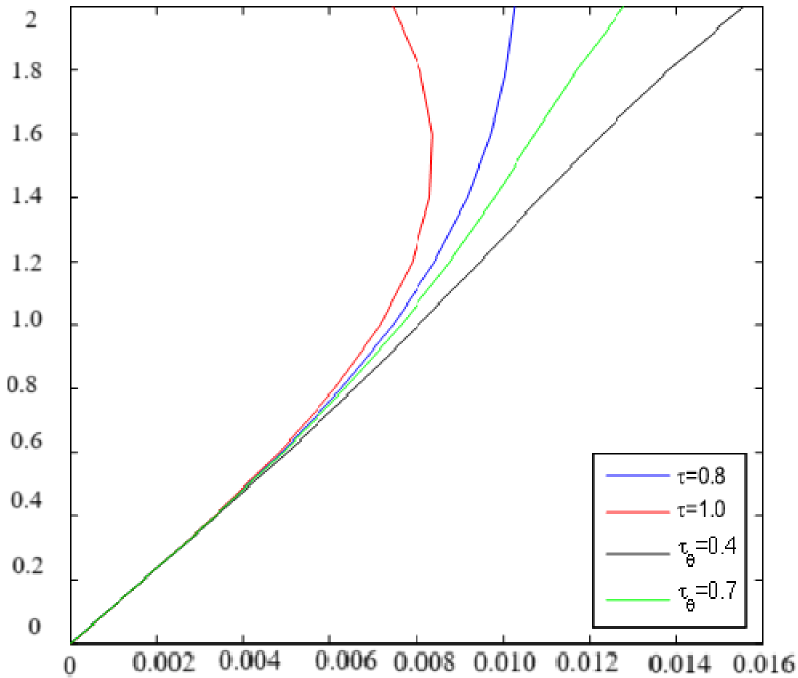

Fig. 15. Displacement variation under LS and DPL model at t=0.2 when α = 1

The important findings of this paper are summarized as follows:

- The proposed results were analyzed to determine the rate of convergence of the series solution for LS and DPL model due to the effect of variations in temperature, displacement, as well as compressive and tensile type stresses, distinct relaxation time and fractional order parameter.

- When the external heat source is applied to the boundary of the half-space, the variable in the field variable functions appears against the space-time domain.

- The compressive and tensile type stresses produced under both LS and DPL model for dis- tinct values of fractional order parameter α = 0, 0.25, 0.5, 0.75, 1, which shows significant effects on the rate of convergence of series solution as shown in Figures 1-5.

- Figures 6-10 show the impact of the CF-fractional order parameter and relaxation time on temperature for both models LS and DPL.

- With the variations in fractional order parameter for various relaxation time the significant changes in displacement have been noted against space-time domain as shown in Figures 11-15.

-The LS and DPL models have been examined and have shown significant effects due to the embedded parameter p, fractional order parameter alpha, tau, and tautheta.

- The results presented here will be more useful in studying the thermal behavior of various bodies in geophysics, real life problems in engineering, mathematical biology and acoustics.

To continue studying this problem, it may be beneficial to explore a half-space with symmetric or axisymmetric heat supply. Moreover, by changing the approach of finding the solution, i.e. instead of homotopy perturbation technique, a use of direct approach or modified homotopy perturbation method can also lead to a new inspiration to continue the current work.

Author Contributions All authors reviewed the manuscript.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing Interests The authors have disclosed that they have no potential conflicts of interest and were not aided by any funding in the preparation of this manuscript.

References

1. C.A. Cattaneo, Form of heat conduction equation which eliminate the paradox of instantaneous propagation, C R, vol. 247, pp. 431-433, 1958.
2. P. Vernotte, Paradox in the continuous theory of heat equation, C R, vol. 246, pp. 3154-3155, 1958.
3. Lord, H.W.; Shulman, Y. A generalized dynamical theory of thermo elasticity, J. Mech. Phys. Solids, vol. 15, pp. 299–309, 1967.
4. K.B. Oldham, J. Spanier, The Fractional Calculus, Academic Press, New York, 1974.
5. Dhaliwal R.S. and Singh A., Dynamic Coupled Thermo-elasticity, Hindustan Publishing Corporation: New Delhi, India, pp. 726, 1980.
6. S.J. Liao, Proposed Homotopy Analysis Technique for the Solution of Nonlinear Problems (Ph.D. thesis), Shanghai Jio Tong University, 1992
7. K. S. Miller and B. Ross, An Introduction to Fractional Calculus and Fractional Differential Equations, Wiley, New York, 1993.
8. D.Y. Tzou, A unified field approach for heat conduction from macro to micro-scale, Journal of heat transfer, vol. 117, pp. 8-16, 1995.
9. S.J. Liao, An approximate solution technique not depending on small parameters: a special example, International Journal of Non-Linear Mech., vol. 30, 3, pp. 371–380, 1995.
10. I. Podlubny, Fractional Differential Equations, Academic Press, NY, 1998.
11. J.H. He, Homotopy perturbation technique, Computer methods in applied mechanics and engineering, vol. 178, (3–4), pp. 257–262, 1999.
12. Hetnarski R.B. and Ignaczak J., Generalized Thermo-elasticity, J. Therm. Stress., vol. 22, pp. 451–476, 1999.
13. R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific Publishing, River Edge, NJ, 2000.
14. S.J. Liao, Beyond Perturbation: Introduction to the Homotopy Analysis Method, Chapman and Hall/CRC, Boca Raton, 2003.
15. J. H. He, Homotopy perturbation method: a new nonlinear analytical technique, Appl. Math. Comput., vol. 1, pp. 73-79, 2003.
16. S. J. Liao, On the homotopy analysis method for nonlinear problems. Applied Mathematics and Computation, vol. 147, 2, pp. 499–513, 2004.
17. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier Science, Amsterdam, 2006.
18. G. Jumarie, Lagrange characteristic method for solving a class of nonlinear partial differential equations of fractional order, Appl. Math. Lett., vol. 19, 3, pp. 873–880, 2006.
19. Lin W., Global existence theory and chaos control of fractional differential equations, J. Math. Anal. Appl., vol. 332, pp. 709–726, 2007.

