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Viscosity Effects On Axisymmetric Viscothermoelastic Diffusion Under Dual-Phase-Lag Heat And Mass Transfer

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The present investigation deals with an axisymmetric problem in a homogeneous isotropic viscothermoelastic diffusive medium within the framework of dual-phase-lag heat transfer (DPLT)and dual-phase-lag diffusion (DPLD) models. The medium is assumed to have thickness 2b, and its upper and lower boundary surfaces are traction-free and subjected to prescribed axisymmetric thermal and chemical potential boundary conditions. The governing equations are formulated in cylindrical coordinates under plane axisymmetric deformation. Laplace and Hankel transform techniques are employed to obtain the solution in the transformed domain. Analytical expressions for the axial displacement, stress components, temperature change, chemical potential, and mass concentration are derived. Numerical inversion techniques are applied to obtain the corresponding results in the physical domain. Numerical results are presented graphically to examine the influence of viscosity on the displacement, stresses, temperature change, chemical potential, and mass concentration. Some particular cases are also deduced from the present formulation.

Keywords: Dual-phase-lags; viscothermoelastic; diffusion; Laplace transform; Hankel transform; axisymmetric problem.

1. Introduction

The development of generalized thermoelastic theories began with the need to remove the physical inconsistency present in classical Fourier heat conduction. According to Fourier's law, a thermal disturbance propagates with infinite speed, which is unrealistic for problems involving short time intervals, rapid heating, high-frequency thermal loading, small length scales, and wave-like thermal transport. Cattaneo [1] proposed a modified form of the heat-conduction equation to eliminate the paradox of instantaneous propagation. Vernotte [2] also discussed the paradoxes associated with the classical continuous theory of heat conduction and introduced a relaxation-type modification. These works provided the basis for finite-speed heat propagation in non-Fourier heat-conduction theory.

On the basis of this relaxation concept, Lord and Shulman [3] developed a generalized dynamical theory of thermoelasticity. Their theory introduced one thermal relaxation time into the heat-conduction equation and allowed thermal disturbances to propagate with finite velocity. Green and Lindsay [4] further developed generalized thermoelasticity by introducing two relaxation parameters. These two generalized thermoelastic theories are widely used for studying thermal shock, transient heat conduction, and thermoelastic wave propagation in elastic solids.

The coupling of thermal, elastic, and diffusive fields was later developed through thermoelastic diffusion theory. Nowacki [5] studied dynamical problems of thermodiffusion in solids and provided an important foundation for the analysis of coupled deformation, temperature, and concentration fields. Sherief, Saleh, and Hamza [6] formulated the theory of generalized thermoelastic diffusion, in which mass diffusion was incorporated into generalized thermoelasticity. Kumar and Kansal [7] presented thermoelastic diffusion theory under Green-Lindsay model and discussed the propagation of Lamb waves in a transversely isotropic thermoelastic diffusion plate. These studies established the theoretical and applied basis for analysing thermoelastic diffusion problems in solid media.

A major extension of non-Fourier heat-conduction theory was introduced by Tzou [8] through the dual-phase-lag model. This model considers two phase lags: one corresponding to the heat flux and the other corresponding to the temperature gradient. It provides a more general description of heat transfer in microscale and ultrafast thermal processes. Quintanilla and Racke [9] studied stability in dual-phase-lag heat conduction. Abdallah [10] investigated dual-phase-lag heat conduction and thermoelastic properties of a semi-infinite medium subjected to ultrashort pulsed laser heating. Chou and Yang [11] analysed two-dimensional dual-phase-lag thermal behaviour in single-layer and multilayer structures. Zhou, Zhang, and Chen [12] developed an axisymmetric dual-phase-lag bioheat model for laser heating of living tissues.

Further developments in generalized thermodiffusive and thermoelastic models were reported by several authors. Sharma [13] studied boundary-value problems in generalized thermodiffusive elastic media. Sharma [14] analysed deformation due to an inclined load in a generalized thermodiffusive elastic medium. Sharma, Kumar, and Ram [15] investigated interactions of generalized thermoelastic diffusion due to inclined loading. Atwa [16] examined generalized thermoelastic diffusion with the effect of a fractional parameter in a temperature-dependent elastic medium. Sharma, Sharma, and Bhargava [17] studied wave propagation in a micropolar thermoelastic solid with two temperatures bordered by layers or half-spaces of inviscid liquid. Kumar and Gupta [18] analysed plane-wave propagation in an anisotropic dual-phase-lag thermoelastic diffusion medium. Sharma, Sharma, and Bhargava [19] investigated plane waves and fundamental solutions in electro-microstretch elastic solids. Sharma and Sharma [20] studied the influence of heat sources and relaxation time on temperature distribution in tissues.

Additional work on the dual-phase-lag theory and generalized heat-conduction models was carried out by El-Karamany and Ezzat [21], who studied the dual-phase-lag thermoelasticity theory. Rukolaine [22] discussed unphysical effects of the dual-phase-lag model of heat conduction and emphasized the need for careful interpretation of phase-lag parameters. Ying and Yun [23] developed a time-fractional dual-phase-lag heat-conduction equation. Tripathi, Kedar, and Deshmukh [24] studied a generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply.

The role of rheological and viscoelastic effects in thermal-stress problems was highlighted by Freudenthal [25], who studied the effect of rheological behaviour on thermal stresses. Kaliski [26] investigated the absorption of magneto-viscoelastic surface waves in a real conductor magnetic field. Iesan and Scalia [27] presented some theorems in the theory of thermoviscoelasticity. Borrelli and Patria [28] analysed discontinuity waves in thermoviscoelastic solids of integral type. Pal [29] studied torsional body forces in a viscoelastic half-space, while Corr, Starr, Vanderby, and Best [30] developed a nonlinear generalized Maxwell fluid model. These studies contributed to the development of viscoelastic and thermoviscoelastic material theories.

