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# A Derivative Topology On Differential Rings

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The objective of this paper is to introduce the notion of derivative topology on a differential ring R with the help of differential subsets of a differential ring. The derivative topology on R is a collection τ of differential subsets of R, which has the structure that ∅, R are in τ and is closed under arbitrary union and arbitrary intersection. The members of τ are called the differential open sets of the differential ring R. Also, we provide some examples of derivative topologies. Further, we introduce subderivative topology in which the elements are written as the intersection of the differential subring of the differential ring R and the elements of τ. Finally, we give the result which will connect the differential open sets in derivative topology and subderivative topology.

**Key words :** Differential ring, Differential Subring, Derivative topology, Subderivative topology

## 1. Introduction

Throughout this paper, We focus only on differential rings. Let R denote the differential ring unless otherwise specified. The notion of the ring with derivation is quite old and plays a significant role in the integration of analysis, algebraic geometry and algebra. In 1950s a new part of algebra called differential algebra was initiated by the works of Ritt and Kolchin. In 1950 Ritt [8] and in 1973 Kolchin [4] wrote the well known books on differential algebra in which differential rings, differential fields and differential algebras are rings, fields and algebras equipped with finitely many derivations. Kaplansky, too, wrote an interesting book on this subject in 1957 [7].

For many years, various authors [1,2,5] constructed some topologies over algebraic structures and they investigated the relations between the algebraic properties of given algebraic structures (such as rings, modules, lattices and fuzzy structures) and topological properties of these topologies.

Our aim, in this paper, is to introduce a new topology, namely, derivative topology $\tau$ which has the structure that $\emptyset$, R are in $\tau$ and is closed under arbitrary union and arbitrary intersection. We observe that a derivative topology is different from usual topology.

Section 2 deals with the preliminary concepts. In section 3, the derivative topology on a differential ring and subderivative topology on a differential subring are introduced and examples are given.

**2. Preliminaries**

In this section, we present some basic definitions and examples of differential algebra that are very useful in the subsequent discussions. Throughout the work, R denote the differential ring.

**Definition 2.1:** A non empty set R together with two binary operations denoted by "+" and "." are called addition and multiplication which satisfy the following axioms is called a ring.

(i) (R,+) is an abelian group
(ii) "." is an associative binary operation on R.
(iii) a.(b+c)= a.b+a.c and (a+b).c= a.c+b.c for all a,b,c $\in$R.

A ring R is said to be Commutative if ab=ba for all a,b $\in$R. R is a ring with identity if there exists an element $1 \in$ R such that $a1 = 1a = a$ for all a $\in$ R.

**Definition 2.2:** Let R be a commutative ring with identity. A derivation on R is a map d: R$\rightarrow$ R such that

(i) d(a+b)= d(a)+d(b)
(ii) d(a.b)= d(a).b+a.d(b)

A differential ring is a commutative ring with identity R together with the distinguished derivation d. If R is a differential ring and x $\in$R, then write $x'$ for dx, $x''$ for $d^2x$ and in general $x^{(n)}$for $d^n x$.

Let R be a commutative ring with identity. A derivation d on R is said to be the trivial derivation if $d(a) = 0$ for all a $\in$ R.

The derivative of a non empty set S is denoted by d(S) and defined as d(S) = $\{$d(a)/a$\in$ S$\}$.

**Definition 2.3:** A subset S of a differential ring R is called a differential subset if it contains the derivative of each of its elements. Equivalently, d(S)$\subseteq$ S, where d(S) is the derivative of S.

**Example 2.4:** Consider the ring of integers $\mathbb{Z}$. Define $d: \mathbb{Z} \rightarrow \mathbb{Z}$ by $d(a) = 0$ for all a $\in \mathbb{Z}$. Then d(a+b)=0=d(a)+d(b). Clearly, d(a.b)=0. Also, d(a).b+a.d(b)=0.b+a.0=0. Therefore, d(a.b)=d(a).b+a d(b). Hence, d is a derivation on $\mathbb{Z}$. Thus $\mathbb{Z}$ is a differential ring.

In general, any commutative ring with identity may be converted into a differential ring by imposing the trivial derivation.

**Example 2.5:** Let R be a ring and R[x] be the polynomial ring over R. Define d: R[x] →R[x] by $d(f(x)) = d(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} i a_i x^{i-1}$, where $a_i \in R$.

Let f(x) = $\sum_{i=0}^{n} a_i x^i$ and g(x) =$\sum_{i=0}^{n} b_i x^i$, where $a_i, b_i \in$ R. Then d(f+g)= $\sum_{i=0}^{n} i(a_i + b_i)x^{i-1}$ and d(f)+d(g)= $\sum_{i=0}^{n} i a_i x^{i-1}$+$\sum_{i=0}^{n} i b_i x^{i-1}$= $\sum_{i=0}^{n} i(a_i + b_i)x^{i-1}$. Also, d(f.g)= d(f).g+ f. d(g). Therefore, d is a derivation and hence R[x] is a differential ring.

