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Axisymmetric Response Of A Thick Viscoelastic Plate Under Dual-Phase-Lag Heat Transfer And Dual-Phase-Lag Diffusion

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The present investigation deals with the axisymmetric response of a thick viscoelastic plate in a thermoelastic diffusive medium within the framework of dual-phase-lag heat transfer (DPLT) and dual-phase-lag diffusion (DPLD) models. The upper and lower surfaces of the plate are assumed to be traction-free and are subjected to an axisymmetric heat supply together with an impulsive axisymmetric chemical potential source distributed over a circular region. The governing equations are formulated in cylindrical polar coordinates and solved using Laplace and Hankel transform techniques. Analytical expressions for the displacement components, stress components, temperature change, chemical potential, and mass concentration are obtained in the transformed domain. A numerical inversion technique is then employed to recover the corresponding solutions in the physical domain. Numerical results are presented graphically to examine the influence of viscosity on the resulting field quantities. Some particular cases are also deduced from the present formulation.

Keywords: Dual-phase-lag heat transfer; dual-phase-lag diffusion; viscothermoelasticity; thick viscoelastic plate; chemical potential source; Hankel transform.

1 Introduction

The development of generalized thermoelastic theories began with the need to remove the physical inconsistency present in classical Fourier heat conduction. According to Fourier's law, a thermal disturbance propagates with infinite speed, which is unrealistic for problems involving short time intervals, rapid heating, high-frequency thermal loading, small length scales, and wave-like thermal transport. Cattaneo [1] proposed a modified form of the heat-conduction equation to eliminate the paradox of instantaneous propagation. Vernotte [2] also discussed the paradoxes associated with the classical continuous theory of heat conduction and introduced a relaxation-type modification. These works provided the basis for finite-speed heat propagation in non-Fourier heat-conduction theory.

On the basis of this relaxation concept, Lord and Shulman [3] developed a generalized dynamical theory of thermoelasticity. Their theory introduced one thermal relaxation time into the heat-conduction equation and allowed thermal disturbances to propagate with finite velocity. Green and Lindsay [4] further developed generalized thermoelasticity by introducing two relaxation parameters. These generalized thermoelastic theories are widely used for studying thermal shock, transient heat conduction, and thermoelastic wave propagation in elastic solids.

The coupling of thermal, elastic, and diffusive fields was later developed through thermoelastic diffusion theory. Nowacki [5] studied dynamical problems of thermodiffusion in solids and provided an important foundation for the analysis of coupled deformation, temperature, and concentration fields. Sherief, Saleh, and Hamza [6] formulated the theory of generalized thermoelastic diffusion, in which mass diffusion was incorporated into generalized thermoelasticity. Kumar and Kansal [7] presented thermoelastic diffusion theory under Green-Lindsay model and discussed the propagation of Lamb waves in a transversely isotropic thermoelastic diffusion plate. These studies established the theoretical and applied basis for analysing thermoelastic diffusion problems in solid media.

A major extension of non-Fourier heat-conduction theory was introduced by Tzou [8] through the dual-phase-lag model. This model considers two phase lags: one corresponding to the heat flux and the other corresponding to the temperature gradient. It provides a more general description of heat transfer in microscale and ultrafast thermal processes. Quintanilla and Racke [9] studied stability in dual-phase-lag heat conduction. Abdallah [10] investigated dual-phase-lag heat conduction and thermoelastic properties of a semi-infinite medium subjected to ultrashort pulsed laser heating. Chou and Yang [11] analysed two-dimensional dual-phase-lag thermal behaviour in single-layer and multilayer structures. Zhou, Zhang, and Chen [12] developed an axisymmetric dual-phase-lag bioheat model for laser heating of living tissues.

Further developments in generalized thermodiffusive and thermoelastic models were reported by several authors. Sharma [13] studied boundary-value problems in generalized thermodiffusive elastic media. Sharma [14] analysed deformation due to an inclined load in a generalized thermodiffusive elastic medium. Sharma, Kumar, and Ram [15] investigated interactions of generalized thermoelastic diffusion due to inclined loading. Atwa [16] examined generalized thermoelastic diffusion with the effect of a fractional parameter in a temperature-dependent elastic medium. Sharma, Sharma, and Bhargava [17] studied wave propagation in a micropolar thermoelastic solid with two temperatures bordered by layers or half-spaces of inviscid liquid. Kumar and Gupta [18] analysed plane-wave propagation in an anisotropic dual-phase-lag thermoelastic diffusion medium. Sharma, Sharma, and Bhargava [19] investigated plane waves and fundamental solutions in electro-microstretch elastic solids. Sharma and Sharma [20] studied the influence of heat sources and relaxation time on temperature distribution in tissues.

Additional work on the dual-phase-lag theory and generalized heat-conduction models was carried out by El-Karamany and Ezzat [21], who studied the dual-phase-lag thermoelasticity theory. Rukolaine [22] discussed unphysical effects of the dual-phase-lag model of heat conduction and emphasized the need for careful interpretation of phase-lag parameters. Ying and Yun [23] developed a time-fractional dual-phase-lag heat-conduction equation. Tripathi, Kedar, and Deshmukh [24] studied a generalized thermoelastic diffusion problem in a thick circular plate with axisymmetric heat supply.

The role of rheological and viscoelastic effects in thermal-stress problems was highlighted by Freudenthal [25], who studied the effect of rheological behaviour on thermal stresses.