Further extensions of viscoelastic and thermo-viscoelastic models included magnetic fields, viscosity, imperfect boundaries, voids, microstretch effects, rotation, non-homogeneity, and fractional-order strain. Abd-Allaa and Mahmoud [31] investigated magneto-thermo-viscoelastic interactions in an unbounded non-homogeneous body with a spherical cavity subjected to periodic loading. Kumar, Chawla, and Abbas [32] studied the effect of viscosity on wave propagation in an anisotropic thermoelastic medium with a three-phase-lag model. Sharma and Bhargava [33] analysed thermoelastic plane waves at an imperfect boundary between a thermally conducting viscous liquid and a generalized thermoelastic solid. Sharma and Kumar [34] investigated plane waves and fundamental solutions in a thermoviscoelastic medium with voids. Sharma, Sharma, and Bhargava [35] studied the effect of viscosity on wave propagation in an anisotropic thermoelastic medium with Green–Naghdi theories of type II and type III. Sharma, Sharma, and Bhargava [36] examined wave motion and fundamental solutions in electro-microstretch viscoelastic solids. Al-Basyouni, Mahmoud, and Alzahrani [37] investigated the effect of rotation, magnetic field, and periodic loading on radial vibrations of thermo-viscoelastic non-homogeneous media. Hilton [38] studied coupled longitudinal thermal and viscoelastic waves in media with temperature-dependent material properties. Yadav, Kalkal, and Deswal [39] considered two-temperature generalized thermoviscoelasticity with fractional-order strain subjected to a moving heat source. Li, Guo, Tian, and Tian [40] studied the transient response of a half-space with variable thermal conductivity and diffusivity under thermal and chemical shock.

Although generalized thermoelasticity, thermoelastic diffusion, and dual-phase-lag heat transfer have been widely studied, fewer works have considered viscosity together with dual-phase-lag diffusion in an axisymmetric setting. The effect of viscosity on the coupled mechanical, thermal, and diffusive fields therefore need further investigation.

In the present work, this problem is studied for a homogeneous isotropic viscothermoelastic diffusive medium using DPLT and DPLD models. The governing equations are formulated in cylindrical coordinates under plane axisymmetric deformation. Laplace and Hankel transform techniques are used to obtain the transformed-domain solutions, and numerical inversion is applied to obtain the physical-domain results. The influence of viscosity on the coupled thermoelastic and diffusive fields is then examined graphically.

2. Basic Equations

The basic equations of motion, heat conduction and mass diffusion in a homogeneous isotropic viscothermoelastic solid with DPLT and DPLD models in the absence of body forces, heat sources and mass diffusion sources are

$$(\lambda_I+\mu_I)\nabla(\nabla.u)+\mu_I\nabla^2 u-\beta_1\nabla T-\beta_2\nabla C=\rho\ddot{u}, \tag{1}$$

$$\left(1+\tau_t\frac{\partial}{\partial t}\right)KT_{,ii}=\left(1+\tau_q\frac{\partial}{\partial t}+{\tau_q}^2\frac{\partial^2}{\partial t^2}\right)[\rho C_E\dot{T}+\beta_1 T_0\dot{e}_{kk}+aT_0\dot{C}], \tag{2}$$

$$\left(1+\tau_p\frac{\partial}{\partial t}\right)(D\beta_2\nabla^2(\nabla.u)+Da\nabla^2T-Db\nabla^2C)+\frac{\partial}{\partial t}\left(1+\tau_\eta\frac{\partial}{\partial t}+\tau_\eta{}^2\frac{\partial^2}{\partial t^2}\right)C=0\,, \quad (3)$$

and the constitutive relations are

$$\sigma_{ij}=2\mu_I e_{ij}+\delta_{ij}(\lambda e_{kk}-\beta_1T-\beta_2C), \quad (4)$$

$$\rho T_0S=\left(1+\tau_q\frac{\partial}{\partial t}+\tau_q{}^2\frac{\partial^2}{\partial t^2}\right)(\rho C_ET+\beta_1T_0e_{kk}+aT_0C), \quad (5)$$

$$P=-\beta_2e_{kk}-aT-bC, \quad (6)$$

where

$$\lambda_I=\lambda+\lambda_\nu\frac{\partial}{\partial t},$$

$$\mu_I=\mu+\mu_\nu\frac{\partial}{\partial t},\ \beta_1=(3\lambda_I+2\mu_I)\alpha_t\,,\ \beta_2=(3\lambda_I+2\mu_I)\alpha_c$$

where λ , μ are Lame's constants , λ_ν , μ_ν are viscosity constants, ρ is the density assumed to be independent of time, D is the diffusivity, P is the chemical potential per unit mass, C is the mass concentration, u_i are components of displacement vector **u**, K is the coefficient of thermal conductivity, C_E is the specific heat at constant strain, $T=\vartheta-T_0$ is small temperature increment, ϑ is the absolute temperature of the medium, T_0 is the reference temperature of the body such that $\left|\frac{T}{T_0}\right|\ll 1$, a and b are respectively , the coefficients describing the measure of thermodiffusion and mass diffusion effect respectively, σ_{ij} and e_{ij} are the components of stress and strain respectively, e_{kk} is dilatation, S is the entropy per unit mass, , $\alpha_c$is the coefficient of linear diffusion expansion and α_t is the coefficient of thermal linear expansion. τ_t, τ_q, τ_p, τ_η are respectively phase lag of temperature gradient, the phase lag of heat flux, the phase lag of chemical potential, and phase lag of diffusing mass flux vector. In above equations, a comma followed by suffix denotes spatial derivative and a superposed dot denotes derivative with respect to time.

