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Oscillatory Behavior of Second Order Canonical Delay Differential Equations with Several Sublinear Neutral Terms

R. Rama, S. Chithambara Bharathy

Quaid-E-Millath Govt College for Women, India.
E-mail: renurama68@gmail.com

The aim of this paper is to define some new oscillation conditions for the second order canonical differential equations with several sublinear neutral terms of the form $\left(b(\phi)\left(w'(\phi)\right)\right)' + f\left(\phi, v\left(\xi(\phi)\right)\right) = 0$, $\phi \geq \phi_0 > 0$, where $w(\phi) = v(\phi) + \sum_{i=1}^{k} q_i(\phi)\, v^{\theta_i}\left(\tau_i(\phi)\right)$ and $k \geq 1$ is an integer, θ_i are ratios of odd positive integers with $0 < \theta_i < 1$ for $i = 1, 2, \ldots, k$. By using the Comparison principle and Riccati approach, we give new conditions for oscillation of the equation. Furthermore, we provide an example to illustrate the significance of the new results.

Keywords: Several Sublinear Neutral Terms, Second order, Oscillatory Behavior, Canonical Form.

1. Introduction

In this paper, we focus on the oscillatory behavior of second order canonical differential equations with several sublinear neutral terms

$$\left(b(\phi)\left(w'(\phi)\right)\right)' + f\left(\phi, v\left(\xi(\phi)\right)\right) = 0, \qquad \phi \geq \phi_0 > 0, \tag{1.1}$$

where $w(\phi) = v(\phi) + \sum_{i=1}^{k} q_i(\phi)\, v^{\theta_i}\left(\tau_i(\phi)\right)$ and $k \geq 1$ is an integer, subject to the following conditions:

H1: θ_i are ratios of odd positive integers with $0 < \theta_i < 1$ for $i = 1, 2, \ldots, k$;

H2: $b \in C^1([\phi_0, \infty), \mathbb{R}^+)$ and Equation (1.1) is in canonical form if

$$I(\phi, \phi_0) = \int_{\phi_0}^{\phi} \frac{1}{b(\alpha)}\, d\alpha \longrightarrow \infty \text{ as } \phi \longrightarrow \infty; \tag{1.2}$$

H3: $q_i \in C([\phi_0, \infty), \mathbb{R}^+)$ and $q_i(\phi) \longrightarrow 0$ as $\phi \longrightarrow \infty$ for $i = 1, 2, \ldots, k$;

H4: $\tau_i \in C([\phi_0, \infty), \mathbb{R})$, $\tau_i(\phi) \leq \phi$ and $\lim_{\phi \to \infty} \tau_i(\phi) = \infty$ for $i = 1, 2, \ldots, k$;

H5: $\xi \in C^1([\phi_0, \infty), \mathbb{R})$, $\xi(\phi) \leq \phi$, $\xi'(\phi) \geq 0$ and $\lim_{\phi \to \infty} \xi(\phi) = \infty$.

H6: $f(\phi, v) \in C([\phi_0, \infty) \times \mathbb{R}, \mathbb{R})$, and there exists γ is a ratio of odd positive integers with $\gamma \geq 1$ and a function $p(\phi) \in C([\phi_0, \infty), \mathbb{R}^+)$ such that $f(\phi, v)/v^\gamma \geq p(\phi)$, for all $v \neq 0$.

Under a proper solution of (1.1), we mean a function $v \in C([\phi_v, \infty), \mathbb{R})$, $\phi_v \geq \phi_0$, which has properties $w(\phi)$, $b(\phi)\left(w'(\phi)\right) \in C^1([\phi_v, \infty), \mathbb{R})$, and which satisfies (1.1) on $[\phi_v, \infty)$. We focus only those solutions v of (1.1) which satisfy

$$\sup\{|v(\phi)| : \phi \geq T\} > 0, \text{ for every } T \geq \phi_v.$$

A proper solution v of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its proper solutions oscillate.

2. Lemmas

This section presents a few important lemmas that will contribute to proving our main results. We only need to provide proofs for the case of eventually positive solutions because the proofs for eventually negative solutions would be comparable due to the assumption and the form of our equation.