Kaliski [26] investigated the absorption of magneto-viscoelastic surface waves in a real conductor magnetic field. Iesan and Scalia [27] presented some theorems in the theory of thermoviscoelasticity. Borrelli and Patria [28] analysed discontinuity waves in thermoviscoelastic solids of integral type. Pal [29] studied torsional body forces in a viscoelastic half-space, while Corr, Starr, Vanderby, and Best [30] developed a nonlinear generalized Maxwell fluid model. These studies contributed to the development of viscoelastic and thermoviscoelastic material theories.

Further extensions of viscoelastic and thermo-viscoelastic models included magnetic fields, viscosity, imperfect boundaries, voids, microstretch effects, rotation, non-homogeneity, modified couple stress, phase-lag effects, and fractional-order strain. Abd-Allaa and Mahmoud [31] investigated magneto-thermo-viscoelastic interactions in an unbounded non-homogeneous body with a spherical cavity subjected to periodic loading. Kumar, Chawla, and Abbas [32] studied the effect of viscosity on wave propagation in an anisotropic thermoelastic medium with a three-phase-lag model. Sharma and Bhargava [33] analysed thermoelastic plane waves at an imperfect boundary between a thermally conducting viscous liquid and a generalized thermoelastic solid. Sharma and Kumar [34] investigated plane waves and fundamental solutions in a thermoviscoelastic medium with voids. Sharma, Sharma, and Bhargava [35] studied the effect of viscosity on wave propagation in an anisotropic thermoelastic medium with Green-Naghdi theories of type II and type III. Sharma, Sharma, and Bhargava [36] examined wave motion and fundamental solutions in electro-microstretch viscoelastic solids.

Al-Basyouni, Mahmoud, and Alzahrani [37] investigated the effect of rotation, magnetic field, and periodic loading on radial vibrations of thermo-viscoelastic non-homogeneous media. Hilton [38] studied coupled longitudinal thermal and viscoelastic waves in media with temperature-dependent material properties. Yadav, Kalkal, and Deswal [39] considered two-temperature generalized thermoviscoelasticity with fractional-order strain subjected to a moving heat source. Kumar [40] analysed the response of a thermoelastic beam due to a thermal source within modified couple stress theory. Li, Guo, Tian, and Tian [41] studied the transient response of a half-space with variable thermal conductivity and diffusivity under thermal and chemical shock. Kumar, Miglani, and Rani [42] investigated the transient analysis of a nonlocal microstretch thermoelastic thick circular plate with phase lags. Kumar and Devi [43] examined deformation in modified couple stress thermoelastic diffusion in a thick circular plate due to heat sources.

Although many studies have been reported on generalized thermoelasticity, thermoelastic diffusion, dual-phase-lag heat conduction, and thermoviscoelastic media, the axisymmetric response of a thick viscoelastic plate under the combined action of heat supply and chemical potential source within DPLT and DPLD models has received comparatively less attention. In particular, the effect of viscosity on displacement, stress, temperature, chemical potential, and mass concentration fields has not been discussed in detail.

In the present paper, this problem is studied for a traction-free thick viscoelastic plate, radially infinite in extent, subjected to an axisymmetric heat supply and an impulsive axisymmetric chemical potential source. The governing equations are solved using Laplace and Hankel transform techniques. Transformed expressions for the field quantities are obtained, numerical inversion is applied to recover the physical-domain results, and the effect of viscosity is discussed graphically. Some particular cases are also deduced from the present formulation.

2 Basic Equations

The basic equations of motion, heat conduction and mass diffusion in a homogeneous isotropic thermoelastic solid with DPLT and DPLD models in the absence of body forces, heat sources and mass diffusion sources are

$$(\lambda_I + \mu_I)\nabla(\nabla.\mathbf{u}) + \mu_I\nabla^2\mathbf{u} - \beta_1\nabla T - \beta_2\nabla C = \rho\ddot{\mathbf{u}} \tag{1}$$

$$\left(1 + \tau_t\frac{\partial}{\partial t}\right)KT_{,ii} = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{{\tau_q}^2}{2}\frac{\partial^2}{\partial t^2}\right)[\rho C_E\dot{T} + \beta_1 T_0\nabla.\dot{\mathbf{u}} + aT_0\dot{C}] \tag{2}$$

$$\left(1 + \tau_p\frac{\partial}{\partial t}\right)(D\beta_2\nabla^2(\nabla.\dot{\mathbf{u}}) + Da\nabla^2 T - Db\nabla^2 C) + \frac{\partial}{\partial t}\left(1 + \tau_\eta\frac{\partial}{\partial t} + \frac{{\tau_\eta}^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0 \tag{3}$$

and the constitutive relations are

$$\sigma_{ij} = 2\mu_I e_{ij} + \delta_{ij}(\lambda u_{k,k} - \beta_1 T - \beta_2 C) \tag{4}$$

$$\rho T_0 S = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{{\tau_q}^2}{2}\frac{\partial^2}{\partial t^2}\right)(\rho C_E T + \beta_1 T_0 e_{kk} + aT_0 C) \tag{5}$$

$$P = -\beta_2 u_{k,k} - aT + bC \tag{6}$$

Where

$$\lambda_I = \lambda + \lambda_\nu\frac{\partial}{\partial t},$$

$$\mu_I = \mu + \mu_\nu\frac{\partial}{\partial t},\ \beta_1 = (3\lambda_I + 2\mu_I)\alpha_t\quad \beta_2 = (3\lambda_I + 2\mu_I)\alpha_c$$