**Example 2.6:** Let K be a field and K[x] be the polynomial ring over K. Define d: K[x] →K[x] by $d(f(x)) = d(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} i a_i x^{i-1}$, where $a_i \in K$.

Let f(x) = $\sum_{i=0}^{n} a_i x^i$ and g(x) = $\sum_{i=0}^{n} b_i x^i$, where $a_i, b_i \in$ K. Then d(f+g)= $\sum_{i=0}^{n} i(a_i + b_i)x^{i-1}$ and d(f)+d(g)= $\sum_{i=0}^{n} i a_i x^{i-1}$+$\sum_{i=0}^{n} i b_i x^{i-1}$= $\sum_{i=0}^{n} i(a_i + b_i)x^{i-1}$. Also, d(f.g)= d(f).g+ f. d(g) . Therefore, d is a derivation and hence K[x] is a differential ring.

**Example 2.7:** Consider the polynomial ring $Z_6$[x]. Let $S_1$ be the set of all linear polynomials in $Z_6$[x]. Then d($S_1$)= the set of all constant polynomials in $Z_6$[x], which is subset of $S_1$. Therefore, $S_1$ is a differential subset of $Z_6$[x].

**Definition 2.8:** A differential subring is a subring of a differential ring that is closed under the derivation of the ring.

**Example 2.9:** $\mathbb{Z}$ [x] is a differential subring of $\mathbb{Q}$ [x]. For, Define d: $\mathbb{Q}$[x]→ $\mathbb{Q}$ [x] by $d(f(x)) = d(\sum_{i=0}^{n} a_i x^i) = \sum_{i=0}^{n} i a_i x^{i-1}$, where $a_i \in \mathbb{Q}$. Then d is a derivation on $\mathbb{Q}$ [x] and hence $\mathbb{Q}$ [x] is a differential ring. Since $\mathbb{Z}$ [x] itself is a ring, $\mathbb{Z}$ [x] is a subring of $\mathbb{Q}$ [x]. If f(x) =$\sum_{i=0}^{n} a_i x^i$ where $a_i \in \mathbb{Z}$ is in $\mathbb{Z}$ [x], then $f'(x) = \sum_{i=0}^{n} i a_i x^{i-1} \in \mathbb{Z}$ [x]. Therefore $\mathbb{Z}$ [x] is closed under derivation of $\mathbb{Q}$ [x]. Hence, $\mathbb{Z}$ [x] is a differential subring of $\mathbb{Q}$ [x].

## 3. Derivative topology

In this section, we introduce a new class of topology, namely, derivative topology and study their properties. First, we begin with the proposition which leads to derivative topology.

**Proposition 3.1:** Let R be a differential ring with derivation d. Let $\tau$ be a collection of differential subsets of R. Then

1. $\emptyset, R \in \tau$
2. Arbitrary union of differential subsets of R is a differential subset of R.
3. Arbitrary intersection of differential subsets of R is a differential subset of R.

**Proof:**

1. Clearly $d(\emptyset) \subseteq \emptyset$. Since R is a differential ring, we have R is closed under derivation d. Therefore, $d(R) \subseteq R$ . Hence $\emptyset, R \in \tau$.
2. Let $\{A_i / i \in I\}$ be an indexed family of elements of $\tau$. We shall show that $\bigcup_{i \in I} A_i \in \tau$. Let $x \in \bigcup_{i \in I} A_i$. Then $x \in A_i$ for some $i \in I$. Since $A_i$ is a differential subset of R, we have $x' \in A_i$. Therefore, $x' \in \bigcup_{i \in I} A_i$. Hence, $\bigcup_{i \in I} A_i$ is a differential subset of R.

3. Let $\{A_i / i \in I\}$ be an indexed family of elements of $\tau$. We shall show that $\bigcap_{i\in I} A_i \in \tau$. Let $x \in \bigcap_{i\in I} A_i$. Then $x \in A_i$ for each $i \in I$. Since each $A_i$ is a differential subset of R, we have $x' \in A_i$ for each $i \in I$. Therefore, $x' \in \bigcap_{i\in I} A_i$. Hence $\bigcap_{i\in I} A_i$ is a differential subset of R.

From proposition 3.1, we observe that the collection of differential subsets on a differential ring R forms a topology on the differential ring R, called the derivative topology. A differential ring R for which a derivative topology $\tau$ has been specified is called a derivative topological space. We denote the derivative topological space by the triplet (R, d, $\tau$) consisting of the differential ring R , a derivation d on R and a derivative topology $\tau$ on R.

If R is a derivative topological space with derivative topology $\tau$, we say that a differential subset U of R is a differential open set of R if U belongs to $\tau$.