Lemma 2.1 [7]. If β is positive and $0 < \varepsilon < 1$, then

$$\beta^\varepsilon \leq \beta\,\varepsilon + (1 - \varepsilon). \tag{2.1}$$

Lemma 2.2 [3]. Assume that condition H5 holds. For $\phi \geq \phi_0$, if the function g satisfies $g > 0$, $g' > 0$ and $g'' \leq 0$, then there exists $\phi_\omega \geq \phi_0$ such that

$$g(\xi(\phi)) \geq \frac{\omega}{\phi}\,\xi(\phi)\,g(\phi), \tag{2.2}$$

for all $\omega \in (0,1)$.

Lemma 2.3 [15]. Let

$$\mu(u) = Au - Bu^{\frac{(\alpha+1)}{\alpha}}, \tag{2.3}$$

where A and B are positive constants, and α is a quotient of odd positive integers. Then, μ reaches its maximum value at $\theta = \left(\frac{A\,\alpha}{B(\alpha+1)}\right)^\alpha$ on $\mathbb{R}$ and

$$\max_{u \in \mathbb{R}} \mu = \mu(\theta) = \frac{A^{\alpha+1}}{B^\alpha}\frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}. \tag{2.4}$$

Lemma 2.4 Suppose that $v(\phi)$ is a positive solution of (1.1) such that

$$\int_{\phi_0}^{\infty} \frac{1}{b(u)} \int_u^{\infty} p(\alpha)\, d\alpha\, du = \infty. \tag{2.5}$$

Then, for $\phi \geq \phi_0$,

(I) $w(\phi) > 0$, $w'(\phi) > 0$ and $\left(b(\phi)\left(w'(\phi)\right)\right)' < 0$; (2.6)

(II) $\frac{w(\phi)}{I(\phi,\phi_1)}$ is decreasing;

(III) $w(\phi) \longrightarrow \infty$ as $\phi \longrightarrow \infty$.

Proof: Assume that $v(\phi)$ is a positive solution of (1.1) on $[\phi_0, \infty)$. Then by the assumptions [H4] and [H5], there exists a $\phi_1 \geq \phi_0$, such that $v\big(\tau_i(\phi)\big) > 0$ for i = 1, 2, …, k and $v\big(\xi(\phi)\big) > 0$ on $[\phi_1, \infty)$. Now, since $w(\phi) = v(\phi) + \sum_{i=1}^{k} q_i(\phi)\, v^{\theta_i}\big(\tau_i(\phi)\big)$, then

$$w(\phi) \geq v(\phi) > 0, \text{ for } \phi \geq \phi_1.$$

From (1.1), we have

$$\Big(b(\phi)\,\big(w'(\phi)\big)\Big)' = -f\Big(\phi, v\big(\xi(\phi)\big)\Big) \leq -p(\phi)\, v^{\gamma}\big(\xi(\phi)\big) < 0. \tag{2.7}$$

This implies

$$\Big(b(\phi)\,\big(w'(\phi)\big)\Big)' < 0.$$

Thus $b(\phi)\,\big(w'(\phi)\big)$ is decreasing.

Now, to show that $w'(\phi) > 0$ on $[\phi_1, \infty)$, assume that it is not true. Then we can find $\phi_2 > \phi_1$, such that $w'(\phi_2) < 0$. But since $b(\phi)\,\big(w'(\phi)\big)$ is decreasing, then

$$b(\phi)\,\big(w'(\phi)\big) < b(\phi_2)\,\big(w'(\phi_2)\big) = c_0 < 0, \text{ for } \phi \geq \phi_2.$$

By integrating the above inequality from ϕ_2 to ϕ, we get

$$w(\phi) < w(\phi_2) + c_0 \int_{\phi_2}^{\phi} \frac{1}{b(\alpha)}\, d\alpha.$$

This with (1.2), leads to $w(\phi) \longrightarrow -\infty$ as $\phi \longrightarrow \infty$ which is a contradiction with the fact that $w(\phi) > 0$ eventually. This completes the proof of $w'(\phi) > 0$.

Thus,

$$\begin{aligned} w(\phi) &= w(\phi_1) + \int_{\phi_1}^{\phi} \frac{b(\alpha)\, w'(\alpha)}{b(\alpha)}\, d\alpha \\ &\geq b(\phi)\, w'(\phi) \int_{\phi_1}^{\phi} \frac{1}{b(\alpha)}\, d\alpha \\ &= b(\phi)\, w'(\phi)\, I(\phi, \phi_1), \end{aligned}$$

and hence

$$\left(\frac{w(\phi)}{I(\phi,\phi_1)}\right)' = \frac{b(\phi)\, w'(\phi)\, I(\phi,\phi_1) - w(\phi)}{b(\phi)\, I^2(\phi,\phi_1)} \leq 0.$$

So $\frac{w(\phi)}{I(\phi,\phi_1)}$ is a decreasing function.