Where λ , μ are Lamé's constants; λ_ν, μ_ν are viscosity constants; ρ is the density; D is the diffusivity; P is the chemical potential per unit mass; C is the mass concentration; u_i are displacement components; K is the thermal conductivity; C_E is the specific heat at constant strain; $T = \vartheta - T_0$ is small temperature increment; ϑ is the absolute temperature of the medium; T_0 is the reference temperature of the body such that $\left|\frac{T}{T_0}\right| \ll 1$.The constants a and b measure the thermodiffusion and mass diffusion effect respectively; σ_{ij} and e_{ij} denote the stress and strain components respectively; e_{kk} is dilatation; S is the entropy per unit mass; α_c and α_t are the coefficients of linear diffusion and thermal expansion respectively. The quantities τ_t, τ_q, τ_p, τ_η denote the phase lag of temperature gradient, heat flux, chemical potential, and diffusing mass flux vector respectively. A comma followed by suffix denotes spatial differentiation, and a superposed dot denotes differentiation with respect to time.

3 Formulation and Solution of the Problem

Consider a homogeneous isotropic viscoelastic plate of finite thickness $2b_1$, unbounded in the radial direction, occupying the region D= $\{(r,z) : 0 \le r < \infty , -b_1 \le z \le b_1\}$. The upper and lower surfaces $z = \pm b_1$, are mechanically traction-free and are subjected to an axisymmetric thermal flux together with a prescribed impulsive chemical potential distribution supported over a circular region $0 \le r \le a_0$. Owing to axisymmetry, all field variables are independent of θ, and the circumferential displacement vanishes, i.e., $u_\theta = 0$, Consequently, the field quantities u_r, u_z, T and C are independent of θ. Thus, the analysis is restricted to the two-dimensional problem:

$$u = (u_r, 0, u_z) \tag{7}$$

Equations (1)-(6) with the aid of (7) take the form

$$(\lambda_I + \mu_I)\frac{\partial e}{\partial r} + \mu_I\left(\nabla^2 - \frac{1}{r^2}\right)u_r - \beta_1\frac{\partial T}{\partial r} - \beta_2\frac{\partial C}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2} \quad (8)$$

$$(\lambda_I + \mu_I)\frac{\partial e}{\partial z} + \mu_I\nabla^2 u_z - \beta_1\frac{\partial T}{\partial z} - \beta_2\frac{\partial C}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2} \quad (9)$$

$$(1+\tau_t\frac{\partial}{\partial t})K\nabla^2 T = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{{\tau_q}^2}{2}\frac{\partial^2}{\partial t^2}\right)[\rho C_E\frac{\partial T}{\partial t} + \beta_1 T_0\frac{\partial e}{\partial t} + aT_0\frac{\partial C}{\partial t}] \quad (10)$$

$$(1+\tau_p\frac{\partial}{\partial t})(D\beta_2\nabla^2 e + Da\nabla^2 T - Db\nabla^2 C) + \frac{\partial}{\partial t}\left(1 + \tau_\eta\frac{\partial}{\partial t} + \frac{{\tau_\eta}^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0 \quad (11)$$

and Constitutive relations

$$\sigma_{rr} = 2\mu_I e_{rr} + \lambda_I e - \beta_1 T - \beta_2 C \quad (12)$$

$$\sigma_{\theta\theta} = 2\mu_I e_{\theta\theta} + \lambda_I e - \beta_1 T - \beta_2 C \quad (13)$$

$$\sigma_{zz} = 2\mu_I e_{zz} + \lambda_I e - \beta_1 T - \beta_2 C \quad (14)$$

$$\sigma_{rz} = 2\mu_I e_{rz},\ \sigma_{r\theta} = 0,\ \sigma_{z\theta} = 0 \quad (15)$$

$$P = -\beta_2 e - aT + bC \quad (16)$$

where,

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z},\ e_{rr} = \frac{\partial u_r}{\partial r},\ \ e_{\theta\theta} = \frac{u_r}{r},\ e_{zz} = \frac{\partial u_z}{\partial z},\ e_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \quad (17)$$

To facilitate the solution, the following dimensionless quantities are introduced

$$r' = \frac{\omega_1^*}{c_1}r,\ z' = \frac{\omega_1^*}{c_1}z,\ (u_r', u_z') = \frac{\omega_1^*}{c_1}(u_r, u_z),\ t' = \omega_1^* t,\ (\sigma_{rr}', \sigma_{\theta\theta}', \sigma_{zz}', \sigma_{rz}') = \frac{1}{\beta_1 T_0}(\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz})$$

$$(T', C') = \frac{1}{\rho c_1^2}(\beta_1 T, \beta_2 C),\ \ (\tau_q', \tau_t', \tau_p', \tau_\eta') = \omega_1^*(\tau_q, \tau_t, \tau_p, \tau_\eta),\ \omega_1^* = \frac{\rho C_E c_1^2}{K},\ c_1^2 = \frac{\lambda+2\mu}{\rho} \quad (18)$$