A derivative topology is a general topology but not conversely.

**Example 3.2:** Consider the differential ring $\mathbb{Q}[x]$, the set of all polynomials with rational coefficients with the derivation d (usual differentiation).

1. Let $S_1 = \emptyset$. Then $S_1$ is a differential subset of $\mathbb{Q}[x]$.

2. Let $S_2 = \{0\}$, the set containing only the zero polynomial. Then $S_2$ is a differential subset of $\mathbb{Q}[x]$.

3. Let $S_3 = \mathbb{Q}[x]$.Then $S_3$ is a differential subset of $\mathbb{Q}[x]$

4. Let $S_4 = \mathbb{Q}$, the set of all constant polynomials of $\mathbb{Q}$ [x]. Then $S_4$ is a differential subset of $\mathbb{Q}[x]$.

5. Let $S_5 =$ the set of all polynomials of degree at most 1 $=\{a_0 + a_1 x / a_0, a_1 \in \mathbb{Q}\}$. Then $d(a_0 + a_1 x) = a_1$, which is a polynomial of degree 0. Therefore, $d(S_5) \subseteq S_5$. Hence, $S_5$ is a differential subset of $\mathbb{Q}[x]$.

Let $\tau = \{S_1, S_2, S_3, S_4, S_5\}$. Now we check $\tau$ is a derivative topology on $\mathbb{Q}[x]$.

i. Clearly, $S_1 = \emptyset$ and $S_3 = \mathbb{Q}[x]$ are in $\tau$.

ii. $S_1 \cup S_2 = S_2$ , $S_1 \cup S_3 = S_3$, $S_1 \cup S_4 = S_4$, $S_1 \cup S_5 = S_5$ , $S_2 \cup S_3 = S_3$, $S_2 \cup S_4 = S_4$, $S_2 \cup S_5 = S_5$, $S_3 \cup S_4 = S_3$, $S_3 \cup S_5 = S_3$ and $S_4 \cup S_5 = S_5$. Hence $\tau$ is closed under arbitrary union.

iii. $S_1 \cap S_2 = S_1$ , $S_1 \cap S_3 = S_1$, $S_1 \cap S_4 = S_1$, $S_1 \cap S_5 = S_1$ , $S_2 \cap S_3 = S_2$, $S_2 \cap S_4 = S_2$, $S_2 \cap S_5 = S_2$, $S_3 \cap S_4 = S_4$, $S_3 \cap S_5 = S_5$ and $S_4 \cap S_5 = S_4$. Hence $\tau$ is closed under arbitrary intersection

Then $\tau$ is a derivative topology on $\mathbb{Q}[x]$.

**Definition 3.3:** Let (R, d,$\tau$) be a derivative topological space. A subset A of R is said to be differential closed if its complement is differential open subset of R.

**Proposition 3.4:** Let (R, d, $\tau$) be a derivative topological space. Then the following properties hold:

1. $\emptyset$ and R are differential closed sets

2. Arbitrary intersection of differential closed set is differential closed set.
3. Arbitrary union of differential closed set is differential closed set.

**Proof:** The proof follows from proposition 3.1. □

Next, we will derive a derivative topology on a differential subring of a differential ring R, which is called as subderivative topology on a differential subring.

**Proposition 3.5:** Let R be a differential ring with derivation d and with the derivative topology $\tau$. Let S be a differential subring of R. Then $\tau_S = \{S \cap U / U \in \tau\}$ forms a derivative topology on S.

**Proof:**

1. Since $S \cap \emptyset = \emptyset$ and $\emptyset \in \tau$, we have $\emptyset \in \tau_S$. Also $S \cap R = S$ and $R \in \tau$. Therefore $R \in \tau_S$.
2. Let $\{S \cap U_i / U_i \in \tau, i \in I\}$ be the family sets in $\tau_S$. Then by distributive law, $\bigcup_{i \in I}(S \cap U_i) = S \cap (\bigcup_{i \in I} U_i)$. Since $U_i \in \tau$, we have $\bigcup_{i \in I} U_i \in \tau$. Therefore, $S \cap (\bigcup_{i \in I} U_i) \in \tau_S$. Hence, $\bigcup_{i \in I}(S \cap U_i) \in \tau_S$. Thus $\tau_S$ is closed under arbitrary union.
3. Let $\{S \cap U_i / U_i \in \tau, i \in I\}$ be the family of sets in $\tau_S$. Then $\bigcap_{i \in I} S \cap U_i = S \cap (\bigcap_{i \in I} U_i)$. Since $U_i \in \tau$, we have $\bigcap_{i \in I} U_i \in \tau$. Therefore, $S \cap (\bigcap_{i \in I} U_i) \in \tau_S$. Hence, $\bigcap_{i \in I} S \cap U_i \in \tau_S$. Thus, $\tau_S$ is closed under arbitrary intersection.
   Hence, $\tau_S = \{S \cap U / U \in \tau\}$ forms a derivative topology on S. □

The derivative topology defined on a differential subring S is called the subderivative topology.