We claim that (2.5) ensures $\boldsymbol{w(\phi)} \longrightarrow \infty$ as $\boldsymbol{\phi} \longrightarrow \infty$. Actually, as $\boldsymbol{w(\phi)}$ is a positive increasing function, we actually know that there is a constant c > 0; therefore

$$\boldsymbol{w(\phi)} \geq 2c > 0. \tag{2.8}$$

Furthermore, it is derived from $\boldsymbol{w(\phi)}$ that

$$\begin{aligned}\boldsymbol{v(\phi)} &= \boldsymbol{w(\phi) - \sum_{i=1}^{i=k} q_i(\phi)\, v^{\theta_i}(\tau_i(\phi))} \\ &\geq \boldsymbol{w(\phi) - \sum_{i=1}^{i=k} q_i(\phi)\, w^{\theta_i}(\tau_i(\phi))} \\ &\geq \boldsymbol{w(\phi) - \sum_{i=1}^{i=k} q_i(\phi)\ [\theta_i\, w(\phi) + (1-\theta_i)]} \\ &= \boldsymbol{w(\phi)\ \left(1 - \sum_{i=1}^{i=k} \theta_i\, q_i(\phi) - \frac{1}{w(\phi)} \sum_{i=1}^{i=k}(1-\theta_i)\, q_i(\phi)\right)},\end{aligned}$$

where the inequality (2.1) was used. So, we have

$$\boldsymbol{v(\phi)} \geq \boldsymbol{w(\phi)\left(1 - \left[q(\phi)\sum_{i=1}^{i=k}\theta_i + \frac{q(\phi)}{w(\phi)}\sum_{i=1}^{i=k}(1-\theta_i)\right]\right)}, \tag{2.9}$$

where $\boldsymbol{q(\phi)} := \boldsymbol{\max_{0\leq i\leq k} q_i(\phi)}$.

Substituting (2.8) into (2.9), we get

$$\boldsymbol{v(\phi)} \geq 2\mathrm{c}\boldsymbol{\left(1 - \left[q(\phi)\sum_{i=1}^{i=k}\theta_i + \frac{q(\phi)}{c}\sum_{i=1}^{i=k}(1-\theta_i)\right]\right)}.$$

When considering [H3], we have

$$\boldsymbol{v(\phi)} \geq \mathrm{c} > 0, \boldsymbol{\phi} \ \geq \boldsymbol{\phi_1}. \tag{2.10}$$

When we use (2.10) in the derived inequality after integrating (2.7) from $\boldsymbol{\phi}$ to ∞, we get

$$\mathrm{w}'\boldsymbol{(\phi)} \geq \boldsymbol{\frac{c^\gamma}{b(\phi)} \int_\phi^\infty p(\alpha)\, d\alpha}.$$

By further integrating the prior inequality from $\boldsymbol{\phi_1}$ to $\boldsymbol{\phi}$, we can show that

$$\boldsymbol{w(\phi)} \geq \mathrm{w}\boldsymbol{(\phi_1) + c^\gamma \int_{\phi_1}^{\phi} \frac{1}{b(u)} \int_u^\infty p(\alpha)\, d\alpha\, du},$$

which, with (2.5), suggests that $\boldsymbol{w(\phi)} \longrightarrow \infty$ as $\boldsymbol{\phi} \longrightarrow \infty$. Thus, the proof of the lemma is complete. ■

3. Oscillation Criteria

In this section, we introduce some new oscillation criteria for solutions of (1.1). In order to provide a clear presentation, we define the following functions:

$$\mathrm{P}\boldsymbol{(\phi)} := \boldsymbol{q(\phi)\sum_{i=1}^{i=k}\theta_i + \frac{q(\phi)}{w(\phi)}\sum_{i=1}^{i=k}(1-\theta_i)},$$

$$\boldsymbol{I_1(\phi)} := \mathrm{I}\boldsymbol{(\phi, \phi_1)} + \boldsymbol{\int_{\phi_1}^{\phi} \frac{p(\alpha)}{\alpha^\gamma}\, \beta^\gamma\, \xi^\gamma(\alpha)\, \rho(\alpha)\, I^2(\alpha, \phi_1)\, d\alpha},$$