3. Basic Equations Formulation and Solution of the Problem:

Consider a homogeneous isotropic viscothermoelastic diffusive medium of thickness 2b, occupying the region D ={(r,z): $0\le r<\infty, -b\le z\le b$}.

The medium is referred to a cylindrical polar coordinate system (r,θ,z), with symmetry about the z–axis. The upper and lower boundary surfaces $z=\pm b$ are assumed to be traction-free.The medium is subjected to prescribed axisymmetric thermal and chemical potential boundary conditions on these surfaces. The initial temperature of the medium is taken to be the uniform reference temperature T_0. In the application considered later, the thermal source is taken as an axisymmetric distributed source, whereas the chemical potential source is taken as a ring-shaped source concentrated at $r=a_0$.

Since the problem is plane axisymmetric, all field variables are independent of the angular coordinate θ, and there is no circumferential displacement, i.e.,

u_θ=0.

Hence, the analysis is restricted to a two-dimensional problem in the r - z plane, and the displacement vector is written as

$$\mathbf{u}=(u_r,0,u_z). \quad (7)$$

Equations (1)-(6) with the aid of (7) take the form

$$(\lambda_I + \mu_I)\frac{\partial e}{\partial r} + \mu_I\left(\nabla^2 - \frac{1}{r^2}\right)u_r - \beta_1\frac{\partial T}{\partial r} - \beta_2\frac{\partial C}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \quad (8)$$

$$(\lambda_I + \mu_I)\frac{\partial e}{\partial z} + \mu_I\nabla^2 u_z - \beta_1\frac{\partial T}{\partial z} - \beta_2\frac{\partial C}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}, \quad (9)$$

$$(1+\tau_t\frac{\partial}{\partial t})K\nabla^2 T = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{\tau_q{}^2}{2}\frac{\partial^2}{\partial t^2}\right)[\rho C_E\dot{T} + \beta_1 T_0\frac{\partial}{\partial t}\mathrm{div}\, u + aT_0\frac{\partial C}{\partial t}], \quad (10)$$

$$(1+\tau_p\frac{\partial}{\partial t})(D\beta_2\nabla^2\mathrm{div}\, u + Da\nabla^2 T - Db\nabla^2 C) + \frac{\partial}{\partial t}\left(1 + \tau_\eta\frac{\partial}{\partial t} + \frac{\tau_\eta{}^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0, \quad (11)$$

and

$$\sigma_{rr} = 2\mu_I e_{rr} + \lambda_I e - \beta_1 T - \beta_2 C, \quad (12)$$

$$\sigma_{\theta\theta} = 2\mu_I e_{\theta\theta} + \lambda_I e - \beta_1 T - \beta_2, \quad (13)$$

$$\sigma_{zz} = 2\mu_I e_{zz} + \lambda_I e - \beta_1 T - \beta_2 C, \quad (14)$$

$$\sigma_{rz} = \mu_I e_{rz},\ \sigma_{r\theta} = 0,\ \sigma_{z\theta} = 0, \quad (15)$$

$$P = -\beta_2 e - aT + bC, \quad (16)$$

wher

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z},\ e_{rr} = \frac{\partial u_r}{\partial r},\ \ e_{\theta\theta} = \frac{u_r}{r},\ e_{zz} = \frac{\partial u_z}{\partial z},\ e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \quad (17)$$

To facilitate the solution, the following dimensionless quantities are introduced

$$r' = \frac{\omega_1}{c_1}r \quad , \quad z' = \frac{\omega_1}{c_1}z \quad , \quad (u_r', u_z') = \frac{\omega_1}{c_1}(u_r, u_z), \quad t' = \omega_1 t \quad , \quad (\sigma_{rr}', \sigma_{\theta\theta}', \sigma_{zz}', \sigma_{rz}') = \frac{1}{\beta_1 T_0}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}),$$

$$(T', C') = \frac{\beta_1}{\rho c_1^2}(T, C),\ \left(\tau_q', \tau_t', \tau_p', \tau_\eta'\right) = \omega_1(\tau_q, \tau_t, \tau_p, \tau_\eta),\ \omega_1 = \frac{\rho C_E c_1^2}{K},\ c_1^2 = \frac{\lambda+2\mu}{\rho}. \quad (18)$$

We define Laplace and Hankel transforms as

$$\bar{f}(r, z, s) = \int_0^\infty f(r, z, t)e^{-st}dt, \quad (19)$$

$$\hat{f}(\xi, z, s) = \int_0^\infty \bar{f}(r, z, s)rJ_n(r\xi)\, dr. \quad (20)$$

Using (18) in equations (8)-(11), suppressing the primes for convenience and then applying Laplace Transform defined by (19) on the resulting quantities and after simplification, we have

$$\nabla^2\bar{T} + \nabla^2\bar{C} - (\nabla^2 - s^2)\bar{e} = 0, \quad (21)$$

$$(\nabla^2 - \frac{\tau_q{}^1}{\tau_t{}^1}ks)\bar{T} - \frac{KaT_0\tau_q{}^1 s}{\rho C_E\beta_2\,\tau_t{}^1}\bar{C} - \frac{K\beta_1{}^2T_0}{\rho^2 C_E c_1^2}\frac{\tau_q{}^1}{\tau_t{}^1}s\,\bar{e} = 0, \quad (22)$$

$$Da\nabla^2\frac{\rho c_1^2}{\beta_1}\bar{T} - \left(Db\frac{\rho c_1^2}{\beta_2}\nabla^2 - \frac{s\tau_\eta{}^1 kc_1^2}{\tau_p{}^1\beta_2 C_E}\right)\bar{C} + D\beta_2\nabla^2\bar{e} = 0, \quad (23)$$

where $\tau_q{}^1 = 1 + s\tau_q + \frac{s^2\tau_q{}^2}{2}$, $\tau_\eta{}^1 = 1 + s\tau_\eta + \frac{s^2\tau_\eta{}^2}{2}$, $\tau_p{}^1 = 1 + s\tau_p$, $\tau_t{}^1 = 1 + s\tau_t$.

