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Edge Domination Decomposition Polynomial Of Graphs

E. Shiny Edmund[1,*] and E. Ebin Raja Merly[2]

[1]*Research Scholar [Reg.No:20123112092026]*, [2]*Associate Professor*
[1,2]*Department of Mathematics, Nesamony Memorial Christian College, Marthandam-629 365.*
[1,2]*Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu,India.*
**Corresponding Author:* [1]*suginshiny2011@gmail.com*
[2]*ebinmerly@gmail.com*

A decomposition $\{G_1, G_2, \cdots, G_n\}$ is said to be an edge domination decomposition of a connected graph G if $\gamma'(G_j) = j, 1 \leq j \leq n$ where $\gamma'(G_j)$ is called the Edge Domination number of G_j. In this paper, we introduce Edge Domination Decomposition Polynomial of a graph. We examine Edge Domination Decomposition Polynomial of various graphs and establish some results based on Edge Domination Decomposition Polynomial.

Keywords: Edge Domination number, Edge Domination Decomposition, Edge Domination Decomposition Polynomial.

AMS Subject Classification 05C12, 05C69

1. Introduction

We consider only finite, simple and undirected graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$ and let $p = |V(G)|$ and $q = |E(G)|$. For standard terminologies and notations, we refer [5]. The concept of edge domination was introduced by Mitchell and Hedetniemi [9]. Ebin Raja Merly and Jeya Jothi [3] introduced Connected Domination path Decomposition polynomial of path and cycle. Motivated by the above we introduce the concept Edge Domination Decomposition polynomial of Graph. In this paper, We examine Edge Domination Decomposition Polynomial of various graphs and establish some results based on Edge Domination Decomposition Polynomial. The basic terminologies which are needed in the article are given below:

Definition 1.1. [8] A subset F of E is called an edge dominating set of G if every edge not in F is adjacent to some edge in F. The edge domination number $\gamma'(G)$ of G is the minimum cardinality taken over all edge dominating sets of G.

Definition 1.2. [3] A decomposition of a graph G is a collection of edge disjoint subgraphs $\{G_1, G_2, \cdots, G_n\}$ of G such that every edge of G belongs to exactly one $G_j, 1 \leq j \leq n$.

Definition 1.3. [4] A decomposition $\{G_1, G_2, \dots G_n\}$ is said to be an edge domination decomposition (EDD) of a connected graph G if $\gamma'(G_j) = j, 1 \leq j \leq n$.

Obviously $\sum_{j=1}^{n} \gamma'(G_j) = \frac{n(n+1)}{2}$.

Definition 1.4. [4] If G admits EDD$\{G_1, G_2, \dots, G_n\}$ then G is said to be an edge domination decomposable graph and is denoted by G_{EDD}.

Definition 1.5. [6] A polynomial is said to be integer monic if all its coefficients are integers and the coefficient of highest power is unity .

Definition 1.6. [6] The polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where $a_0, a_1, \dots a_n$ are integers is said to be primitive if the greatest common divisor of $a_0, a_1, \dots a_n$ is 1 .

Definition 1.7. [6] The content of the Polynomial $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ where the a's are integers is the greatest common divisor of the integers $a_0, a_1, \dots a_n$.

Definition 1.8. Each finite friendship graph F_n which consists of n edge disjoint triangles such that all $n > 1$ triangles have one vertex in common (F_1 is a triangle i.e. the complete graph with three vertices). Thus F_n has $2n + 1$ vertices, 2 n of them being of degree two and the remaining one (the common vertex of n triangles if $n > 1$) being of degree 2n.

Theorem 1.9. [4] The graph P_p admits EDD $\{G_1, G_2, \dots, G_n\}$ if and only if $G_j = P_{3j+1}$, $j = 1,2, \dots n-1$ and $G_n = P_k$ where $3n - 1 \le k \le 3n + 1$.

Theorem 1.10. [4] The graph $C_t \odot K_{1,m}$ admits EDD$\{G_1, G_2, \dots, G_n\}$ if and only if $t = \frac{n(n+1)}{2}$.

Remark 1.11. A Path having either $3j - 2, 3j - 1$ or $3j$ edges has the edge domination number j.

2. Edge Domination Decomposition Polynomial

Definition 2.1. Let G be a connected graph which admits EDD$\{G_1, G_2, \cdots, G_n\}$. Then the polynomial of G is defined as $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)}$ where each coefficients are non negative integers.

