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Bi Conditional Cordial Labeling Of Some Special Graphs

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In this article we propose to conduct a study on Bi conditional Cordial Labeling of Some special graphs. A simple graph G(R,S) is said to be Bi conditional Cordial labeling if there exists a mapping $\lambda: N \rightarrow \{0,1\}$ such that for every $a, b \in R$ we find $ab \in S$ for which

$\lambda^*(ab) = \begin{cases} 1, \text{if } \lambda(a) = \lambda(b) \\ 0, \text{if } \lambda(a) \neq \lambda(b) \end{cases}$. We introduce a combination of different graph structures and have proved the graphs are Bi Conditional Cordial Labeling graphs.

Keywords : Path related graphs, Bi conditional Cordial Labeling,

1. Introduction

For our discussion we consider simple finite graph G(R,S) with vertex set R having p vertices and edge set S having q edges. In understanding the graph labeling concepts by the researchers we found that various labeling techniques have been studied for different graphs and the types of labeling and graphs is given by Gallian J. A[2}. Cahit[1] introduced the cordial labeling technique for a graph and we find many more types of cordial labeling techniques. One among them is Bi Conditional Cordial labeling introduced by Murali, Thirusangu K, Madura Meenakshi [4] . Many researchers have carried out their research in proving various graphs has Bi Conditional Cordial labeling and similarly other cordial labeling varieties. We in this paper discuss Bi Conditional Cordial labeling for some special graphs where we combine the comb graph with a cycle graph , combining path union of cycle graph with ladder graph. For basic terminology we refer Harary .F [3]. Path union of graphs were discussed by P.Jeyanthi et.al [5]. Rosa in his discussion on valuation of graphs is more specific and we have utilized and have used the techniques.[6]

2. Preliminaries

Definition 2.1 : A simple graph G(R,S) is said to be Bi conditional Cordial labeling if there exists a mapping $\lambda: N \rightarrow \{0,1\}$ such that for every $a, b \in R$ we find $ab \in S$ for which

$$\lambda^*(ab) = \begin{cases} 1, \text{if } \lambda(a) = \lambda(b) \\ 0, \text{if } \lambda(a) \neq \lambda(b) \end{cases}.$$

Definition.2.2

Let $G_1, G_2, \ldots, G_n, n \geq 2$ be copies of a graph G. Let v_i be the vertices corresponding to the vertex $v \in V(G)$ on the i^{th} copy of G_i. We denote by $P(n, G^v)$ the graph obtained by adding the edge $v_i v_{i+1}$ to G_i and G_{i+1}, $1 \leq i \leq n-1$ and we call $P(n, G^v)$ the path union of n copies of the graph G.

3. Main Results

Theorem 3.1 : Let $r \geq 3$ and $n \geq 2$ be integers. The path union of Comb cycle $(P(n, C_r), K_m)$ graph is Bi Conditional Cordial labeling graph where $r = 4i, (i = 1,2,3, \ldots)$ and m is even

Proof: Consider $G = (P(n, C_r), K_m)$, the path union of Comb cycle graph with vertex set $R = \left\{a_r^j, 1 \leq r \leq 4i\right\}, \left\{b_i^j\right\}$ where $(i = 1,2,3, \ldots, j = 1,2,3, \ldots)$ and edge set $S = \left\{\left(a_i^j a_{i+1}^j\right), \left(a_{4i-1}^j a_{4i}^j\right)\right\} \cup \left\{\left(a_1^j b_k^j\right)\right\}$ where $(k = 1,2,3, \ldots, m)$. Define a function $\lambda: N \to \{0,1\}$. Now let us label the vertices as follows

$$\lambda(a_{4i-3}^{4j-3}) = 0$$
$$\lambda\left(a_{4i-2}^{4j-3}\right) = 0$$
$$\lambda\left(a_{4i-1}^{4j-3}\right) = 1$$
$$\lambda\left(a_{4i}^{4j-3}\right) = 1$$
$$\lambda(a_{4i-3}^{4j-2}) = 0$$
$$\lambda\left(a_{4i-2}^{4j-2}\right) = 0$$
$$\lambda\left(a_{4i-1}^{4j-2}\right) = 1$$
$$\lambda\left(a_{4i}^{4j-2}\right) = 1$$
$$\lambda\left(a_{4i-3}^{4j-1}\right) = 1$$
$$\lambda\left(a_{4i-2}^{4j-1}\right) = 0$$
$$\lambda\left(a_{4i-1}^{4j-1}\right) = 0$$
$$\lambda\left(a_{4i}^{4j-1}\right) = 1$$
$$\lambda\left(a_{4i-3}^{4j}\right) = 1$$
$$\lambda\left(a_{4i-2}^{4j}\right) = 0$$
$$\lambda\left(a_{4i-1}^{4j}\right) = 0$$
$$\lambda\left(a_{4i}^{4j}\right) = 1$$

$$\lambda\left(b_k^j\right) = \begin{cases} 1 \text{ if } k \equiv 1 \pmod 2 \\ 0 \text{ if } k \equiv 0 \pmod 2 \end{cases}$$