Eliminating $\bar{T}, \bar{C}, \bar{e}$ from equations (21)-(23), yield

$$(\nabla^2 - k_1^2)(\nabla^2 - k_2^2)(\nabla^2 - k_3^2)(\bar{T}, \bar{C}, \bar{e}) = 0. \quad (24)$$

The solutions of the equation (24) can be written in the form

$$\bar{T} = \sum_{i=1}^3\bar{T}_i,\ \bar{e} = \sum_{i=1}^3\bar{e}_i,\ \bar{C} = \sum_{i=1}^3\bar{C}_i, \quad (25)$$

where $\bar{T}_i$, $\bar{e}_i$,and $\bar{C}_i$ are solutions of the following equation

$$(\nabla^2 - k_i^2)(\bar{T}_i, \bar{e}_i, \bar{C}_i) = 0,\ i = 1,2,3. \quad (26)$$

On taking Hankel transform defined by (20) on the equation (26), we obtain

$$\left(D^2 - \xi^2 - k_i^2\right)\left(\hat{T}, \hat{e}, \hat{C}\right) = 0, \quad (27)$$

$$\hat{T} = \sum_{i=1}^3 B_i\cos h(\varepsilon_i z), \quad (28)$$

$$\hat{C} = \sum_{i=1}^{3} r_i B_i cosh(\varepsilon_i z) , \tag{29}$$
$$\hat{e} = \sum_{i=1}^{3} p_i B_i cosh(\varepsilon_i z) , \tag{30}$$
$$r_i = \frac{\zeta_{34}\varepsilon_i^4 - 6\zeta_{21}\zeta_{34}\varepsilon_i^2 + \zeta_{23}\zeta_{31}}{-\varepsilon_i^2(\zeta_{22}\zeta_{34}+\zeta_{23}\zeta_{32})+\zeta_{23}\zeta_{33}} , \tag{31}$$
$$p_i = \frac{-\zeta_{32}\varepsilon_i^4 + (\zeta_{33}+\zeta_{21}\zeta_{32})\varepsilon_i^2 - \zeta_{21}\zeta_{33}}{-\varepsilon_i^2(\zeta_{22}\zeta_{34}+\zeta_{23}\zeta_{32})+\zeta_{23}\zeta_{33}} , \tag{32}$$
and
$$\varepsilon_i = \sqrt{\xi^2 + k_i^2} , \zeta_{21} = \frac{\tau_q{}^1}{\tau_t{}^1} Ks, \zeta_{22} = \frac{KaT_0\tau_q{}^1 s}{\rho C_E \beta_2 \tau_t{}^1}, \zeta_{23} = \frac{K{\beta_1}^2 T_0}{\rho^2 C_E c_1^2}\frac{\tau_q{}^1}{\tau_t{}^1} s , \zeta_{31} = \frac{Da\rho c_1^2}{\beta_1} , \zeta_{32} = \frac{\rho c_1^2}{\beta_2} ,$$
$$\zeta_{33} = \frac{s\tau_\eta{}^1 k c_1^2}{\tau_p{}^1 \beta_2 C_E} , \zeta_{34} = D\beta_2 . \tag{33}$$
Using (12)-(20) and (28)-(30), we obtain the components of displacement, stress components and chemical potential in the transformed domain as
$$\hat{u}_r = \xi \left[R_1 cosh(qz) + \sum_{i=1}^{3} \frac{\lambda_i}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} cosh(\varepsilon_i z) \right] , \tag{34}$$
$$\hat{u}_z = \left[R_2 \, sinh(\varepsilon z) - \sum_{i=1}^{3} \frac{\lambda_i}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} sinh(\varepsilon_i z) \right] , \tag{35}$$
$$\hat{\sigma}_{\theta\theta} = \frac{2\mu}{\beta_1 T_0} \xi \left[R_1 cosh(\varepsilon z) + \sum_{i=1}^{3} \frac{\lambda_i}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} cosh(\varepsilon_i z) \right] + \sum_{i=1}^{3} \eta_i \, cosh(\varepsilon_i z), \tag{36}$$
$$\hat{\sigma}_{rr} = \frac{2\mu\xi^2}{\beta_1 T_0} \left[R_1 cosh(\varepsilon z) + \sum_{i=1}^{3} \frac{\lambda_i}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} cosh(\varepsilon_i z) \right] d\xi + \sum_{i=1}^{3} \eta_i \, cosh(\varepsilon_i z) , \tag{37}$$
$$\hat{\sigma}_{zz} = \frac{2\mu}{\beta_1 T_0} \left[R_2 \varepsilon \, cosh(\varepsilon z) + \sum_{i=1}^{3} \frac{\lambda_i \varepsilon_i^2}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} cosh(\varepsilon_i z) \right] d\xi + \sum_{i=1}^{3} \eta_i \, cosh(\varepsilon_i z), \tag{38}$$
$$\hat{\sigma}_{rz} = \frac{\mu\xi}{2\beta_1 T_0} \left[\left(\frac{\varepsilon^2 - \xi^2}{\varepsilon}\right) R_1 \, q sinh(\varepsilon z) + 2\sum_{i=1}^{3} \frac{\lambda_i}{\left(\frac{\mu\varepsilon_i^2}{\rho c_1^2} - s^2\right)} \varepsilon_i \, sinh(\varepsilon_i z) \right] , \tag{39}$$
$$\hat{P} = \sum_{i=1}^{3} \zeta_i \, cosh(\varepsilon_i z) , \tag{40}$$
where
$$\lambda_i = \left(\frac{\lambda+\mu}{\rho c_1^2} p_i - 1 - r_i\right) B_i , \; R_2 = \frac{\xi^2 R_1}{\varepsilon} , \varepsilon = \sqrt{\xi^2 + \frac{\rho c_1^2}{\mu} s^2}$$
$$\eta_i = \left(\frac{p_i - \rho c_1^2 - \rho c_1^2 r_i}{\beta_1 T_0}\right) B_i, \; \zeta_i = \left(-\beta_2 p_i - \frac{\rho c_1^2}{\beta_1} + \frac{b\rho c_1^2}{\beta_2} r_i\right) B_i$$