Example 2.2. Figure 1 illustrates the edge domination decomposition polynomial of a graph G :

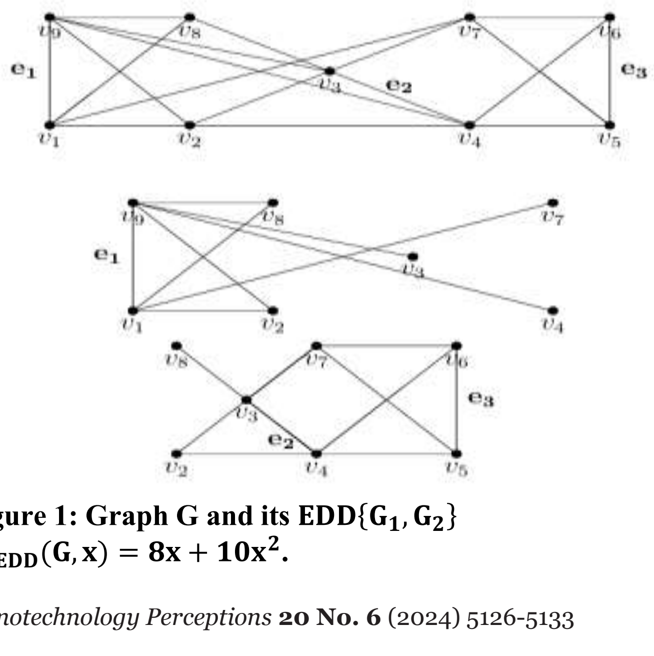


Figure 1: Graph G and its EDD$\{G_1, G_2\}$

$\mathbf{M_{EDD}(G, x) = 8x + 10x^2}$.

Theorem 2.3. For any graph G which admits $EDD\{G_1, G_2, \cdots, G_n\}$, the degree of $M_{EDD}(G, x)$ is n.

Proof. Suppose G admits $EDD\{G_1, G_2, \cdots, G_n\}$. Then $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)} = |E(G_1)|x^{\gamma'(G_1)} + |E(G_2)|x^{\gamma'(G_2)} + \cdots + |E(G_n)|x^{\gamma'(G_n)} = q_1x^1 + q_2x^2 + \cdots + q_nx^n$ where $q_j = |E(G_j)|$. Clearly, $M_{EDD}(G, x)$ is a polynomial of degree n with nonnegative integer coefficients $q_1, q_2, \ldots q_n$. Hence the theorem.

Theorem 2.4. If G admits $EDD\{G_1, G_2, \cdots, G_n\}$. Then (i) $M_{EDD}(G, x)$ has no constant term (ii) $\deg M_{EDD}(G, x) \geq \deg M_{EDD}(G_j, x)$ for any $j = 1,2, \ldots, n$ (iii) $M_{EDD}(G, x)$ has a zero with multiplicity 1.

Proof. (i) Obviously $M_{EDD}(G, x)$ has no constant term, Since $\gamma'(G_j) \geqslant 1$.

(ii) We have $\gamma'(G_j) = j, 1 \leq j \leq n$. Also $M_{EDD}(G_j, x) = |E(G_j)| x^{\gamma'(G_j)} = q_jx^j, \deg M_{EDD}(G_j, x) = j \leq n = \deg M_{EDD}(G, x), j = 1,2,3 \ldots n$.

(iii) By (i) $M_{EDD}(G, x)$ has no constant term. Hence x is a factor of $M_{EDD}(G, x)$. Therefore $M_{EDD}(G, x)$ has a zero with multiplicity 1.

Remark 2.5. The derivative of an edge domination decomposition polynomial of G is not an Edge domination decomosition polynomial.

Proof. We know that edge domination decomposition polynomial has no constant term. But the derivative of edge domination decomposition polynomial should contain a constant term. Hence the derivative of an edge domination decomposition polynomial is not an edge domination decomposition polynomial.

Theorem 2.6. If G admits $EDD\{G_1, G_2, \cdots, G_n\}$ then

$$\frac{d^s}{dx^s}[M_{EDD}(G, x)]\bigg|_{x=0} = s!\,|E(G_s)|, \text{ for } s = 1,2, \ldots, n$$

Proof. Let G be the given graph with $EDD\{G_1, G_2, \cdots, G_n\}$. We have $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)} = |E(G_1)|x^{\gamma'(G_1)} + |E(G_2)|x^{\gamma'(G_2)} + \cdots + |E(G_n)|x^{\gamma'(G_n)}$.