Then we find the induced edge labeling as follows

$$\lambda^*\left(a_i^{4j-3}a_{i+1}^{4j-3}\right) = \begin{cases} 1 \text{ if } i \equiv 1 \pmod 2 \\ 0 \text{ if } i \equiv 0 \pmod 2 \end{cases}$$

$$\lambda^*\left(a_i^{4j-2}a_{i+1}^{4j-2}\right) = \begin{cases} 0 \ \ if\ i \equiv 1\ (mod\ 2) \\ 1 \ \ if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$\lambda^*\left(a_i^{4j-1}a_{i+1}^{4j-1}\right) = \begin{cases} 0 \ \ if\ i \equiv 1\ (mod\ 2) \\ 1\ if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$\lambda^*\left(a_i^{4j}a_{i+1}^{4j}\right) = \begin{cases} 1 \ \ if\ i \equiv 1\ (mod\ 2) \\ 0 \ \ if\ i \equiv 0\ (mod\ 2) \end{cases}$$

$$\lambda^*(a_t^j b_k^j) = \begin{cases} 0\ if\ i \equiv 1\ (mod\ 2) \\ 1\ if\ i \equiv 0\ (mod\ 2) \end{cases} for\ \ t = \ 4i-3, 4i-2 \text{ and } 1 \le k \le m$$

$$\lambda^*(a_t^j b_k^j) = \begin{cases} 1\ if\ i \equiv 1\ (mod\ 2) \\ 0\ if\ i \equiv 0\ (mod\ 2) \end{cases} for\ \ t = \ 4i-1, 4i \text{ , and } 1 \le k \le m$$

We find the induced edge labeling satisfies the condition namely $|v_\lambda(1) - v_\lambda(0)| \le 1$ and $|e_\lambda(1) - e_\lambda(0)| \le 1$ as we find the following table

	Number of Vertices Labeled with 1	Number of Vertices Labeled with 0	Number of Edges Labeled with 1	Number of Edges Labeled with 0
For the cycle Graph of even degree	$\frac{r}{2}$	$\frac{r}{2}$	$\frac{r}{2}$	$\frac{r}{2}$
For the Path attached to the Cycle	-----	-----	Number of Edges labeled with 1 is equal to Number of Edges labeled with or Number of Edges Labeled with 0 is one less than the Number of Edges Labeled with 1	
For K_m	------	----------	Number of Edges Labeled with 1 is equal to Number of Edges Labeled with 0 as m is even.	

Hence the graph path union of Comb cycle graph is Bi Conditional Cordial Labeling graph. Hence the proof.