4. Boundary Conditions

The upper and lower surfaces $z = \pm b$ are assumed to be stress-free. An axisymmetric thermal source and a chemical potential source are prescribed on these boundary surfaces. Mathematically, the boundary conditions are written as

$$\frac{\partial T}{\partial z} = \pm g_0 G(r, z), \qquad (41)$$

$$\sigma_{zz} = 0, \qquad (42)$$

$$\sigma_{rz} = \sigma_{rz} = 0, \qquad (43)$$

$$P = F(r, t)\,. \qquad (44)$$

5. Application

As an application, we consider the following axisymmetric thermal source and ring-shaped chemical potential source:

$$G(r, z) = z^2 e^{-\omega r}\,, \qquad (45)$$

$$F(r, t) = \frac{1}{2\pi r}\delta(r - a_0)\ e^{-b_0 t} H(t)\,, \qquad (46)$$

where $\delta(r - a_0)$ denotes the Dirac delta distribution, $H(t)$ is the Heaviside unit step function, a_0 is the radius of the ring source, and b_0 is the temporal decay parameter.

Applying Laplace transform and Hankel transform defined by (19)-(20), on the equations (41)-(46), we obtain

$$\frac{d\hat{T}}{dz} = \pm g_0 \hat{G}(\xi\,, z)\,, \qquad (47)$$

$$\hat{\sigma}_{zz} = 0, \qquad (48)$$

$$\hat{\sigma}_{rz} = 0\,, \qquad (49)$$

$$\hat{P} = \hat{F}\,(\xi, s)\ . \qquad (50)$$

The transformed thermal and chemical potential source functions are given by

$$\hat{G}(\xi, z) = \frac{z^2 \omega}{(\xi^2 + \omega^2)^{3/2}}\,, \qquad (51)$$

$$\hat{F}(\xi, s) = \frac{1}{2\pi(s + b_0)} J_0(\xi a_0)\,. \qquad (52)$$

Substituting the values of $\hat{T}$, $\hat{P}$, $\hat{\sigma}_{zz}$, $\hat{\sigma}_{rz}$ from (28), (38)-(40) in the boundary conditions (47)-(50) and with the aid of (51) and (52), we obtain a set of non-homogeneous equations in four unknowns and solving the resulting equations, we obtain the values of unknown parameters as

$$B_1 = \frac{\Delta_1}{\Delta}\,, B_2 = \frac{\Delta_2}{\Delta}\,,\ B_3 = \frac{\Delta_3}{\Delta}\,, R_1 = \frac{\Delta_4}{\Delta}\,, \qquad (53)$$

where

$$\Delta = \begin{vmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} & 0 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \dfrac{-2\mu}{\beta_1 T_0}\xi^2 \varepsilon cos(\varepsilon h) \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \dfrac{\varepsilon^2 - \xi^2}{\varepsilon} sinh(\varepsilon h) \\ \Delta_{41} & \Delta_{42} & \Delta_{43} & 0 \end{vmatrix}$$

$\Delta_{1i}= \varepsilon_i\, sinh(\varepsilon_i h)$, $\Delta_{2i}= \left(\mu_i\varepsilon_i^2+\eta_i\right) cosh(\varepsilon_i h)$, $\Delta_{3i}= \mu_i\varepsilon_i\, sinh(\varepsilon_i h)$, $\Delta_{4i}= \zeta_i\, cosh(\varepsilon_i h)$, i=1,2,3 and Δ_i is obtained from Δ, by interchanging ith column with $\left[g_0\hat{G}(\xi,h)\ \ 0\ \ \ 0\ \ \ \frac{1}{2\pi(s+b_0)}\, J(\xi a_0)\right]^{t_r}$, where t_r denotes transpose.

Substituting the values of A_1, A_2, A_3 and R_1 and the values of $\hat{G}(\xi,z)$ and $\hat{F}(\xi,s)$ in the equations (34)-(40), we obtain the components of displacement, stress components, temperature change and chemical potential in the transformed domain subjected to thermal source and chemical potential source.

6. Basic Equations Inversion of Double Transform

Due to the complexity of the solution in the Laplace transform domain, the inverse of the Laplace transform is obtained by using the Gaver-Stehfast algorithm. Graver [41] and Stehfast ([42],[43]) derived the formula given below. By this method, the inverse $f(t)$ of Laplace transform $\bar{f}(s)$ is approximated by

$$f(t)=\frac{log2}{t}\sum_{j=1}^{k} D(j,K)F\left(j\frac{log2}{t}\right),$$

with

$$D(j,K)=(-1)^{j+M}\sum_{n=m}^{min(j,M)}\frac{n^M(2n)!}{(M-n)!n!(n-1)!(j-n)!(2n-j)!},$$

where K is an even integer, whose value depends on the word length of computer used. M=K/2, and m is an integer part of $(j+1)/2$. The optimal value of K was chosen as described in Gaver-Stehfast algorithm, for the fast convergence of results with desired accuracy. The Romberg numerical integration technique [44] with variable step size was used to evaluate the results involved.