$$\boldsymbol{I_2(\phi)} := \exp\boldsymbol{\left(-\int_{\xi(\phi)}^{\phi} \frac{1}{\tilde{I}(\alpha)\, b(\alpha)}\, d\alpha\right)},$$

and

$$\boldsymbol{\rho}(\boldsymbol{\phi}) := \begin{cases} \boldsymbol{\lambda}, & \boldsymbol{if\ \gamma > 1}; \\ \boldsymbol{1}, & \boldsymbol{if\ \gamma = 1}, \end{cases}$$

where $\boldsymbol{\lambda}$ and $\boldsymbol{\beta} \in (0, 1)$.

Now using the comparison principle, we obtain the following theorem.

Theorem 3.1. Assume that the condition (2.5) holds. If the first order delay differential equation

$$\boldsymbol{y}'(\boldsymbol{\phi}) + \boldsymbol{p}(\boldsymbol{\phi})\, \boldsymbol{\beta}^{\gamma}\, \boldsymbol{y}^{\gamma}(\boldsymbol{\xi}(\boldsymbol{\phi}))\, \boldsymbol{I_1}^{\gamma}(\boldsymbol{\xi}(\boldsymbol{\phi})) = 0, \tag{3.1}$$

is oscillatory, then every solution of (1.1) is oscillatory.

Proof. Assume on the contrary that $\boldsymbol{v}$ is a nonoscillatory solution of (1.1). Then there exists $\boldsymbol{\phi} \geq \boldsymbol{\phi_1} \geq \boldsymbol{\phi_0}$ such that $\boldsymbol{v}(\boldsymbol{\phi}) > 0$, $\boldsymbol{v}(\boldsymbol{\xi}(\boldsymbol{\phi})) > 0$, $\boldsymbol{v}(\boldsymbol{\tau_i}(\boldsymbol{\phi})) > 0$, for $i = 1, 2, \ldots, k$, and (2.6) holds for $\boldsymbol{\phi} \geq \boldsymbol{\phi_1}$. By taking [H3] and the characteristics of $\boldsymbol{w}(\boldsymbol{\phi})$, we obtain

$$\mathrm{P}(\boldsymbol{\phi}) < \boldsymbol{\eta},$$

for any $\boldsymbol{\eta} \in (0, 1)$. Considering the prior inequality and (2.9) that

$$\boldsymbol{v}(\boldsymbol{\phi}) \geq \boldsymbol{\beta}\, \boldsymbol{w}(\boldsymbol{\phi}), \tag{3.2}$$

where $\boldsymbol{\beta} = \boldsymbol{1} - \boldsymbol{\eta} \in (0, 1)$. Substituting (3.2) into (2.7), we get

$$\left(\boldsymbol{b}(\boldsymbol{\phi})\left(\boldsymbol{w}'(\boldsymbol{\phi})\right)\right)' + \boldsymbol{p}(\boldsymbol{\phi})\, \boldsymbol{\beta}^{\gamma}\, \boldsymbol{w}^{\gamma}(\boldsymbol{\xi}(\boldsymbol{\phi})) \leq 0.$$

That is

$$\left(\boldsymbol{b}(\boldsymbol{\phi})\left(\boldsymbol{w}'(\boldsymbol{\phi})\right)\right)' \leq -\, \boldsymbol{p}(\boldsymbol{\phi})\, \boldsymbol{\beta}^{\gamma}\, \boldsymbol{w}^{\gamma}(\boldsymbol{\xi}(\boldsymbol{\phi})). \tag{3.3}$$

By the result of monotony of $\boldsymbol{b}(\boldsymbol{\phi})\, \boldsymbol{w}'(\boldsymbol{\phi})$ that

$$\begin{aligned} \boldsymbol{w}(\boldsymbol{\phi}) &= \mathrm{w}(\boldsymbol{\phi_1}) + \int_{\boldsymbol{\phi_1}}^{\boldsymbol{\phi}} \frac{\boldsymbol{b}(\boldsymbol{\alpha})\, \boldsymbol{w}'(\boldsymbol{\alpha})}{\boldsymbol{b}(\boldsymbol{\alpha})}\, \boldsymbol{d\alpha} \\ &\geq \boldsymbol{b}(\boldsymbol{\phi})\, \boldsymbol{w}'(\boldsymbol{\phi}) \int_{\boldsymbol{\phi_1}}^{\boldsymbol{\phi}} \frac{\boldsymbol{1}}{\boldsymbol{b}(\boldsymbol{\alpha})}\, \boldsymbol{d\alpha} \\ &= \boldsymbol{b}(\boldsymbol{\phi})\, \boldsymbol{w}'(\boldsymbol{\phi})\, \mathrm{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1}). \end{aligned}$$