The following example illustrates the path union of Comb cycle graph

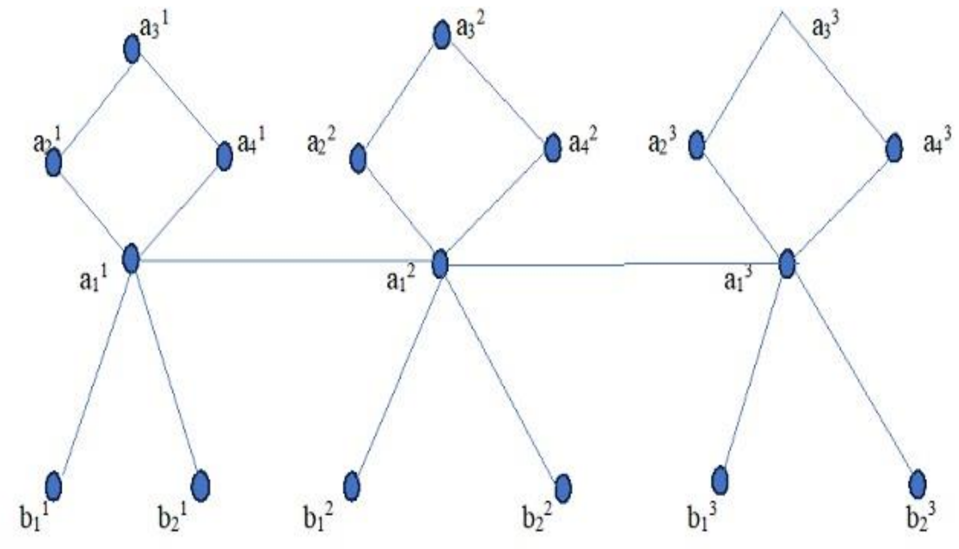


Figure.3.1 : Graph path union of Comb cycle $(P(3, 4_r), K_2)$

3.2 Construction of Two Path Union of Graphs of Comb Cycle Graph

Now let us construct two path union of Comb Cycle $(P(n, C_r), K_m)$ where $r = 4i, (i = 1,2,3, \dots)$and m is even.

We construct a two path union of Comb Cycle $(P(n, C_r), K_m)$ from the path union of Comb Cycle $(P(n, C_r), K_m)$ by considering the following schema of vertices and edges. For the first structure we have
$R^1 = \{a_r^{j1}, 1 \le r \le 4i\}, \{b_i^{j1}\}$ where $(i = 1,2,3, \dots, j = 1,2,3, \dots)$ and $S^1 = \{(a_i^{j1} a_{i+1}^{j1}), (a_{4i-1}^{j1} a_{4i}^{j1})\} \cup \{(a_1^{j1} b_k^{j1})\}$ where $(k = 1,2,3, \dots, m)$ and for the second structure we have
$R^2 = \{a_r^{j2}, 1 \le r \le 4i\}, \{b_i^{j2}\}$ where $(i = 1,2,3, \dots, j = 1,2,3, \dots)$ and $S^2 = \{(a_i^{j2} a_{i+1}^{j2}), (a_{4i-1}^{j2} a_{4i}^{j2})\} \cup \{(a_1^{j2} b_k^{j2})\}$ where $(k = 1,2,3, \dots, m)$
Now for the graph two path union of Comb Cycle $(P(n, C_r), K_m)$ that to be obtained is by connecting the vertices of each K_m of the first structure b_i^{j1} to the vertices of each K_m of the second structure b_k^{j2} by edges . Let us consider the new edge set to be $b_k^{j1} b_k^{j2}$. Since we have considered m to be even we have on connecting even number of edges. Hence a new graph is obtained and we call as two path union of Comb Cycle $(P(n, C_r), K_m)$.
Theorem.3.2 : Let $r \ge 3$ and $n \ge 2, m$ be integers. The two path union of Comb Cycle $(P(n, C_r), K_m)$ graph is Bi Conditional Cordial labeling graph where $r = 4i, (i = 1,2,3, \dots)$and m is even.

Proof: Let G be a graph two path union of Comb Cycle $(P(n, C_r), K_m)$ as constructed above. Define a function $\lambda: N \to \{0,1\}$ .Now let us label the vertices of the first structure as follows

$$\lambda(a_{4i-3}^{(4j-3)1}) = 0$$
$$\lambda\left(a_{4i-2}^{(4j-3)1}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j-3)1}\right) = 1$$
$$\lambda\left(a_{4i}^{(4j-3)1}\right) = 1$$
$$\lambda(a_{4i-3}^{(4j-2)1}) = 0$$
$$\lambda\left(a_{4i-2}^{(4j-2)1}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j-2)1}\right) = 1$$
$$\lambda\left(a_{4i}^{(4j-2)1}\right) = 1$$
$$\lambda\left(a_{4i-3}^{(4j-1)1}\right) = 1$$
$$\lambda\left(a_{4i-2}^{(4j-1)1}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j-1)1}\right) = 0$$
$$\lambda\left(a_{4i}^{(4j-1)1}\right) = 1$$
$$\lambda\left(a_{4i-3}^{(4j)1}\right) = 1$$
$$\lambda\left(a_{4i-2}^{(4j)1}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j)1}\right) = 0$$
$$\lambda\left(a_{4i}^{(4j)1}\right) = 1$$

$$\lambda\left(b_k^{j1}\right) = \begin{cases} 1 \ if \ k \equiv 1 \ (mod\ 2) \\ 0 \ if \ k \equiv 0 \ (mod\ 2) \end{cases}$$