Using Eq. (18) in Eqs. (7)–(16), and then suppressing the primes, we obtain

$$A_1\frac{\partial e}{\partial r} + A_2\nabla^2 u_r - \frac{\partial T}{\partial r} - \gamma\frac{\partial C}{\partial r} = \frac{\partial^2 u_r}{\partial t^2}, \quad (19)$$

$$A_1\frac{\partial e}{\partial z} + A_2\nabla^2 u_z - \frac{\partial T}{\partial z} - \gamma\frac{\partial C}{\partial z} = \frac{\partial^2 u_z}{\partial t^2}, \quad (20)$$

$$(1+\tau_t\frac{\partial}{\partial t})\nabla^2 T = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{{\tau_q}^2}{2}\frac{\partial^2}{\partial t^2}\right)[\frac{\partial T}{\partial t} + +\varepsilon\frac{\partial e}{\partial t} + \chi\frac{\partial C}{\partial t}], \quad (21)$$

$$(1+\tau_p\frac{\partial}{\partial t})(\delta_1\nabla^2 e + \delta_2\nabla^2 T - \delta_3\nabla^2 C) + \delta_4\frac{\partial}{\partial t}\left(1 + \tau_\eta\frac{\partial}{\partial t} + \frac{{\tau_\eta}^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0 \quad (22)$$

Here,

$A_1 = (\lambda_I + \mu_I)/(\rho c_1^2)$, $A_2 = \mu_I/(\rho c_1^2)$, $\gamma = \beta_2/\beta_1$,

$\varepsilon = \beta_1^2 T_0/(\rho C_E c_1^2)$, $\chi = aT_0/(\rho C_E)$, $\nabla^2 = \partial^2/\partial r^2 + (1/r)\,\partial/\partial r + \partial^2/\partial z^2$.

$\delta_1 = \beta_2^2/(\rho c_1^2)$, $\delta_2 = a\beta_2/\beta_1$, $\delta_3 = b$, $\delta_4 = K/\rho C_E D$.

The non-dimensional constitutive relations are

$$\sigma_{rr} = 2A_2 e_{rr} + (A_1 - A_2)e - T - \gamma C, \quad (23)$$

$$\sigma_{\theta\theta} = 2A_2\frac{u_r}{r} + (A_1 - A_2)e - T - \gamma C \quad (24)$$

$$\sigma_{zz} = 2A_2\frac{\partial u_z}{\partial z} + (A_1 - A_2)e - T - \gamma C \quad (25)$$

$$\sigma_{rz} = A_2\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right), \quad (26)$$

$$P = -\gamma e - \gamma_1 T + \gamma_2 C, \quad (27)$$

Where, $\gamma_1 = a/\beta_1$, $\gamma_2 = b/\beta_2$.

To simplify the displacement equations, we introduce the potential functions φ(r,z,t) and ψ(r,z,t) as

$u_r = \partial\varphi/\partial r - \partial\psi/\partial z,$ (28)

$u_z = \partial\varphi/\partial z + (1/r)\partial(r\psi)/\partial r.$ (29)

Taking the divergence of Eqs. (19) and (20), and using Eqs. (28) and (29), we obtain

$[M\nabla^2 - \partial^2/\partial t^2]\, e - \nabla^2 T - \gamma\nabla^2 C = 0,$ (30)

$[A_2\nabla^2 - \partial^2/\partial t^2]\psi = 0,$ (31)

where

$M = A_1 + A_2,\ e = \nabla^2\varphi$

Using Eqs. (28) and (29), Eqs. (21) and (22) reduce to

$(1+\tau_t\frac{\partial}{\partial t})\nabla^2 T = \left(1+\tau_q\frac{\partial}{\partial t}+\frac{\tau_q{}^2}{2}\frac{\partial^2}{\partial t^2}\right)[\frac{\partial T}{\partial t} + +\varepsilon\frac{\partial e}{\partial t} + \chi\frac{\partial C}{\partial t}]$, (32)

$(1+\tau_p\frac{\partial}{\partial t})(\delta_1\nabla^2 e + \delta_2\nabla^2 T - \delta_3\nabla^2 C) + \delta_4\frac{\partial}{\partial t}\left(1+\tau_\eta\frac{\partial}{\partial t}+\frac{\tau_\eta{}^2}{2}\frac{\partial^2}{\partial t^2}\right)C = 0$ (33)

Define the Laplace and Hankel transforms as

$$\bar{f}(r,z,s) = \int_0^\infty f(r,z,t)e^{-st}dt \quad (34)$$

$$\hat{f}(\xi,z,s) = \int_0^\infty \bar{f}(r,z,s)rJ_n(r\xi)\,dr \quad (35)$$

Applying the Laplace and Hankel transforms defined by Eq. (34) and (35) to Eqs. (30)–(33), we obtain

$[M\,L - s^2]\,\hat{e} - L\,\hat{T} - -\gamma L\hat{C} = 0,$ (36)

$[A_2 L - s^2]\hat{\psi} = 0,$ (37)

$(L - a_1)\hat{T} - a_1\varepsilon\,\hat{e} - a_1\chi\hat{C} = 0$ (38)

$\delta_1 L\ \hat{e} + \delta_2 L\hat{T} + (a_2 - \delta_3 L)\hat{C} = 0$ (39)

where

$L = (d^2/dz^2) - \xi^2,\ \hat{e} = L\,\hat{\varphi}$, $a_1 = s\left(1+\tau_q s+\frac{\tau_q{}^2}{2}s^2\right)/(1+\tau_t s)$,

$a_2 = \delta_4 s\left(1+\tau_\eta s+\frac{\tau_\eta{}^2}{2}s^2\right)/(1+\tau_p s)$, (40)

Assuming the solution of Eqs. (36), (38) and (39) in the form

$(\hat{e}, \hat{T}, \hat{C}) = (E_i, F_i, G_i)\cosh(q_i z)$, i=1,2,3. (41)

and substituting Eq. (41) into Eqs. (36), (38) and (39), we obtain

$[Mk_i^2 - s^2]\,E_i - k_i^2\,F_i - \gamma k_i^2\,G_i) = 0,$ (42)

$-a_1\varepsilon E_i + (k_i^2 - a_1)F_i - a_1\chi G_i = 0$ (43)

$\delta_1\,k_i^2\,E_i + \delta_2 k_i^2 F_i + (a_2 - \delta_3 k_i^2)G_i = 0$ (44)

where $q_i^2 = k_i^2 + \xi^2$, (i=1,2,3).