That is $M_{EDD}(G, x) = |E(G_1)|x^1 + |E(G_2)|x^2 + \cdots + |E(G_n)|x^n$

$$\frac{d}{dx}[M_{EDD}(G, x)] = |E(G_1)| + 2|E(G_2)|x + \cdots + n|E(G_n)|x^{n-1}$$

$$\frac{d}{dx}[M_{EDD}(G, x)]\bigg|_{x=0} = |E(G_1)|$$

In the same manner, $\frac{d^2}{dx^2}[M_{EDD}(G, x)]\Big|_{x=0} = 2!\,|E(G_2)|$. Continuing in this way

$\frac{d^s}{dx^s}[M_{EDD}(G, x)]\Big|_{x=0} = s!\,|E(G_s)|$ for $s = 1,2, \ldots n$.

Corollary 2.7. If G admits $EDD\{G_1, G_2, \cdots, G_n\}$ then

$$\frac{d^n}{dx^n}[M_{EDD}(G, x)] = n!\,|E(G_n)|$$

Proof. Let G be the given graph with $EDD\{G_1, G_2, \cdots, G_n\}$. We have $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)} = |E(G_1)|x^{\gamma'(G_1)} + |E(G_2)|x^{\gamma'(G_2)} + \cdots + |E(G_n)|x^{\gamma'(G_n)}$.

By definition $\gamma'(G_j) = j, j = 1,2, \ldots n$. Now, $M_{EDD}(G, x) = |E(G_1)|x^1 + |E(G_2)|x^2 + \cdots + |E(G_n)|x^n$

$\frac{d}{dx}[M_{EDD}(G,x)] = |E(G_1)| + 2|E(G_2)|x + \cdots + n|E(G_n)|x^{n-1}$
$\frac{d^2}{dx^2}[M_{EDD}(G,x)] = 1.2|E(G_2)| + \cdots + n(n-1)|E(G_n)|x^{n-2}$ Continuing in this way
$\frac{d^n}{dx^n}[M_{EDD}(G,x)] = n!\,|E(G_n)|$.

Result 2.8. If G admits EDD$\{G_1, G_2, \cdots, G_n\}$ then $[M_{EDD}(G,x)]|_{x=1} = q$

Proof. $M_{EDD}(G,x) = q_1x^1 + q_2x^2 + \cdots + q_nx^n$. $[M_{EDD}(G,x)]|_{x=1} = q_1 + q_2 + \cdots + q_n = q$ where the q_j^s are the edges of the graph G_j.

Result 2.9. If G admits $EDD\{G_1, G_2, \cdots, G_n\}$ then
$\frac{d}{dx}[M_{EDD}(G,x)]\Big|_{x=1} = \sum_{j=1}^{n}|E(G_j)|\gamma'(G_j)$

Proof. $\frac{d}{dx}[M_{EDD}(G,x)] = |E(G_1)| + 2|E(G_2)|x + \cdots + n|E(G_n)|x^{n-1}$
$\frac{d}{dx}[M_{EDD}(G,x)]\Big|_{x=1} = |E(G_1)| + 2|E(G_2)| + \cdots + n|E(G_n)| = \sum_{j=1}^{n}|E(G_j)|\gamma'(G_j)$

Theorem 2.10. The Edge Domination Decomposition Polinomial of a graph G is integer monic iff G is K_2.

Proof. Assume that $M_{EDD}(G,x)$ is integer monic. Therefore $q_n = 1$ and $\gamma'(k_2) = 1$. Therefore degree of $M_{EDD}(K_2,x)$ is also 1 . Obviously $n = 1$. Hence $G \cong K_2$. Conversely, Suppose G is $K_2, \gamma'(G) = 1$ and $|E(K_2)| = 1$. So $M_{EDD}(G,x) = 1.x^1 = x$. Therefore edge domination decomposition polynomial of a graph G is integermonic.