Let us label the vertices of the second structure in a similar way as like the first structure as follows

$$\lambda(a_{4i-3}^{(4j-3)2}) = 0$$
$$\lambda\left(a_{4i-2}^{(4j-3)2}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j-3)2}\right) = 1$$
$$\lambda\left(a_{4i}^{(4j-3)2}\right) = 1$$
$$\lambda(a_{4i-3}^{(4j-2)2}) = 0$$
$$\lambda\left(a_{4i-2}^{(4j-2)2}\right) = 0$$
$$\lambda\left(a_{4i-1}^{(4j-2)2}\right) = 1$$

$$\lambda\left(a_{4i}^{(4j-2)2}\right)=1$$
$$\lambda\left(a_{4i-3}^{(4j-1)2}\right)=1$$
$$\lambda\left(a_{4i-2}^{(4j-1)2}\right)=0$$
$$\lambda\left(a_{4i-1}^{(4j-1)2}\right)=0$$
$$\lambda\left(a_{4i}^{(4j-1)2}\right)=1$$

$$\lambda\left(a_{4i-3}^{(4j)2}\right)=1$$

$$\lambda\left(a_{4i-2}^{(4j)2}\right)=0$$
$$\lambda\left(a_{4i-1}^{(4j)2}\right)=0$$
$$\lambda\left(a_{4i}^{(4j)2}\right)=1$$

$$\lambda\left(b_k^{j2}\right)=\begin{cases}1\ if\ k\equiv 1\ (mod\ 2)\\ 0\ if\ k\equiv 0\ (mod\ 2)\end{cases}$$

Then we find the induced edge labeling as follows for the first structure

$$\lambda^*\left(a_i^{(4j-3)1}a_{i+1}^{(4j-3)1}\right)=\begin{cases}1\ \ if\ i\equiv 1\ (mod\ 2)\\ 0\ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j-2)1}a_{i+1}^{(4j-2)1}\right)=\begin{cases}0\ \ if\ i\equiv 1\ (mod\ 2)\\ 1\ \ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j-1)1}a_{i+1}^{(4j-1)1}\right)=\begin{cases}0\ \ if\ i\equiv 1\ (mod\ 2)\\ 1\ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j)1}a_{i+1}^{(4j)1}\right)=\begin{cases}1\ \ if\ i\equiv 1\ (mod\ 2)\\ 0\ \ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$\lambda^*(a_t^{j1}b_k^{j1})=\begin{cases}0\ if\ i\equiv 1\ (mod\ 2)\\ 1\ if\ i\equiv 0\ (mod\ 2)\end{cases}$ $for\ \ t=\ 4i-3,4i-2$ and $1\le k\le m$

$\lambda^*(a_t^{j1}b_k^{j1})=\begin{cases}1\ if\ i\equiv 1\ (mod\ 2)\\ 0\ if\ i\equiv 0\ (mod\ 2)\end{cases}$ $for\ \ t=\ 4i-1,4i$, and $1\le k\le m$

Also we find the induced edge labeling for the second structure

$$\lambda^*\left(a_i^{(4j-3)2}a_{i+1}^{(4j-3)2}\right)=\begin{cases}1\ \ if\ i\equiv 1\ (mod\ 2)\\ 0\ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j-2)2}a_{i+1}^{(4j-2)2}\right)=\begin{cases}0\ \ if\ i\equiv 1\ (mod\ 2)\\ 1\ \ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j-1)2}a_{i+1}^{(4j-1)2}\right)=\begin{cases}0\ \ if\ i\equiv 1\ (mod\ 2)\\ 1\ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$$\lambda^*\left(a_i^{(4j)2}a_{i+1}^{(4j)2}\right)=\begin{cases}1 \ if \ i\equiv 1\ (mod\ 2)\\ 0\ \ if\ i\equiv 0\ (mod\ 2)\end{cases}$$

$\lambda^*(a_t^{j2}b_k^{j2})=\begin{cases}0\ if\ i\equiv 1\ (mod\ 2)\\ 1\ if\ i\equiv 0\ (mod\ 2)\end{cases}$ $for\ \ t=\ 4i-3{,}4i-2$ and $1\le k\le m$