After some algebraic simplification, the solution of Eqs. (42)–(44) can be written as

$$\hat{T} = \sum_{i=1}^{3} F_i cosh(q_i z)\ , \quad (45)$$

$$\hat{C} = \sum_{i=1}^{3} d_i F_i cosh(q_i z)\ , \quad (46)$$

$$\hat{e} = \sum_{i=1}^{3} f_i F_i cosh(q_i z) \quad (47)$$

$$\hat{\varphi} = \sum_{i=1}^{3}(f_i/k_i^2)F_i cosh(q_i z)\ , \quad (48)$$

where

$$f_i = \frac{(k_i^2 \; - a_1)(a_2 - \delta_3 k_i^2) + a_1 \chi \delta_2 k_i^2}{a_1[\varepsilon \; (a_2 - \delta_3 k_i^2) - \chi \delta_1 \; k_i^2]},$$

$$d_i = -\frac{[\delta_1 (k_i^2 \; - a_1) + a_1 \varepsilon \delta_2] k_i^2}{a_1[\varepsilon \; (a_2 - \delta_3 k_i^2) - \chi \delta_1 \; k_i^2]}. \quad (49)$$

The solution of Eq. (37) can be expressed as

$\hat{\psi}$= $B_2 \sinh(q_0 z)$, (50)

where

$q_0^2 = \xi^2 + s^2/A_2$

Applying the Laplace and Hankel transforms defined by Eqs. (34) and (35) to Eqs. (28), (29), and (23)–(27), and using Eqs. (45)–(50), we obtain the transformed expressions for displacement components, stress components, and chemical potential

$$\hat{u}_r = -\xi \sum_{i=1}^{3} (f_i/k_i^2) F_i cosh(q_i z) - B_2 q_0 \cosh(q_0 z), \quad (51)$$

$$\hat{u}_z = \sum_{i=1}^{3} (q_i f_i/k_i^2) F_i sinh(q_i z) + \xi B_2 \sinh(q_0 z). \quad (52)$$

$$\hat{\sigma}_{rr} = \sum_{i=1}^{3} \; [-2A_2 \xi^2 f_i/k_i^2 + (A_1 - A_2) f_i - 1 - \gamma d_i)] F_i cosh(q_i z) - 2A_2 \xi \; q_0 B_2 cosh(q_0 z). \quad (53)$$

$$\hat{\sigma}_{\theta\theta} = \sum_{i=1}^{3} \; [\,(A_1 - A_2) f_i - 1 - \gamma d_i)] F_i cosh(q_i z) + 2A_2 \int_{\xi}^{\infty} \hat{u}_r(\eta\,, z, s)\, d\eta \quad (54)$$

$$\hat{\sigma}_{zz} = \sum_{i=1}^{3} [\,2A_2 q_i^2 (f_i/k_i^2) + (A_1 - A_2) f_i - 1 - \gamma d_i] F_i cosh(q_i z) \; + \; 2A_2 q_0 \, \xi B_2 cosh(q_0 z),$$
(55)

$$\hat{\sigma}_{rz} = A_2 \, [-\, 2\, \xi \sum_{i=1}^{3} (q_i f_i/k_i^2) F_i sinh(q_i z) - ({q_0}^2 + \xi^2) B_2 \sinh(q_0 z)]. \quad (56)$$

$$\hat{P} = \sum_{i}^{3} (-\gamma f_i - \gamma_1 + \gamma_2 d_i) \; F_i cosh(q_i z) \quad (57)$$

4 Boundary Conditions

We consider an axisymmetric thermal source and a chemical potential source of unit magnitude applied on the stress-free upper and lower surfaces of the thick viscoelastic plate. The stress components vanish on the surfaces$z = \pm b_1$. Mathematically, these are written as

$$\frac{\partial T}{\partial z} = \pm g_0 F(r, z), \quad (58)$$

$$\sigma_{zz} = 0, \quad (59)$$

$$\sigma_{rz} = 0, \quad (60)$$

$$P = f(r, t) \quad (61)$$

5. Application

As an application, we consider the following source functions:

$$F(r, z) = z^2 e^{-\omega r} \quad (62)$$

$$f(r, t) = \delta(t) H(a_0 - r) \quad (63)$$

Where $\delta(t)$ is the Dirac delta function, $H(a_0 - r)$ is the Heaviside unit step function representing a circular source region of radius a_0.