7. Particular Cases

(i) If the diffusion effect is neglected, i.e., $\beta_2, a, b = 0$, in equations (34)-(40), then the expressions for the displacement components, stress components, temperature change, chemical potential, and mass concentration are obtained for an isotropic viscothermoelastic medium without diffusion.

(ii) Taking $\tau_q=\tau_T=0$in equations (34)–(40), then the corresponding expressions are obtained for a viscothermoelastic medium with the dual-phase-lag diffusion model only.

(iii) Considering $\tau_p=\tau_\eta=0$ in equations (34)–(40), then the corresponding expressions are obtained for a viscothermoelastic medium with the dual-phase-lag heat transfer model only.

(iv) If $\tau_q=0$ and $\tau_p=0$ in equations (34)–(40), then the corresponding expressions reduce to those for the single-phase-lag heat transfer model and the single-phase-lag diffusion model.

8. Numerical Results and Discussion

The numerical computations are carried out using the material constants taken from Dhaliwal and Singh [45].

$\lambda=7.76\times10^{10}Nm^{-2}$, $\mu=3.86\times10^{10}Nm^{-2}$, $K=386JK^{-1}m^{-1}s^{-1}$, $\beta_1=5.518\times10^{6}\ Nm^{-2}deg^{-1}$, $\rho=8954\,Kgm^{-3}$, $a=1.2\ \times10^{4}m^2/s^2k$, $b=$

$0.9 \times 10^6 m^5/kgs^2$, $D = 0.88 \times 10^{-8} kgs/m^3$, $\beta_2 = 61.38 \times 10^6\ Nm^{-2}deg^{-1}$, $T_0 = 293$K, $C_E = 383.1\, Jkg^{-1}K^{-1}$,

The graphs are plotted to study the effect of viscosity on the various field quantities in the range

$0 \leq r \leq 10$.

For the viscothermoelastic medium, the viscosity parameters are taken as

$Q_1 = 0.5,\ Q_2 = 1.0$,

whereas for the medium without viscosity, they are taken as

$Q_1 = Q_2 = 0$.

In the figures:

(i) The solid line represents the thermoelastic diffusive medium with viscosity (VS).

(ii) The small-dashed line represents the thermoelastic diffusive medium without viscosity (WVS).