**Example 3.6:** Consider the differential ring Q[x], set of all polynomials with rational co efficients with the derivation d (usual differentiation) .

1. Let $S_1 = \emptyset$. Then $S_1$ is a differential subset of $\mathbb{Q}[x]$.
2. Let $S_2$={0}, the set containing only the zero polynomials. Then $S_2$ is a differential subset of Q[x].
3. Let $S_3$= Q[x]. Then $S_3$ is a differential subset of Q[x].
4. Let $S_4$ = Q, the set of all constant polynomials of Q[x]. Then $S_4$ is a differential subset of Q[x].
5. Let $S_5$ = the set of all polynomials of degree at most 1 $= \{a_0 + a_1 x / a_0, a_1 \in Q\}$. Then $d(a_0 + a_1 x) = a_1$, which is a polynomial of degree 0. Therefore, $d(S_5) \subseteq S_5$. Hence, $S_5$ is a differential subset of Q[x].

Let $\tau = \{S_1, S_2, S_3, S_4, S_5\}$. Then $\tau$ is a derivative topology on Q[x]. Consider, Z[x], the set of all polynomials with integer coefficients. Then Z[x] is a differential subring of Q[x]. Let $\tau_S = \{S_1 \cap Z[x], S_2 \cap Z[x], S_3 \cap Z[x], S_4 \cap Z[x], S_5 \cap Z[x]\}$

Then $S_1 \cap Z[x] = S_1$ , $S_2 \cap Z[x]$= the zero polynomial =$S_2$, $S_3 \cap Z[x] = Z[x]$, $S_4 \cap Z[x] = Z$ , $S_5 \cap Z[x]$= the set of all polynomials of degree atmost 1 with integer coefficients. Hence, $\tau_s$ is a subderivative topology on Z[x].

The following proposition gives the relation between the differential open sets of derivative topology and subderivative topology.

**Proposition 3.7:** Let S be a subderivative topological space of a derivative topological space $(R, d, \tau)$. If X is differential open in S and S is differential open in R, then X is differential open in R.

**Proof:** If X is differential open in S, then X= S∩U, where U is differential open in R. Since S is differential open in R and U is differential open in R, we have S∩U in differential open in R. Hence, X is differential open in R.

**Proposition 3.8:** Let S be a subderivative topological space of a derivative topological space $(R, d, \tau)$. Then X is differential closed set in S if and only if it equals the intersection of differential closed set of a differential ring with S.
**Proof:** The proof follows from proposition 3.5
**Proposition 3.9:** Let S be a subderivative topological space of a derivative topological space $(R, d, \tau)$. If X is differential closed in S and S is differential closed in R, then X is differential closed in R.
**Proof:** The proof follows from proposition 3.7.

## Conclusion

In this paper, we have introduced the derivative topology on a differential ring and subderivative topology on a differential subring of a differential ring. Also, we have studied some properties of differential open sets and differential closed sets of a differential ring.

## References

[1] Abuhlail, J. A Zariski topology for modules. Commun. Algebra 39(11):4163–4182, (2011).
[2] Ansari-Toroghy, H., Keyvani, S., Farshadifar, F. ,The Zariski topology on the Second spectrum of a module (II). Bull. Malays. Math. Sci. Soc. 39(3):1089–1103, (2016).
[3] Djavvat Khadjiev Fethi Callial P, On the Differential Prime Radical of a Differential Ring, Turkish Journal of mathematics, Vol 22 No. 4, 355-368, (1998).
[4] Hadji-Abadi, H., Zahedi, M. M. ,Some results on fuzzy prime spectrum of a ring. Fuzzy Sets Syst. 77(2):235–240, (1996).
[5] I.N. Herstein, Topics in algebra, second edition, Wiley india Pvt. Ltd, New Delhi, Reprint 2017.
[6] James R. Munkres , Topology, 2nd edition, Prentice- hall of India Private limited New Delhi.
[7] I. Kaplansky, An introduction to differential algebra, Actualities Sei. Indust. 1251 (1957), 9-63.
[8] E. Kolchin," Differential Algebra and Algebraic Groups." Academic Press, New York, 1973.
[9] J.F. Ritt, Differential Algebra, Amer. Math. Soc. Coll. Pub. Vol 33, New York, 1950.
[10] William F. Keigher, Prime differential ideals in differential rings, Contribution to algebra, 239-249, (1977).

[11] William F. Keigher, Spectra of differential rings Cahiers de topologie et géométrie différentielle catégoriques, tome 24, no 1, p. 47-56, (1983).