Applying Laplace and Hankel transform given by (34)-(35) to (58)-(63), we obtain

$$\frac{d\hat{T}}{dz} = \pm g_0 \hat{F}(\xi\,, z), \quad (64)$$

$$\hat{\sigma}_{zz} = 0, \quad (65)$$

$$\hat{\sigma}_{rz} = 0 \quad (66)$$

$$\hat{P} \; = \; \hat{f}\,(\xi, s) \quad (67)$$

The transformed source functions are

$\hat{F}(\xi\ ,z),=\frac{z^2\omega}{(\xi^2+\omega^2)^{3/2}},$ (68)

$\hat{f}\ (\xi,s)=\frac{a_0J_1(\xi a_0)}{\xi}$. (69)

Using Eqs, (45),(55)-(57) in (64)-(67) ,we obtain a set of four non homogeneous equations in the four unknowns $F_1, F_2\ ,F_3$ and B_2. Solving this system, the unknown parameters are obtained as

$F_1=\frac{\Delta_1}{\Delta}\ , F_2=\frac{\Delta_2}{\Delta},\ F_3=\frac{\Delta_3}{\Delta}\ , B_2=\frac{\Delta_4}{\Delta}$ (70)

where

$$\Delta=\begin{vmatrix} H_1 & H_2 & H_3 & 0 \\ Z_1 & Z_2 & Z_3 & Z_4 \\ R_1 & R_2 & R_3 & R_4 \\ P & P_2 & P_3 & 0 \end{vmatrix},$$

and $\Delta_1, \Delta_2, \Delta_3, \Delta_4$, are obtained by replacing $\left[\frac{z^2\omega}{(\xi^2+\omega^2)^{3/2}} \quad 0 \quad 0 \quad \frac{a_0J_1(\xi a_0)}{\xi}\right]^{tr}$ in first, second, third and fourth column of Δ respectively by $\left[\frac{{b_1}^2\omega}{(\xi^2+\omega^2)^{3/2}} \quad 0 \quad 0 \quad \frac{a_0J_1(\xi a_0)}{\xi}\right]^{tr}$.

Here tr denotes transpose.

Also,

$H_i= q_i sinh(q_ib_1\ )$

$Z_i=[2A_2q_i^2(f_i/k_i^2)+ +(A_1-A_2)f_i-1-\gamma d_i]cosh(q_ib_1\ )$

$Z_4=2A_2q_0\ \xi cosh(q_0b_1\ )$

$R_i=\ -2\ \ A_2\ \xi\ (q_if_i/k_i^2)sinh(q_ib_1\ )$

$R_4=({q_0}^2+\ \xi^2)A_2\sinh(q_0\ b_1\ ).$

$P_i=(-\gamma f_i-\gamma_1+\gamma_2d_i)\ cosh(q_ib_1\ )$

Making use the values of F_i ($i=1,2,3$) and B_2 from Eq. (70) in Eqs. (51)-(55), yield the expressions for the displacement components, stress components and chemical potential in the transformed domain

6 Inversion of the Transforms

The solution is obtained in physical domain, we must invert the transforms in (51)-(57), for all the theories. Here the displacements, stresses, temperature change, chemical potential and mass concentration are functions of z, the parameters of Laplace and Hankel transforms s and η respectively and hence are of the form $\hat{f}(\eta,z,s)$. First we invert the Hankel transform, which gives the Laplace transform expression $\bar{f}(r,z,s)$ of the function $f(r,z,t)$ as

$$\bar{f}(r,z,s)=\int_0^\infty \eta\,\hat{f}(\eta,z,s)J_n(\eta r)d\eta\,. \tag{71}$$

Now for fixed values of η, r and z, the function $\bar{f}(r,z,s)$ in (71) can be considered as the

Laplace transform $\bar{g}(s)$ of the same function $g(t)$. Following Honig and Hirdas [44], the Laplace transformed function $\bar{g}(s)$ can be inverted as given below:

$$g(t)=\frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} e^{st}\,\bar{g}(s)ds, \tag{72}$$

where C is an arbitrary real number greater than all the real parts of the singularities of $\bar{g}(s)$. Taking $s=C+iy$, we get

$$g(t)=\frac{e^{Ct}}{2\pi}\int_{-\infty}^{\infty} e^{ity}\,\bar{g}(C+iy)dy, \tag{73}$$

The last step is to calculate the integral in equation (71). The method for calculating this integral is described by Press et al. [45]. It involves the use of Romberg's integration with adaptive step size. This also uses the results from successive refinements of the extended trapezoidal rule followed by extrapolation of the results to the limit when the step size tends to zero.

7 Particular Cases

(i) If the diffusion effect is neglected, i.e., $\beta_2, a, b=0$, then Eqs. (51)–(57), along with Eq. (70), give the expressions for the displacement components, stress components, temperature change, and chemical potential in a thermoviscoelastic isotropic medium.

(ii). If $\tau_q=\tau_t=a=0$, then Eqs (51)-(57) along with Eq. (70) , give the corresponding expressions in thermoviscoelastic with the DPLD model.

(iii) If $\tau_\eta = \tau_p=0 = a$, then Eqs (51)-(57) along with (70) give the corresponding expressions for DPLT model.

(iv) If $\tau_t=0$ and $\tau_p=0$, then Eqs (51)-(57) along with (70) give the corresponding expressions for single- phase-lag heat model (SPLT) and single- phase-lag diffusion model (SPLD)

8 Numerical results and Discussion

The material constants used for numerical computation are taken from Dhaliwal and Singh [46]:

$\lambda=7.76\times10^{10}Nm^{-2}$, $\mu=3.86\times10^{10}Nm^{-2}$, $K=386JK^{-1}m^{-1}s^{-1}$, $\beta_1=5.518\times10^{6}Nm^{-2}deg^{-1}$, $\rho=8954\,Kgm^{-3}$, $a=1.2\times10^{4}m^{2}s^{-2}\ k^{-1}$, $b=0.9\times10^{6}m^{5}kg^{-1}\ s^{-2}$, $D=0.88\times10^{-8}kgsm^{-3}$, $\beta_2=61.38\times10^{6}\ Nm^{-2}deg^{-1}$, $T_0=293$K, $C_E=383.1\,Jkg^{-1}K^{-1}$

The graphs are plotted to study the effect of viscosity on the various field quantities in the range $0\le r\le10$ at z=1,With the parameter

$\tau_t=0.02, \tau_q=0.04, \tau_p=0.03, \tau_\eta$=0.05, $\omega=0.5$, t=0.1, $a_0=0.4$, b_1= 1.