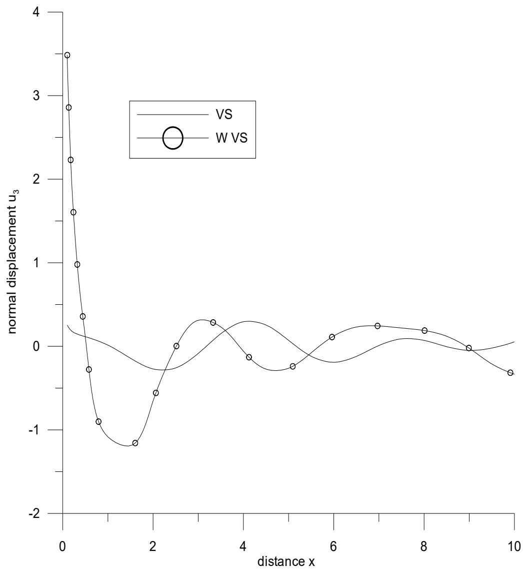

Fig.1 variations of axial displacement uzwith distance r

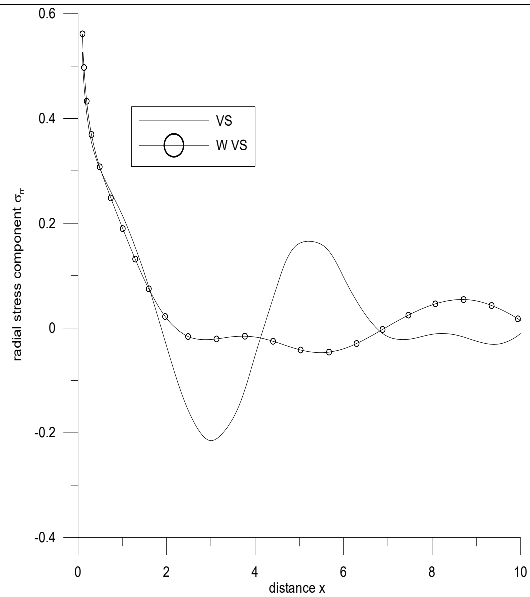

Fig.2 variations of radial stress component σ_{rr} with displacement r

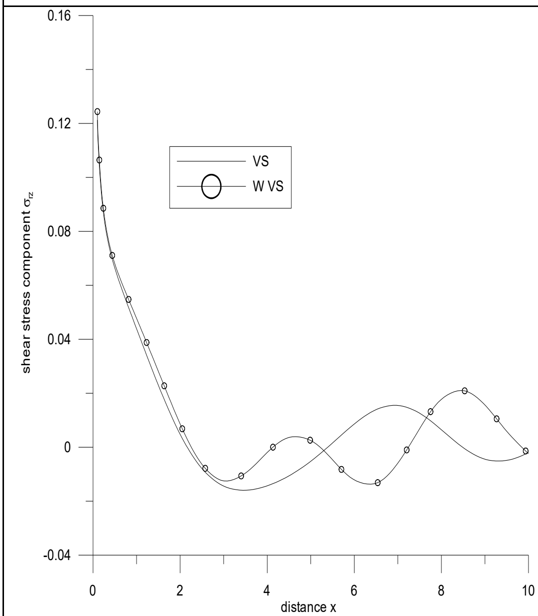

Fig.3 Variations of shear stress component σ_{rz} with displacement r

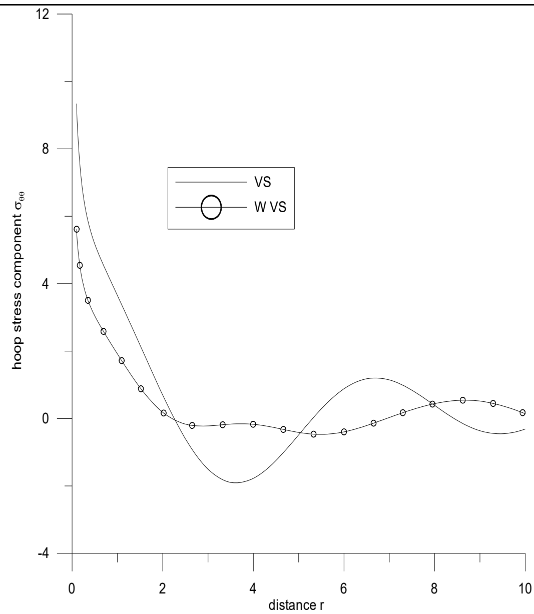

Fig.4 Variations of hoop stress component $\sigma_{\theta\theta}$with distance r

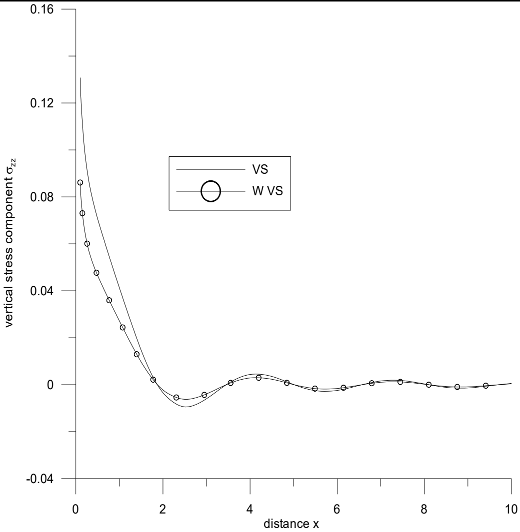

Fig.5 Variations of vertical stress component σ_{zz} with distance r

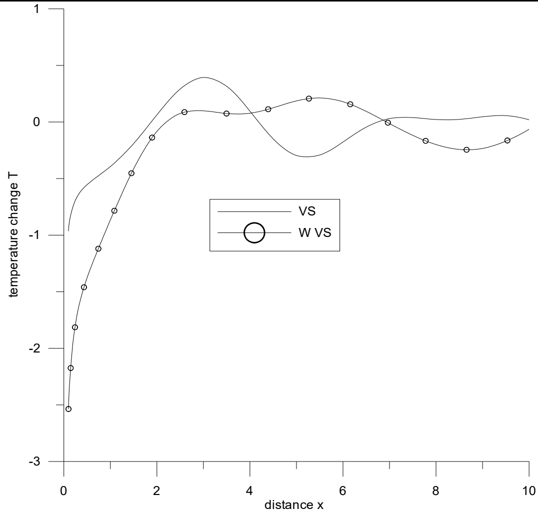

Fig.6 Variations of temperature change T with distance r

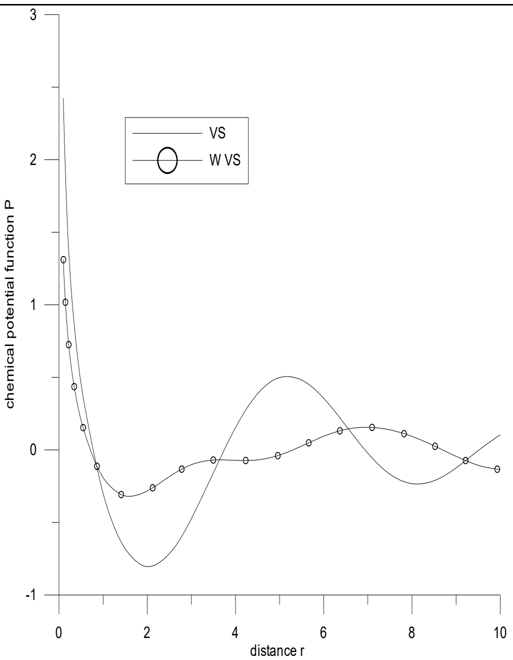

Fig.7 Variations of chemical potential function P with distance r

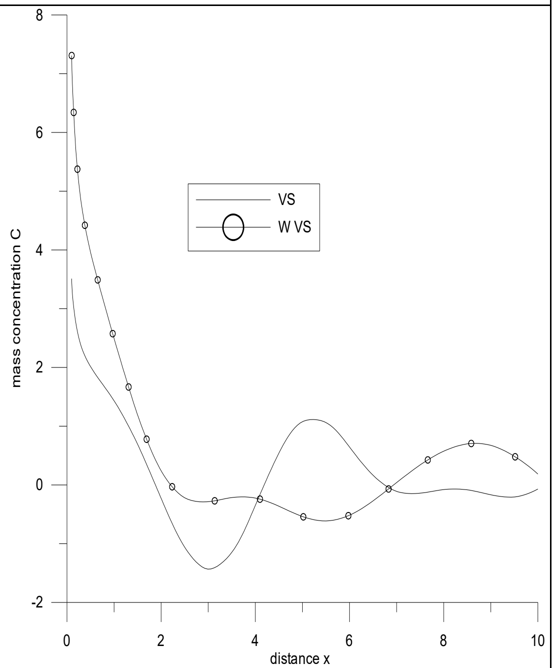

Fig.8 Variations of mass concentration C with distance r

Fig. 1 shows the variation of axial displacement uz with distance r. It is observed that, in the presence of viscosity, the axial displacement exhibits an oscillatory behaviour throughout the considered range. In the absence of viscosity, the displacement initially decreases sharply in the range $0 \leq r \leq 1$, and thereafter it follows an oscillatory trend. This indicates that viscosity has a significant influence on both the amplitude and nature of the axial displacement.
Fig. 2 illustrates the variation of the radial stress component σ_{rr} with distance r. For both viscous and non-viscous cases, the radial stress decreases sharply near the loading region, particularly in the range $0 \leq r \leq 2$. Beyond this range, the response becomes oscillatory. However, the amplitudes of oscillation are different in the two cases, indicating that viscosity considerably affects the radial stress distribution.
Fig. 3 depicts the variation of the shear stress component σ_{rz} with distance r. For both cases, the shear stress decreases smoothly in the initial region, approximately $0 \leq r \leq 3$. After this region, the response becomes oscillatory. The viscous and non-viscous curves show different oscillatory patterns, indicating that the inclusion of viscosity changes the nature of shear stress propagation in the medium.
Fig. 4 presents the variation of the hoop stress component $\sigma_{\theta\theta}$ with radial distance r. In the absence of viscosity, the variation of hoop stress is similar to the behaviour observed for the shear stress component in Fig. 3. In the presence of viscosity, the hoop stress exhibits a distinct oscillatory pattern and shows a noticeable variation near the boundary surface. This demonstrates that viscosity plays an important role in modifying the hoop stress distribution.
Fig. 4 presents the variation of the hoop stress component $\sigma_{\theta\theta}$with radial distance r. The hoop stress shows a rapid change near the loading region, indicating a strong circumferential stress response close to the source. As the radial distance increases, the response becomes oscillatory and gradually changes in amplitude. The presence of viscosity modifies the oscillatory behaviour and produces a noticeable difference between the viscous and non-viscous cases. This indicates that viscosity has a significant influence on the distribution and attenuation of hoop stress in the medium.
Fig. 5 illustrates the variation of the vertical stress component σ_{zz} with distance r. It is observed that, for both viscous and non-viscous cases, σ_{zz} decreases sharply near the loading region, particularly in the range $0 \leq r \leq 2$. Beyond this range, the magnitude of the vertical stress becomes very small and shows a mild oscillatory behaviour about the zero level. The effect of viscosity is more noticeable near the loading surface, where the difference between the two curves is significant. As the radial distance increases, both curves approach zero, indicating that the influence of the source becomes weaker away from the loading region.