That is

$$\boldsymbol{w}(\boldsymbol{\phi}) \geq \boldsymbol{b}(\boldsymbol{\phi})\, \boldsymbol{w}'(\boldsymbol{\phi})\, \mathrm{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1}). \tag{3.4}$$

Now a simple computation shows that

$$\left(\boldsymbol{w}(\boldsymbol{\phi}) - \boldsymbol{b}(\boldsymbol{\phi})\, \boldsymbol{w}'(\boldsymbol{\phi})\, \boldsymbol{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1})\right)' = -\left(\boldsymbol{b}(\boldsymbol{\phi})\left(\boldsymbol{w}'(\boldsymbol{\phi})\right)\right)' \mathrm{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1}). \tag{3.5}$$

Hence,

$$-\left(\boldsymbol{b}(\boldsymbol{\phi})\left(\boldsymbol{w}'(\boldsymbol{\phi})\right)\right)' \mathrm{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1}) \geq \boldsymbol{p}(\boldsymbol{\phi})\, \boldsymbol{\beta}^{\gamma}\, \boldsymbol{w}^{\gamma}(\boldsymbol{\xi}(\boldsymbol{\phi}))\, \mathrm{I}(\boldsymbol{\phi}, \boldsymbol{\phi_1}). \tag{3.6}$$

When we combine (3.5) with (3.6), we get

$$(w(\phi) - b(\phi)\, w'(\phi)\, I(\phi, \phi_1))' \geq p(\phi)\, \beta^\gamma\, w^\gamma(\xi(\phi))\, \mathrm{I}(\phi, \phi_1).$$

From (2.2), we get

$$(w(\phi) - b(\phi)\, w'(\phi)\, I(\phi, \phi_1))' \geq p(\phi)\, \beta^\gamma \left(\frac{\omega}{\phi}\ \xi(\phi)\, w(\phi)\right)^\gamma \mathrm{I}(\phi, \phi_1)$$

$$= \frac{p(\phi)}{\phi^\gamma}\, \beta^\gamma\, \omega^\gamma\, \xi^\gamma(\phi)\, w^{\gamma-1}(\phi)\, w(\phi)\, \mathrm{I}(\phi, \phi_1).$$

By using (3.4),

$$(w(\phi) - b(\phi)\, w'(\phi)\, I(\phi, \phi_1))' \geq \frac{p(\phi)}{\phi^\gamma}\, \beta^\gamma\, \omega^\gamma\, \xi^\gamma(\phi)\, w^{\gamma-1}(\phi)\, \mathrm{b}(\phi)\, w'(\phi)\, I^2(\phi, \phi_1). \tag{3.7}$$

Given that $w(\phi)$ is positive and increasing, we now have that

$$w(\phi) \geq \mathrm{w}(\phi_2) \geq \mu > 0,$$

for $\phi \geq \phi_2 \geq \phi_1$. Then

$$w^{\gamma-1}(\phi) \geq \begin{cases} \lambda, & if\ \gamma > 1; \\ 1, & if\ \gamma = 1, \end{cases} \tag{3.8}$$

for $\phi \geq \phi_2$, where $\lambda = \mu^{\gamma-1}$. Thus,

$$w^{\gamma-1}(\phi) \geq \rho(\phi), \tag{3.9}$$

for some $\lambda \in (0, 1)$. Combining (3.7) and (3.9), we obtain

$$(w(\phi) - b(\phi)\, w'(\phi)\, I(\phi, \phi_1))' \geq \frac{p(\phi)}{\phi^\gamma}\, \beta^\gamma\, \omega^\gamma\, \xi^\gamma(\phi)\, \rho(\phi)\, \mathrm{b}(\phi)\, w'(\phi)\, I^2(\phi, \phi_1).$$