For the viscoelastic medium, the values Q_1=0.5 and $Q_2=1.0$ are taken, whereas for the non-viscoelastic medium, $Q_1=Q_2=0$ are considered.

In the figures, the solid line represents the thermoelastic diffusive plate with viscosity (VS), while the small dashed line corresponds to the thermoelastic diffusive plate without viscosity (WVS).

Fig. 1 shows the variation of the radial displacement component u_r with distance r. It is observed that u_r has a maximum positive value near the loaded region and decreases sharply as r increases. After the initial decrease, both the viscous (VS) and non-viscous (WVS) cases exhibit oscillatory behaviour. The viscous case shows larger fluctuations and attains more pronounced positive and negative values than the non-viscous case.

Fig. 2 depicts the variation of the axial displacement component u_z with distance r.The displacement u_z shows oscillatory behaviour for both the VS and WVS cases throughout the considered range. In the viscous case, the oscillations are more prominent and their amplitude is comparatively larger. In the absence of viscosity, the variation is smoother and the amplitude remains smaller.

Fig. 3 shows the variation of the radial stress component σ_{rr} with distance r. It is observed that near the loaded region, the values of σ_{rr} are positive for both cases and decrease as r increases. The variation corresponding to the viscous medium (VS) shows pronounced oscillatory behaviour with larger amplitude, attaining both positive and negative values. In contrast, the curve corresponding to the medium without viscosity (WVS) varies more smoothly and exhibits smaller oscillations.