Fig. 6 depicts the variation of temperature change T with radial distance r. The temperature field shows a sharp rise near the loading region, indicating a strong localized thermal response close to the applied thermal source. In the initial region, approximately $0 \leq r \leq 2$, both viscous and non-viscous cases exhibit a rapid increase from negative values toward the equilibrium level. At larger radial distances, both curves oscillate about the zero level, showing that the thermal disturbance weakens away from the source.
Fig. 7 depicts the variation of P with radial distance r. The chemical potential is initially high near the source region and then decreases rapidly as the radial distance increases. In the viscous case, the curve exhibits a larger oscillation in the intermediate region, whereas the non-viscous

case varies more smoothly with smaller amplitude. At larger radial distances, the two responses gradually approach the equilibrium level. This shows that viscosity affects the redistribution of the chemical potential in the medium and modifies both the amplitude and phase of the diffusive response.

Fig. 8 illustrates the variation of mass concentration C with distance r. The concentration attains its maximum value close to the source and decreases rapidly with increasing r, showing that the mass diffusion effect is dominant near the region of excitation. At larger radial distances, the concentration response no longer decays monotonically; instead, it exhibits small oscillations about the equilibrium level. The viscous medium shows a more pronounced oscillatory response in the intermediate region, while the non-viscous case varies more gradually with comparatively smaller fluctuation

8. Conclusion

The present investigation examines the effect of viscosity on the axisymmetric response of a viscothermoelastic diffusive medium within the framework of dual-phase-lag heat transfer and dual-phase-lag diffusion models. The analytical expressions for displacement, stresses, temperature change, chemical potential, and mass concentration are obtained by applying Laplace and Hankel transform techniques, and the corresponding physical-domain results are evaluated through numerical inversion.

The numerical results show that viscosity has a significant influence on all the field quantities considered. A comparison between the viscous and non-viscous cases reveals noticeable differences in amplitude, attenuation behaviour, and oscillatory response. The mechanical, thermal, and diffusive fields are found to be more pronounced near the source region and gradually weaken with increasing radial distance. The inclusion of viscosity modifies the redistribution of stresses, temperature, chemical potential, and mass concentration in the medium. Depending on the field variable, the viscous and non-viscous responses exhibit either similar or opposite oscillatory trends, confirming the important role of viscosity in the coupled dynamic behaviour of the medium.

The incorporation of diffusion phase lags in the mass diffusion equation provides a more realistic representation of mass transport than the classical Fickian diffusion model, as it accounts for the delayed response between the mass flux vector and the chemical potential gradient. Similarly, the dual-phase-lag heat transfer model improves the description of heat conduction by considering the phase-lag effects between heat flux and temperature gradient.

The present results may be useful in the analysis of two-dimensional axisymmetric problems involving coupled viscous, thermal, elastic, and mass diffusion effects, with potential applications in geophysical, industrial, and engineering systems.

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