20. S. K. RoyChoudhuri, One-dimensional thermoelastic waves in elastic half-space with dual-phase-lag effects, J. Mech. Mater. Struct., vol. 2, pp. 489–503, 2007.
21. S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A., vol. 365, (5–6), pp. 345–350, 2007.
22. S. Momani, Z. Odibat, and V.S. Erturk, Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation, Phys. Lett. A, vol. 370, (5–6), pp. 379–387, 2007.
23. A. Golbabai and M. Javidi, Application of homotopy perturbation method for solving eighth-order boundary value problems, Applied mathematics and computation, vol. 191, 2, pp. 334-346, 2007.
24. M. Ghasemi, M. Tavassoli Kajani, and E. Babolian, Application of He's homotopy perturbation method to nonlinear integro-differential equation, Applied mathematics and computation, vol. 188, 1, pp. 538-548, 2007.
25. Z. Odibat, A new modification of homotopy perturbation method for linear and nonlinear operators, Applied mathematics and computation, vol. 189, 1, pp. 746-753, 2007.
26. W. Deng, Finite element method for the space and time fractional Fokker–Planck equation, SIAM J. Numer. Anal., vol. 47, 1, pp. 204–226, 2008.
27. Z. Odibat and S. Momani, A generalized differential transform method for linear partial differential equations of fractional order, Appl. Math. Lett., vol. 21, 2, pp. 194–199, 2008.
28. Y. Hu, Y. Luo and Z. Lu, Analytical solution of the linear fractional differential equation by Adomian decomposition method, J. Comput. Appl. Math., vol. 215, 1, pp. 220–229, 2008.
29. Z. Odibat and S. Momani, The variational iteration method: An efficient scheme for handling fractional partial differential equations in fluid mechanics, Comput. Math. Appl., vol. 58, (11–12), pp. 2199–2208, 2009.
30. Xicheng Li, Mingyu Xu and Xiaoyun Jiang, Homotopy perturbation method to time fractional diffusion equation with a moving boundary condition, Applied mathematics and computation, vol. 208, 2, pp. 434-439, 2009.
31. J.H. He, An elementary introduction to the homotopy perturbation method, Computers and Mathematics with Applications, vol. 57, 3, pp. 410–412, 2009.
32. A. M. A. El-Sayed, S. H. Behiry and W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection–dispersion equation, Comput. Math. Appl., vol. 59, 5 pp. 1759–1765, 2010.
33. S. Mukhopadhyay, S. Kothari and R. Kumar, On the Representation of Solutions for the Theory of Generalized Thermo-elasticity with Three Phase Lags, Acta Mech., vol. 214, pp. 305–314, 2010.
34. P. K. Gupta and M. Singh, Homotopy perturbation method for fractional Fornberg-Whitham equation, Comput. Math. Appl., vol. 61, 2, pp. 250–254, 2011.
35. S. Zhang and H. Q. Zhang, Fractional sub-equation method and its applications to nonlinear fractional PDEs, Phys. Lett. A., vol. 375, 7, pp. 1069–1073, 2011.
36. K. R. Gaikwad and K. P. Ghadle, Nonhomogeneous heat conduction problem and its thermal deflection due to internal heat generation in a thin hollow circular disk, Journal of Thermal stresses, vol. 35, 6, pp. 485-498, 2012.
37. D. Baleanu, K. Diethelm, E. Scalas and J.J. Trujillo, Fractional calculus models and numerical methods, in: Nonlinearity and Chaos, in: Series on Complexity, World Scientific, 2012.
38. G.H. Gao, Z.Z. Sun, Y.N. Zhang, A finite difference scheme for fractional sub-diffusion equations on an unbounded domain using artificial boundary conditions , J. Comput. Phys., vol. 231, 7, pp. 2865–2879, 2012.
39. A. M. A. El-Sayed, A. Elsaid, I. L. El-Kalla and D. Hammad, A homotopy perturbation technique for solving partial differential equations of fractional order in finite domains, Applied mathematics and computation, vol. 218, 17, pp. 8329-8340, 2012.
40. S. K. Rana and P. K. Sharma, Application of Homotopy Perturbation Method to study wave propagation in Transversely Isotropic Thermoelastic Three Dimensional Plate, Asian journal of advanced basic sciences, vol. 1, 1, pp. 30-39, 2013.
41. H. H. Sherief and A. M. El-Latief, Application of Fractional Order Theory of Thermo-elasticity to a 1D Problem for A Half-Space, ZAMM, vol. 94, pp. 509-515, 2014a.
42. Sudhakar Yadav and Anil Kumar, A homotopy analysis approach to a thermoelastic interactions under