Theorem 2.11. If the path P_p admits $EDD\{G_1, G_2, \cdots, G_n\}$ then $M_{EDD}(P_p,x) = 3\sum_{j=1}^{n-1} jx^j + q_nx^n$ where $q_n = \begin{cases} 3n-1 \text{ if } p \equiv 0(mod3), \\ 3n \text{ if } p \equiv 1(mod3), \\ 3n-2 \text{ if } p \equiv 2(mod3). \end{cases}$

Proof. Let G be a path P_p graph. Suppose that the path P_p admits $EDD\{G_1, G_2, \cdots, G_n\}$. Then $\gamma'(G_j) = j, 1 \le j \le n$. Hence $M_{EDD}(G,x) = \sum_{j=1}^{n}|E(G_j)|x^{\gamma'(G_j)} - \cdots - \cdots - \cdots (i)$ By Theorem 1.8 if p_p admits EDD $\{G_1, G_2, \cdots, G_n\}$ then $G_j = P_{3j+1}, j = 1,2, \dots n-1$ and $G_n = P_k$ where $3n-1 \le k \le 3n+1$, $|E(G_j)| = 3j$ for $j = 1,2, \dots n-1$ and

$$|E(G_n)| = \begin{cases} 3n-1 & \text{if} \quad p \equiv 0(mod3), \\ 3n & \text{if} \quad p \equiv 1(mod3), \\ 3n-2 & \text{if} \quad p \equiv 2(mod3), \end{cases}$$

Thus we have $M_{EDD}(P_p,x) = 3\sum_{j=1}^{n-1} jx^j + q_nx^n$ where

$$q_n = \begin{cases} 3n-1 & \text{if} \quad p \equiv 0(mod3), \\ 3n & \text{if} \quad p \equiv 1(mod3), \\ 3n-2 & \text{if} \quad p \equiv 2(mod3). \end{cases}$$

Theorem 2.12. $M_{EDD}(P_p,x)$ is Primitive if $p \equiv 0(mod3)$ or $p \equiv 2(mod3)$

Proof. By theorem 2.8 we have If the path P_p admits EDD $\{G_1, G_2, \cdots, G_n\}$ then $M_{EDD}(P_p,x) = 3\sum_{j=1}^{n-1} jx^j + q_nx^n$ where $q_n = \begin{cases} 3n-1 & \text{if} \quad p \equiv 0(mod3), \\ 3n & \text{if} \quad p \equiv 1(mod3), \\ 3n-2 & \text{if} \quad p \equiv 2(mod3), \end{cases}$

Case (i) $p \equiv 0(mod3)$
In the polynomial $M_{EDD}(P_p, x) = 3\sum_{j=1}^{n-1} jx^j + (3n-1)x^n$, the coefficients have no common divisors. Therefore $M_{EDD}(P_p, x)$ is primitive.
Case (ii) $p \equiv 2(mod3)$
In the polynomial $M_{EDD}(P_p, x) = 3\sum_{j=1}^{n-1} jx^j + (3n-2)x^n$, the coefficients have no common divisors. Therefore $M_{EDD}(P_p, x)$ is primitive.
Remark 2.13. 1. If $p \equiv 1(mod3)$ then the content of $M_{EDD}(P_p, x)$ is 3. The content of M_{EDD} polynomial of friendship graph is 3.
Theorem 2.14. If the path C_p admits $EDD\{G_1, G_2, \cdots, G_n\}$ then $M_{EDD}(C_p, x) = 3\sum_{j=1}^{n-1} jx^j + lx^n$ where $l = \begin{cases} 3n \text{ if } p \equiv 0(mod3), \\ 3n+1 \text{ if } p \equiv 1(mod3), \\ 3n-1 \text{ if } p \equiv 2(mod3). \end{cases}$
Proof. Let G be a path C_p graph. Suppose that the path C_p admits $EDD\{G_1, G_2, \cdots, G_n\}$. Then $\gamma'(G_j) = j, 1 \le j \le n$. Hence $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)}$. By theorem if C_p admits EDD $\{G_1, G_2, \cdots, G_n\}$ then $G_j = P_{3j+1}, j = 1,2, \dots n-1$ and $G_n = P_k$ where $3n-1 \le k \le 3n+1$, $|E(G_j)| = 3j$ for $j = 1,2, \dots n-1$ and

$$|E(G_n)| = \begin{cases} 3n-1 & \text{if} \quad p \equiv 0(mod3), \\ 3n & \text{if} \quad p \equiv 1(mod3), \\ 3n-2 & \text{if} \quad p \equiv 2(mod3), \end{cases}$$