Integrating the last inequality from ϕ_1 to ϕ, we have

$$w(\phi) \geq b(\phi)\, w'(\phi)\, I(\phi, \phi_1) + \omega^\gamma \int_{\phi_1}^{\phi} \frac{p(\alpha)}{\alpha^\gamma}\, \beta^\gamma\, \xi^\gamma(\alpha)\, \rho(\alpha)\, b(\alpha)\, w'(\alpha)\, I^2(\alpha, \phi_1)\ d\alpha.$$

By the result of monotony $b(\phi)\, w'(\phi)$ and (3.4), it gives

$$w(\phi) \geq b(\phi)\, w'(\phi) \left[I(\phi, \phi_1) + \int_{\phi_1}^{\phi} \frac{p(\alpha)}{\alpha^\gamma}\, \beta^\gamma\, \xi^\gamma(\alpha)\, \rho(\alpha)\, I^2(\alpha, \phi_1)\ d\alpha\right]. \tag{3.10}$$

Consequently, we deduce that

$$\mathrm{w}(\xi(\phi)) \geq b(\xi(\phi))\, w'(\xi(\phi))\, I_1(\xi(\phi)). \tag{3.11}$$

Using (3.11) in (3.3), it is clearly seen that $\mathrm{y}(\phi) := b(\phi)\, w'(\phi)$ is a positive solution of the first order delay differential inequality

$$y'(\phi) + \mathrm{p}(\phi)\, \beta^\gamma\, y^\gamma(\xi(\phi))\, {I_1}^\gamma(\xi(\phi)) \leq 0. \tag{3.12}$$

But by (Theorem 1 [10]), the following associated delay differential equation

$$y'(\phi) + \mathrm{p}(\phi)\, \beta^\gamma\, y^\gamma(\xi(\phi))\, {I_1}^\gamma(\xi(\phi)) = 0,$$

must also have a positive solution, which is a contradiction. Hence, the proof is complete. ■

Using the results in [5] and [13], the following corollary to Theorem 3.1 can be easily obtained.

Corollary 3.1. Assume that the conditions of Theorem 3.1 hold. If

$$\liminf_{\phi\to\infty} \int_{\xi(\phi)}^{\phi} p(\alpha)\, I_1^{\gamma}(\xi(\alpha))\, d\alpha > \frac{1}{e}, \tag{3.13}$$

for every λ and $\beta \in (0, 1)$, then every solution of (1.1) is oscillatory.

The following theorem is now obtained by applying the Riccati approach.

Theorem 3.2. Assume that condition (1.2) holds, and there exists a function $\chi \in C^1([\phi_0, \infty), \mathbb{R}^+)$ such that for all sufficiently large S $\geq \phi_0$,

$$\limsup_{\phi\to\infty} \int_{S}^{\phi} \left(\beta^{\gamma}\, \rho(\alpha)\, p(\alpha)\, I_2(\alpha)\, \chi(\alpha) - \frac{\left(\chi'_{+}(\alpha)\right)^2 b(\alpha)}{4\, \chi(\alpha)} \right) d\alpha = \infty, \tag{3.14}$$

where $\chi'_{+}(\alpha) = \max\{0, \chi'(\alpha)\}$ and for every λ and $\beta \in (0, 1)$, then every solution of (1.1) is oscillatory.

Proof. Assume on the contrary that (1.1) has a nonoscillatory solution v on $[\phi_0, \infty)$. Then there exists $\phi \geq \phi_1 \geq \phi_0$ such that $v(\phi) > 0$, $v(\xi(\phi)) > 0$, $v(\tau_i(\phi)) > 0$, for i = 1, 2, …, k, and
(2.5) holds for $\phi \geq \phi_1 \geq \phi_0$. The following definition is a Riccati type transformation

$$\omega(\phi) = \chi(\phi)\, \mathrm{b}(\phi) \left(\frac{w'(\phi)}{w(\phi)}\right), \tag{3.15}$$

for $\phi \geq \phi_0$. Then,

$$\omega(\phi) > 0 \text{ for all } \phi \geq \phi_1.$$

By differentiating (3.15), we get

$$\omega'(\phi) = \frac{\chi'(\phi)}{\chi(\phi)}\, \omega(\phi) + \frac{\left(b(\phi)\left(w'(\phi)\right)\right)'}{w(\phi)}\, \chi(\phi) - \frac{1}{\chi(\phi)\, b(\phi)}\, \omega^2(\phi). \tag{3.16}$$

