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On Toeplitz Operators with Poly-Quasihomogenous Symbol

Raja'a Al-Naimi

Department of Mathematics, University of Petra, Amman, Jordan,
rajaa.alnaimi@uop.edu.jo

In this paper, we give some basic results concerning Toeplitz operators whose symbol is of the form $f(\theta)\phi$, where ϕ is a radial function and $f(\theta)$ is a polynomial in $e^{i\theta}$, then use these results to characterize all Toeplitz operators which commute with them on the Bergman space.

Keywords: Toeplitz operators, Bergman space,Poly-Quasihomogenous.

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and dA be the normalized Lebesgue area measure on $\mathbb{D}$. The space of all complex-valued measurable functions f on $\mathbb{D}$ such that

$$||f||_2 = \left(\int_{\mathbb{D}} |f(z)|^2 \, dA(z)\right)^{1/2} < \infty,$$

is denoted, as usual, by $L^2(\mathbb{D})$. The Bergman space $L_a^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ consisting of holomorphic functions.

Let P be the orthogonal projection from $L^2(\mathbb{D})$ onto $L_a^2(\mathbb{D})$. For a function $\phi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator $T: L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ defined by

$$T_\phi f = P(\phi f), \; for \; all \; f \in L_a^2(\mathbb{D}).$$

It is clear that $||T_\phi|| \leq ||\phi||_\infty$ and T_ϕ is bounded.

The Bergman space is a reproducing kernel Hilbert space, the reproducing kernel is given by $k_z(w) = \frac{1}{(1-\bar{z}w)^2}$, and the normalized reproducing kernel is $\frac{k_z(w)}{||k_z(w)||_2} = \frac{(1-|z|^2)}{(1-\bar{z}w)^2}$.

The class of quasihomogenous symbols is one of the most interesting class of symbols of Toeplitz operators, founded by Louhichi and Zakariasy in 2005. In this artical, we define Toeplitz operator like the quasihomogenous Toeplitz operator and call it poly-quasihomogenous Toeplitz operator, and we give some basic results concerning such operator.

2. Preliminaries

A function f is said to be quasihomogeneous of degree p, where p is an integer, if it is of the form $e^{ip\theta}\phi$, where ϕ is a radial function. In this case the associated Toeplitz operator T_f is also called quasihomogeneous Toeplitz operator of degree p. Such functions were studied in [?] and [?]. The reason that we study such family of symbols is that any function f in $L^2(\mathbb{D})$ has the following polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], rdr)$.

Now, we need to introduce the Mellin transform that has been a very useful tool in obtaining many results. The Mellin transform $\hat{f}$ of a radial function f in $L^1([0,1], rdr)$ is defined by

$$\hat{f}(z) = \int_0^1 f(r) r^{z-1}\, dr.$$

The following known lemma, is also helpful.

Lemma 2.1 *Let $k, p \in \mathbb{N}$ and let f be an integrable radial function. Then*

$$T_{e^{ip\theta}f}(z^k) = 2(k+p+1)\hat{f}(2k+p+2)z^{k+p}$$

and

$$T_{e^{-ip\theta}f}(z^k) = \begin{cases} 0 & if 0 \le k \le p-1 \\ 2(k-p+1)\hat{f}(2k-p+2)z^{k-p} & if k \ge p. \end{cases}$$

Now, we list a set of theories and facts about quasihomogenous Toeplitz operators, that we used in our results

Theorem 2.2 *A bounded function f is quasihomogenous of degree $p \in \mathbb{Z}$ if and only if, for all integers $n \ge 0$, there exists $\alpha_n \in \mathbb{C}$ such that*

$$T_f(z^n) = \begin{cases} 0 & if\ n < max(-p,0), \\ \alpha_n z^{n+p} & if\ n \ge max(-p,0). \end{cases}$$ (

It is important and helpful to know that the Mellin transform is uniquely determined by its values on any arithmetic sequence of integers, In fact we have the following theorem.

Theorem 2.3 *[Remmert, 1998] Suppose that f is a bounded analytic function on $\{z: Re(z) > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2 \ldots$, where $inf\{|z_n|\} > 0$ and*

$\sum_{n\ge1} Re(\frac{1}{z_n}) = \infty$. Then f vanishes identically on $\{z: Re(z) > 0\}$.