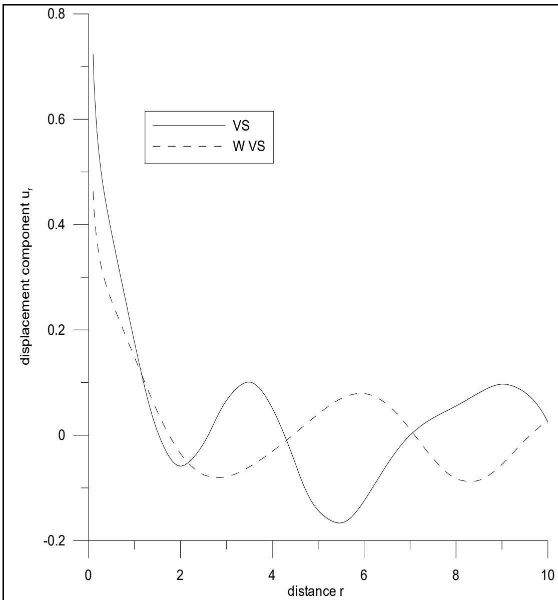

fig.1 variations of displacement component $u_r$with distance r

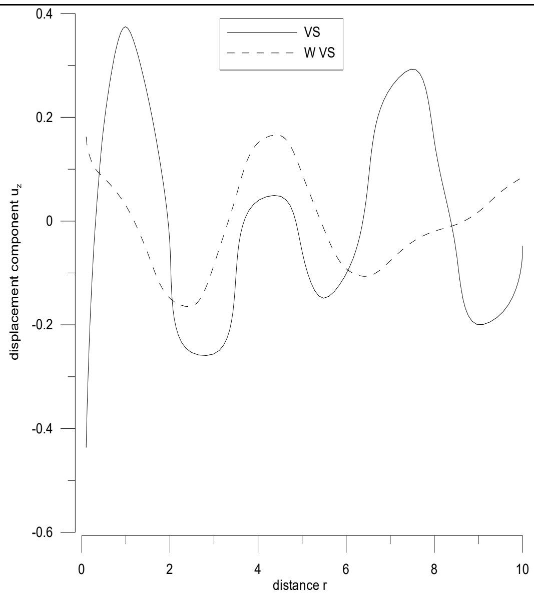

Fig.2. variations of displacement component $u_z$with distance r

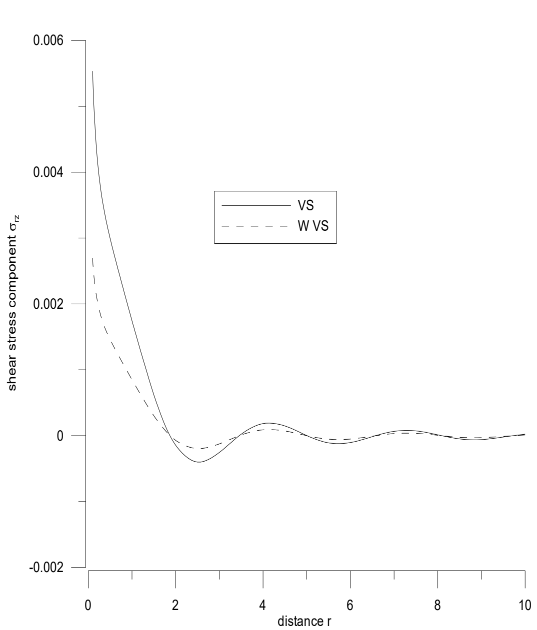

Fig.5. variations of shear stress component σ_{rz} with distance r

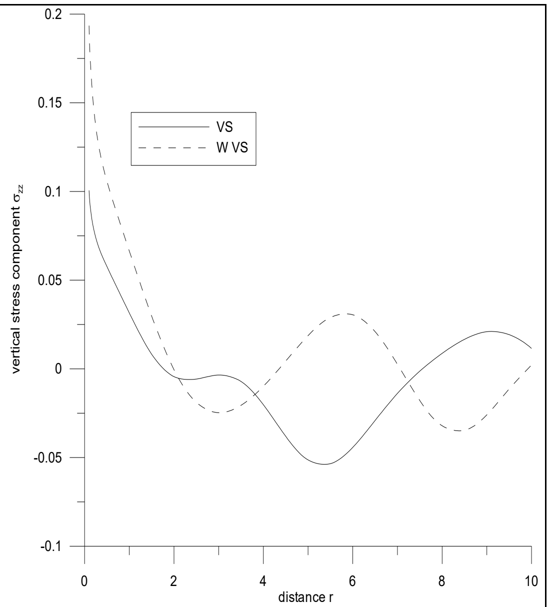

Fig.6 variations of vertical stress component σ_{zz} with distance r

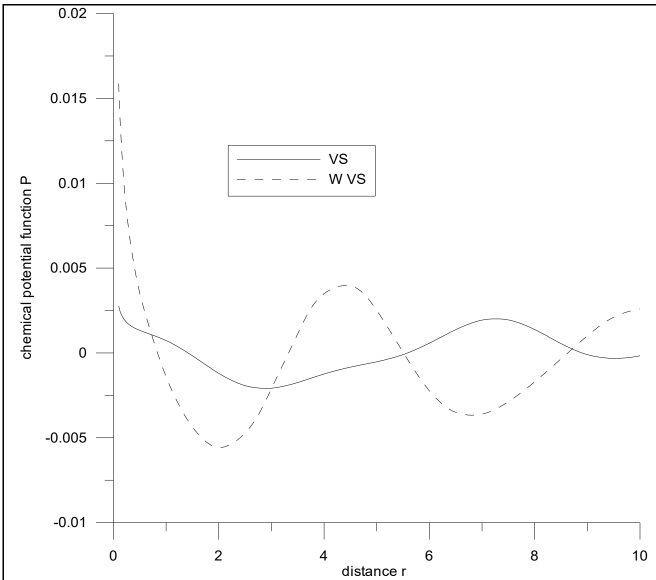

Fig.7. variations of chemical potential function P with distance r

Fig. 4 depicts the variation of the hoop stress component $\sigma_{\theta\theta}$ with distance r. The behaviour of $\sigma_{\theta\theta}$ is similar to that of the radial stress component σ_{rr}. For the viscous case, the hoop stress shows larger oscillations and changes sign over the considered range of r. For the case without viscosity, the variation is comparatively smoother and the amplitude remains smaller.

Fig. 5 shows the variation of the shear stress component σ_{rz} with distance r. Near the loaded region, σ_{rz} decreases sharply for both the viscous (VS) and non-viscous (WVS) cases. As r increases, the magnitude of the shear stress σ_{rz} becomes very small, and the curves show weak oscillatory behaviour about zero. The difference between the VS and WVS cases is more noticeable near the loaded region, whereas away from the source region, both variations become nearly negligible.

Fig. 6 depicts the variation of the normal stress component σ_{zz} with distance r. It is observed that σ_{zz} has a maximum positive value near the loaded region and decreases sharply in the initial range of r. After this decrease, both the VS and WVS cases exhibit oscillatory behaviour with different amplitudes and phases. The viscous case shows a more pronounced negative variation in the middle region, whereas the non-viscous case exhibits a comparatively smoother oscillatory trend.

Fig. 7 shows the variation of the chemical potential P with distance r. Near the loaded region, the value of P is positive and decreases rapidly for both the viscous (VS) and non-viscous (WVS) cases. As r increases, both curves exhibit oscillatory behaviour about zero. The variation corresponding to the non-viscous case shows larger oscillations, whereas the viscous case varies more smoothly with comparatively smaller amplitude.

Conclusion

In the present study, the axisymmetric response of a thick viscoelastic plate has been investigated within the framework of dual-phase-lag heat transfer (DPLT) and dual-phase-lag diffusion (DPLD) models. The upper and lower surfaces of the plate are traction-free and are subjected to an axisymmetric heat supply together with an impulsive axisymmetric chemical potential source distributed over a circular region. The governing equations are formulated in cylindrical polar coordinates and solved using Laplace and Hankel transform techniques. Analytical expressions for the displacement components, stress components, temperature change, chemical potential, and mass concentration are obtained in the transformed domain, and the corresponding physical-domain results are recovered through numerical inversion.

The numerical results show that viscosity has a significant influence on the coupled thermoelastic-diffusive response of the plate. The radial and axial displacement components exhibit oscillatory behaviour with distance r, and the inclusion of viscosity generally increases the amplitude of their variations. Similar effects are observed for the radial, hoop, and normal stress components, for which the viscous medium exhibits more pronounced oscillations than the medium without viscosity. The shear stress decreases rapidly near the loaded region and becomes nearly negligible away from the source region for both cases. The chemical potential also exhibits oscillatory behaviour; however, in this case, the

presence of viscosity reduces the amplitude of oscillation compared with the non-viscous medium.
In addition, some particular cases are obtained by suppressing the diffusion effect or by selecting appropriate phase-lag parameters. These cases show that the present formulation reduces to the corresponding thermoviscoelastic, DPLT, DPLD, and single-phase-lag heat transfer and diffusion models.

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