Because of (3.10), we get

$$\frac{w'(\phi)}{w(\phi)} \leq \frac{1}{I_1(\phi)\, b(\phi)}. \tag{3.17}$$

By integrating the latter inequality from $\xi(\phi)$ to ϕ, we get

$$\frac{w(\xi(\phi))}{w(\phi)} \geq \exp\left(-\int_{\xi(\phi)}^{\phi} \frac{1}{I_1(\alpha)\, b(\alpha)}\, d\alpha\right). \tag{3.18}$$

Combining (3.3) and (3.8), we get

$$\frac{\left(b(\phi)\left(w'(\phi)\right)\right)'}{w(\phi)} \leq -\beta^{\gamma}\, \rho(\phi)\, p(\phi)\, \frac{w(\xi(\phi))}{w(\phi)}.$$

By applying (3.18),

$$\frac{\left(\boldsymbol{b(\phi)}\left(\boldsymbol{w'(\phi)}\right)\right)'}{\boldsymbol{w(\phi)}} \leq -\boldsymbol{\beta^{\gamma}}\,\boldsymbol{\rho(\phi)}\,\boldsymbol{p(\phi)}\exp\left(-\int_{\boldsymbol{\xi(\phi)}}^{\boldsymbol{\phi}}\frac{\boldsymbol{1}}{\boldsymbol{I_1(\alpha)}\,\boldsymbol{b(\alpha)}}\,\boldsymbol{d\alpha}\right)$$

$$= -\boldsymbol{\beta^{\gamma}}\,\boldsymbol{\rho(\phi)}\,\boldsymbol{p(\phi)}\,\boldsymbol{I_2(\phi)}. \tag{3.19}$$

By letting A $:= \frac{\boldsymbol{\chi'(\phi)}}{\boldsymbol{\chi(\phi)}}$ and B $:= \frac{\boldsymbol{1}}{\boldsymbol{\chi(\phi)}\,\boldsymbol{b(\phi)}}$ into (2.3) and (2.4), we obtain

$$\boldsymbol{\mu}(\mathrm{u}) = \frac{\boldsymbol{\chi'(\phi)}}{\boldsymbol{\chi(\phi)}}\,\boldsymbol{\omega(\phi)} - \frac{\boldsymbol{1}}{\boldsymbol{\chi(\phi)}\,\mathbf{b(\phi)}}\,\boldsymbol{\omega^2(\phi)}, \tag{3.20}$$

and

$$\max_{\mathrm{u}\in\mathbb{R}} \mu = \frac{\left(\chi'_+(\phi)\right)^2 \mathrm{b}(\phi)}{4\,\chi(\phi)}. \tag{3.21}$$

Applying (3.19), (3.20) and (3.21) in (3.16), we get

$$\omega'(\phi) \leq -\left[\beta^{\gamma}\,\rho(\phi)\,\mathrm{p}(\phi)\,\mathrm{I}_2(\phi)\,\chi(\phi) - \frac{\left(\chi'_+(\phi)\right)^2 \mathrm{b}(\phi)}{4\,\chi(\phi)}\right]. \tag{3.22}$$

Let $\mathrm{S} \geq \phi_1$ be sufficiently large and integrating (3.22) from S to ϕ to obtain

$$\int_{\mathrm{S}}^{\phi}\left(\beta^{\gamma}\,\rho(\alpha)\,\mathrm{p}(\alpha)\,\mathrm{I}_2(\alpha)\,\chi(\alpha) - \frac{\left(\chi'_+(\alpha)\right)^2 \mathrm{b}(\alpha)}{4\,\chi(\alpha)}\right)\mathrm{d}\alpha \leq \omega(\mathrm{S}),$$

which contradicts (3.14). Hence, the proof is complete. ■

4. Examples

In this section, we present an example to illustrate our main results.