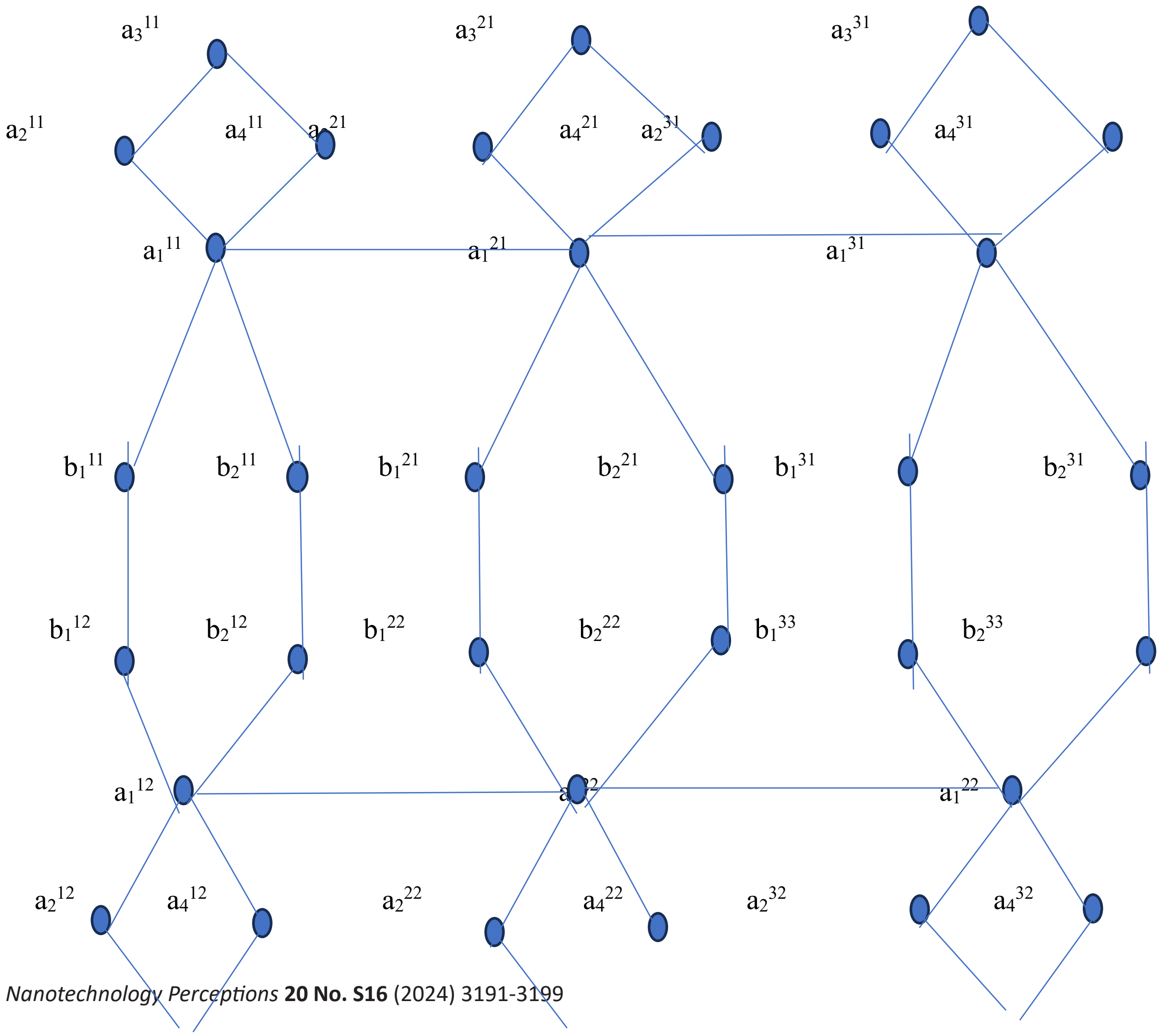
$\lambda^*(a_t^{j2}b_k^{j2})=\begin{cases}1\ if\ i\equiv 1\ (mod\ 2)\\ 0\ if\ i\equiv 0\ (mod\ 2)\end{cases}$ $for\ \ t=\ 4i-1{,}4i$, and $1\le k\le m$

Also we find that the edges connecting the first structure with edges connecting the second structure as

$\lambda^*(b_k^{j1}b_k^{j2})=\begin{cases}0\\ 1\end{cases}$, Accordingly, we consider the vertices of b_k^{j1}, b_k^{j2} alternatively as 0 and 1.

Hence we find that the two path union of Comb Cycle $(P(n,C_r),K_m)$ satisfies the condition $|v_\lambda(1)-v_\lambda(0)|\le 1$ and $|e_\lambda(1)-e_\lambda(0)|\le 1$ and proves that the graph is Bi Conditional Cordial labeling graph. Hence the proof.