Thus we have $M_{EDD}(C_p, x) = 3\sum_{j=1}^{n-1} jx^j + q_n x^n$ where

$$q_n = \begin{cases} 3n & \text{if} \quad p \equiv 0(mod3), \\ 3n+1 & \text{if} \quad p \equiv 1(mod3), \\ 3n-1 & \text{if} \quad p \equiv 2(mod3). \end{cases}$$

Theorem 2.15. The Wheel graph $W_p, p \ge 4$ admits EDD $\{G_1, G_2, \cdots, G_n\}$ then $M_{EDD}(W_p, x) = (p+1)x + 3\sum_{j=1}^{n-1} jx^j + q_n x^n$ where $q_n = \begin{cases} 3n \text{ if } p \equiv 0(mod3), \\ 3n-2 \text{ if } p \equiv 1(mod3), \\ 3n-1 \text{ if } p \equiv 2(mod3), \end{cases}$
Proof. Let $V(W_p) = \{v_i / 1 \le i \le p\}$. Let v_p be the central vertex of G. Let $\{G_1, G_2, \cdots, G_n\}$ be the Subgraphs of G. Let $E(G_1) = \{v_i v_p / 1 \le i \le p-1\} \cup \{v_1 v_2, v_1 v_{p-1}\}$ with edge domination number 1. That is $\gamma'(G_1) = 1$. $E(G \setminus G_1)$ will be a path of length p-3 = P_{p-2}. Let $G_2, G_3, \dots G_n$ be the graph obtained from the path P_{p-2} with edge domination number $\gamma'(G_j) = j, j = 1,2, \dots n$. By result if P_p admits $EDD\{G_1, G_2, \cdots, G_n\}$ then $G_j = P_{3j+1}, j = 1,2, \dots n-1$ and $G_n = P_k$ where $3n-1 \le k \le 3n+1$, $|E(G_j)| = 3j$ for $j = 1,2, \dots n-1$ and

$$|E(G_n)| = \begin{cases} 3n-1 & \text{if} \quad p \equiv 0(mod3), \\ 3n & \text{if} \quad p \equiv 1(mod3), \\ 3n-2 & \text{if} \quad p \equiv 2(mod3), \end{cases}$$

Thus we have $M_{EDD}(W_p, x) = (p+1)x + 3\sum_{j=1}^{n-1} jx^j + q_n x^n$ where

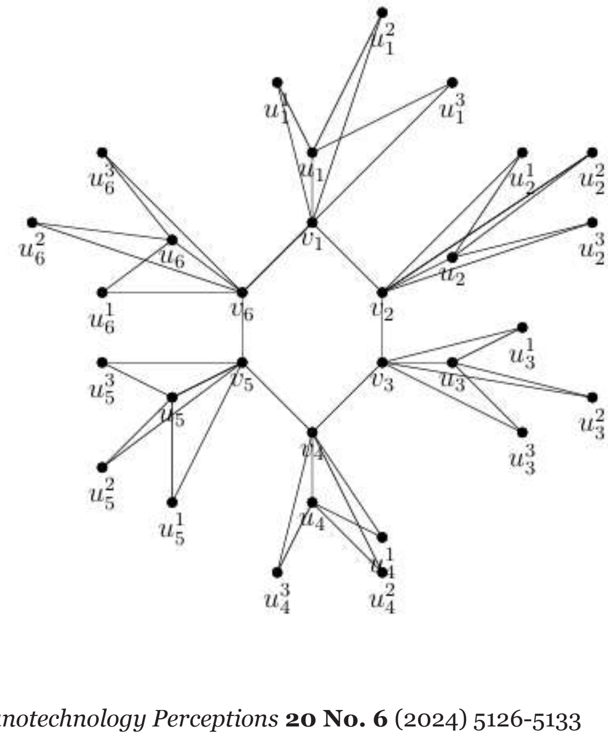
$$q_n = \begin{cases} 3n & \text{if} \quad p \equiv 0(\text{mod}3), \\ 3n-2 & \text{if} \quad p \equiv 1(\text{mod}3), \\ 3n-1 & \text{if} \quad p \equiv 2(\text{mod}3). \end{cases}$$

Theorem 2.16. The graph $C_t \odot K_{1,m}$ admits $EDD\{G_1, G_2, \ldots, G_n\}$ then $M_{EDD}(C_t \odot K_{1,m}, x) = \{m(m+1)+2\}x + \left[\sum_{j=1}^{n-1} j[m(m+1)+1]x^j\right] + n[m(m+1)+(n-1)]$.