Example 4.1. Consider the following second order differential equation with couple of sublinear neutral terms

$$\left(\mathrm{v}(\phi) + \frac{1}{\phi^2}\,\mathrm{v}^{\frac{1}{5}}\left(\frac{\phi}{4}\right) + \frac{1}{\phi^4}\,\mathrm{v}^{\frac{1}{7}}\left(\frac{\phi}{6}\right)\right)'' + \frac{\mathrm{a}}{\phi^2}\,\mathrm{v}^1\left(\frac{\phi}{3}\right) = 0,\ \phi > 0, \tag{4.1}$$

where $\mathrm{b}(\phi) = 1$, $\mathrm{w}(\phi) = \mathrm{v}(\phi) + \frac{1}{\phi^2}\,\mathrm{v}^{\frac{1}{5}}\left(\frac{\phi}{4}\right) + \frac{1}{\phi^4}\,\mathrm{v}^{\frac{1}{7}}\left(\frac{\phi}{6}\right)$, $\mathrm{k} = 2$, $\mathrm{q}_1(\phi) = \frac{1}{\phi^2}$, $\theta_1 = \frac{1}{5}$, $\tau_1(\phi) = \frac{\phi}{4}$, $\mathrm{q}_2(\phi) = \frac{1}{\phi^4}$, $\theta_2 = \frac{1}{7}$, $\tau_2(\phi) = \frac{\phi}{6}$, $\mathrm{f}\left(\phi, \mathrm{v}\left(\xi(\phi)\right)\right) = \mathrm{p}(\phi)\,\mathrm{v}^{\gamma}\left(\xi(\phi)\right)$, $\mathrm{p}(\phi) = \frac{\mathrm{a}}{\phi^2}$, $\mathrm{a} > 1$, $\gamma = 1$, $\xi(\phi) = \frac{\phi}{3}$.

With our equation, it is simple to verify that

$$\lim_{\phi\to\infty} \mathrm{q}_\mathrm{i}(\phi) = 0 \text{ for i = 1,2,}$$

$$\lim_{\phi\to\infty} \tau_\mathrm{i}(\phi) = \infty \text{ for i = 1,2,}$$

and

$$\lim_{\phi\to\infty} \xi(\phi) = \infty.$$

Along with

$$I(\phi, \phi_1) = \phi \text{ and } I_1(\phi) = \left(1 + \frac{a}{3}\,\beta\right)\phi.$$

For Theorem3.2, we have

$$I_2(\phi) = 3^c, \text{ where } c = -\frac{1}{\left(1+\frac{a}{3}\beta\right)}.$$

Letting $\chi(\phi) = \phi$, condition (3.14) becomes

$$\begin{aligned}\lim_{\phi\to\infty} \sup \int_S^\phi \left(\beta^\gamma\, \rho(\alpha)\, p(\alpha)\, I_2(\alpha)\, \chi(\alpha) - \frac{\left({\chi'}_+(\alpha)\right)^2 b(\alpha)}{4\,\chi(\alpha)}\right) d\alpha \\ = \lim_{\phi\to\infty} \sup \int_S^\phi \left(\beta\, \frac{a}{\alpha^2}\, 3^c\, \alpha - \frac{1}{4\,\alpha}\right) d\alpha \\ = \lim_{\phi\to\infty} \sup \int_S^\phi \left(\beta\, a\, 3^c - \frac{1}{4}\right) \frac{1}{\alpha}\, d\alpha \\ = \lim_{\phi\to\infty} \sup \left(\beta\, a\, 3^c - \frac{1}{4}\right) \ln\left[\frac{\phi}{S}\right] \\ = \infty. \qquad (4.2)\end{aligned}$$

Therefore, if condition (4.2) is satisfied, Equation (4.1) is oscillatory.

Example 4.2. The differential equation (4.1) is once again considered.

For corollary 3.1, condition (3.13) becomes

$$\begin{aligned}\lim_{\phi\to\infty} \inf \int_{\xi(\phi)}^{\phi} p(\alpha)\; {I_1}^\gamma(\xi(\alpha))\, d\alpha = \lim_{\phi\to\infty} \inf \int_{\frac{\phi}{3}}^{\phi} \frac{a}{\alpha^2} \left(1 + \frac{a}{3}\,\beta\right) \frac{\alpha}{3}\, d\alpha \\ > \frac{a}{3}\left(1 + \frac{a}{3}\,\beta\right) \ln 3 \\ > \frac{1}{e}. \qquad (4.3)\end{aligned}$$

Therefore, if condition (4.3) is satisfied, Equation (4.1) is oscillatory.

5. Conclusion

In this paper, we studied the oscillatory behavior of second order quasi-linear differential equations with several sublinear neutral terms and obtained new conditions for the oscillation. The Comparison principle and Riccati approach are two distinct techniques used to achieve the oscillation of the studied equation. Thus, many known results in the literature are extended, improved, and complemented by the data reported in this study.

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