The following example illustrates the two path union of Comb Cycle graph.



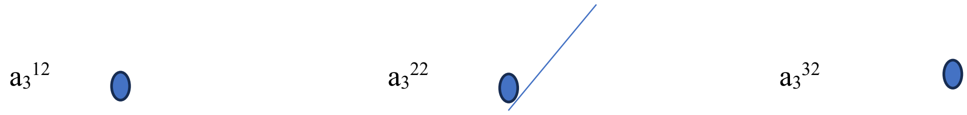


Figure: 3.2 : Two Path Union of Comb Cycle $(P(3, C_4), K_2)$ graph

We now generalize the above structure and we can prove the following theorem from the above schema of labeling of vertices.

Theorem.3.3 : Let $r \geq 3$ and $n \geq 2, m$ be integers. The multi path union of Comb Cycle $(P(n, C_r), K_m)$ graph is Bi Conditional Cordial labeling graph where $r = 4i, (i = 1,2,3, \ldots)$ and m is even.

3.3 **Construction of path union of Comb cycle ladder $(\boldsymbol{P(n, (C_{r,}, L_r))}, \boldsymbol{K_m})$ graph.**

In the procedure adopted in theorem .3.1 we construct on each vertex of a cycle a ladder graph L_r Then the vertex set of the new graph $(P(n, (C_{r,}, L_r))), K_m)$ so obtained is as follows $R = \{a_r^j, 1 \leq r \leq 4i\}, \{b_i^j\}, \{c_i^j\}, \{d_i^j\}$ where $(i = 1,2,3, \ldots, j = 1,2,3, \ldots)$ and the edge set is as follows $S = \{(a_i^j a_{i+1}^j\ ), (a_{4i-1}^j a_{4i}^j\ )\} \cup \{(a_1^j b_k^j)\} \cup \{c_i^j c_{i+1}^j\}$ where $(k = 1,2,3, \ldots, m)$. We fuse the vertex of the cycle a_i^j with the vertex of the ladder graph c_i^j for which the cycle graph vertex is labeled with 0 . We obtain the new graph.. On labeling as in theorem 3.1 the vertices of the cycle graph and the vertices of K_m and the new vertices of the ladder graph attached in the order 0,0,1,1 for the vertices of c_i^j and 1,0,0,1 for the vertices of d_i^j we find that the vertices of the new graph is labeled. We can also find that the induced edge labeling of the new edges formed by the ladder graph satisfies the Bi Conditional Cordial labeling condition as we are considering even number of vertices for the ladder graph.

Theorem.3.4 : Let $r \geq 3$ and $n \geq 2, m$ be integers. The path union of Comb cycle ladder $(P(n, (C_{r,}, L_r))), K_m)$ graph is Bi Conditional Cordial labeling graph where $r = 4i, (i = 1,2,3, \ldots)$ and m is even.

Proof: From the above schema of construction of the graph we can find that the graph path union of Comb cycle ladder $(P(n, (C_{r,}, L_r))), K_m)$ is Bi Conditional Cordial labeling graph.

We can generalize the structure by attaching a multi path to the path union of Comb cycle ladder graph and the following theorem can be proved .

Theorem.3.5 : Let $r \geq 3$ and $n \geq 2, m$ be integers. The multi path union of Comb cycle ladder $(P(n, (C_{r,}, L_n))), K_{2m})$ graph is Bi Conditional Cordial labeling graph where $r = 4i - 1, (i = 1,2,3, \ldots)$.

Conclusion: In this paper we study on Bi Conditional Cordial Labeling of path union of Comb cycle $(P(n, C_r), K_m)$ graph, two path union of Comb cycle $(P(n, C_r), K_m)$ graph and extend the same to multi path union of Comb cycle $(P(n, C_r), K_m)$ graph. We also identify that on attaching a ladder graph of same size of the cycle graph to the vertex of the cycle of path union of Comb cycle $(P(n, C_r), K_m)$ we obtain a new graph path union of Comb cycle ladder $(P(n, (C_{r,}, L_r))), K_m)$ which is also satisfies the Bi Conditional Cordial labeling condition . We extend the same to ta multi path union of Comb cycle ladder path union of Comb cycle ladder $(P(n, (C_{r,}, L_r))), K_m)$. We further like to continue our study on identifying graphs which are suitable for labeling so as to prove that the graphs as Bi Conditional Cordial labeling.

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