Proof. By Theorem 1.10 the graph $C_t \odot K_{1,m}$ admits $EDD\{G_1, G_2, \ldots, G_n\}$ if and only if $t = \frac{n(n+1)}{2}$. Let $G = C_t \odot K_{1,m}$. Then $M_{EDD}(G, x) = \sum_{j=1}^{n} |E(G_j)| x^{\gamma'(G_j)} = |E(G_1)| x^{\gamma'(G_1)} + |E(G_2)| x^{\gamma'(G_2)} + \cdots + |E(G_n)| x^{\gamma'(G_n)}$ Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of C_t. Let u_t be the root vertex of the i^{th} copy of $K_{1,m}$, $i = 1,2, \ldots t$. Let $\left\{u_i^j / j = 1,2, \ldots m\right\}$ be the remaining vertices in the i^{th} copy of $K_{1,m}$. Let $E(G) = \left\{e_i = u_i v_i, e_i^j = u_i^j v_i, h_i^j = u_i u_i^j, l_i / i = 1,2, \ldots t; j = 1,2, \ldots m\right\}$. Let $E(G_1) = \{u_1 v_1, u_1^k v_1, u_1 u_1^k, v_1 v_t, v_1 v_2 / k = 1,2, \ldots m;\}$ Then $|E(G_1)| = m(m+1)+2$.

For $j = 2,3, \ldots n-1$, take $E(G_j) = \{u_i v_i, u_i^k v_{t-(n-1)}, u_i u_i^k, v_i v_{(i+1)}, u_i^k v_{t-(n-1)} / k = 1,2, \ldots m; i = 2,3, \ldots t-(n-1)\}$. Therefore $|E(G_j)| = \sum_{j=2}^{n-1} j[m(m+1)+1]$ and $E(G_n) = \{u_i v_i, u_i^k v_i, u_i^k u_i / k = 1,2, \ldots m; i = t-(n-1) \ldots t\}$. Then $|E(G_n)| = n[m(m+1] + n - 1$. Thus we have $M_{EDD}(C_t \odot K_{1,m}, x) = \{m(m+1)+2\}x + \sum_{j=1}^{n-1} j[m(m+1)+1]x^j + n[m(m+1)+(n-1)]$.

Example 2.17. In the following example we give the Edge Domination Decomposition of $C_6 \odot K_{1,3}$



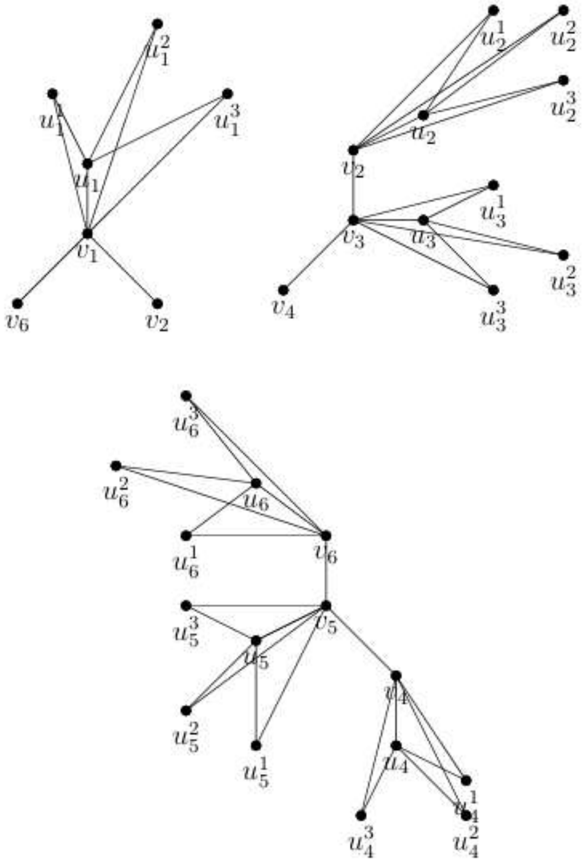


Figure 2: Graph $C_6 \odot K_{1,3}$ and its EDD $\{G_1, G_2, G_3\}$

3. Conclusion

In this paper, we have introduced the new concept edge domination decomposition polynomial of Graphs. Here we examined edge domination decomposition polynomial of some graphs. Further this concept can be expanded to establish the results on edge domination decomposition polynomial of graphs.

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