the boundary condition: Heat source varying exponentially with time and zero stress, Interna- tional Journal of Science and Research, vol. 4, 10, pp. 2319-7064, 2014.

43. A. Allahverdizadeh, R. Oftadeh, M. J. Mahjoob and M. H. Noei, Homotopy perturbation solution and periodicity analysis of nonlinear vibration of thin rectangular functionally graded plates, Acta Mechanica Solida Sinica, vol. 27, 2, pp. 210-220, 2014.
44. M. Caputo and M. Fabrizio, A New Definition of Fractional Derivative without Singular Kernel, Progress in Fractional Differentiation and Applications, vol. 1, 2, pp. 73-85, 2015.
45. Sudhakar Y. and Kumar A., A Homotopy Analysis Approach to Thermoelastic In-teractions under the Boundary Condition: Heat Source Varying Exponentially with Time and Zero Stress, Int. J. Sci. Res., vol. 4, pp. 2126–2129, 2015.
46. Yang X.-J., H. M. Srivastava, and Jose A. T. Machado, A New Fractional Derivative without Singular Kernel: Application to The Modelling of the Steady Heat Flow, Thermal Science, vol. 20, 2, pp. 753-756, 2016.
47. Roul, P., On the numerical solution of singular two-point boundary value problems: A domain decomposition homotopy perturbation approach, Math. Methods Appl. Sci., vol. 40, pp. 7396–7409, 2017.
48. H. M. Youssef, and N. A. Alghamdi, The Exact Analytical Solution of the Dual-Phase-Lag Two-Temperature Bio-heat Transfer of a Skin Tissue Subjected to Constant Heat Flux, Scientific Reports, vol. 10, pp. 1-29, 2020.
49. S. G. Khavale, K. R. Gaikwad, Generalized theory of magneto-thermo-viscoelastic Spherical cavity problem under Fractional order derivative: State Space Approach, Advances in Mathematics: Scientific Journal, vol. 9, pp. 9769-9780, 2020.
50. K. R. Gaikwad, Y. U. Naner, Analysis of transient thermoelastic temperature distribution of a thin circular plate and its thermal deflection under uniform heat generation, Journal of thermal stresses, vol. 44, 1, 2020.
51. K. R. Gaikwad, V. G. Bhandwalkar, Fractional order thermoelastic problem for finite piezoelectric rod subjected to different types of thermal loading - direct approach, Journal of the Korean Society for Industrial and Applied Mathematics, Vol. 25, pp. 117–131, 2021.
52. S. G. Khavale and K. R. Gaikwad, Fractional order thermoelastic problem of thin hollow circular disk and its thermal stresses under axi-symmetric heat supply, Design Engineering, vol. 12, 9, pp. 13851-13862, 2021.
53. Sayed Abo-Dahab, Abdelmooty Abd-Alla, A. A. Kilany, Homotopy perturbation method on wave propagation in a transversely isotropic thermoelastic two-dimensional plate with gravity field, Numerical heat transfer applications, vol. 82, 2, pp. 1-13, 2022.
54. Adedapo Ismaila Alaje, Morufu Oyedunsi Olayiwola, Kamilu Adewale Adedokun, Joseph Adeleke Adedeji and Asimiyu Olamilekan Oladapo, Modifed homotopy perturbation method and its application to analytical solitons of fractional-order Korteweg–de Vries equation, Beni-Suef University Journal of Basic and Applied Sciences, vol. 11, pp. 139, 2022.
55. Althobaiti N., et.al., Solution of a Half-Space in Generalized Thermoelastic Problem in the Context of Two Models Using the Homotopy Perturbation Method, Axioms, vol. 12, pp. 827, 2023.
56. Payam Jalili, Bahram Jalili , Irshad Ahmad , Ahmed S. Hendy , Mohamed R. Ali and Davood Domiri Ganji, Python approach for using homotopy perturbation method to investigate heat transfer problems, Case studies in thermal engineering, vol. 54, pp. 104049, 2024.