Remark 2.4 *[Louhichi, 2007] We can apply the above theorem to prove that if $f \in L^1([0,1], rdr)$ and if there exist $k_o, p \in \mathbb{N}$ such that*

$$\hat{f}(pn + k_o) = 0 \ \ for\ all\ n \in \mathbb{N},$$

then $\hat{f}(z) = 0$ for all $z \in \{z: Re(z) > 2\}$ and so $f = 0$.

Theorem 2.5 *[Louhich and Zakariasy, 2005] Let ϕ be a bounded radial function and $e^{ip\theta}\psi$ be a quasihomogenous bounded function of degree $p > 0$. If $T_\phi T_{e^{ip\theta}\psi} = T_{e^{ip\theta}\psi} T_\phi$ then, $\psi = 0$ or ϕ is constant.*

Theorem 2.6 *[Louhich and Zakariasy, 2005] Let $p \geq k > 0$ be two integers and ϕ and ψ two bounded radial functions. If*

$$T_{e^{ip\theta}\phi} T_{e^{-ik\theta}\psi} = T_{e^{-ik\theta}\psi} T_{e^{ip\theta}\phi},$$

then $\phi = 0$ or $\psi = 0$.

Theorem 2.7 *[Louhich and Zakariasy, 2005] Let $p, s > 0$ be two integers and $\phi \neq 0$ a bounded radial function. If there exists a bounded radial function ψ not identically zero, such that*

$$T_{e^{is\theta}\phi} T_{e^{ip\theta}\psi} = T_{e^{ip\theta}\psi} T_{e^{is\theta}\phi}$$

then, ψ is unique up to a constant.

Lemma 2.8 *[Louhich and Zakariasy, 2005] Let $e^{ik\theta}\psi$ be a bounded quasihomogenous function of degree $k \geq 0$ and let*

$$\phi(re^{i\theta}) = \sum_{p \in \mathbb{Z}} e^{ip\theta}\phi_p(r) \in L^\infty(\mathbb{D}, dA).$$

Then,

$$T_\phi T_{e^{ik\theta}\psi} = T_{e^{ik\theta}\psi} T_\phi \iff T_{e^{ip\theta}\phi_p} T_{e^{ik\theta}\psi} = T_{e^{ik\theta}\psi} T_{e^{ip\theta}\phi_p}, \ \forall p \in \mathbb{Z}$$

Commutants of T is the set of all those operators that commutes with it and bicommutants is the set of all operators that commute with all operators in the commutant.

In (2008), Louhichi and Rao gave a nice relation between commutants and bicommutants of a quasihomogenous Toeplitz operator; they proved the commutants of a quasihomogenous Toeplitz operator is equal to its bicommutants. In other words, they proved that the following theorem.

Theorem 2.9 *Let $\phi, \psi \in L^\infty(\mathbb{D}, dA)$. If T_ϕ and T_ψ are Toeplitz operators which commute with a quasihomogenous Toeplitz operator, then they commute with each other.*

3. Main Results

We introduce new definition and some nice properties about it.

Definition 3.1 *A function Ψ is said to be poly-quasihomogenous function of degree $m \geq 0$, if Ψ can be written as $\Psi(z) = \Psi(re^{i\theta}) = f(e^{i\theta})\phi(r)$, where $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ is a polynomial function in $e^{i\theta}$ of degree m, and $\phi(r)$ is a radial function.*

If such a function $\Psi(re^{i\theta}) = f(e^{i\theta})\phi(r)$ is the symbol of a Toeplitz operator then we will say that the Toeplitz operator T_Ψ is poly-quasihomogenous of degree m.

The following lemma gives the value of the poly-quasihomogenous Toeplitz operator of degree m and the adjoint of the poly-quasihomogenous Toeplitz operator of degree m at the elements of the orthogonal basis of the Bergman space. It is a very useful formula and we will use very often.

Lemma 3.2 *Let $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ be a polynomial function in $e^{i\theta}$ of degree m, and $\phi(r)$ a bounded radial function., then $\forall n \geq 0$:*

$$T_{f(e^{i\theta})\phi}(z^n) = T_{\sum_{p=0}^{m} a_j e^{ij\theta}\phi}(z^n) = \sum_{j=0}^{m} 2a_j(n+j+1)\hat{\phi}(2n+j+2)z^{n+j};$$

$$T^*_{f(e^{i\theta})\phi}(z^n) = T_{\overline{f(e^{i\theta})}\phi}(z^n) = \begin{cases} \sum_{j=0}^{n} 2\overline{a_j}(n-j+1)\hat{\phi}(2n-j+2)z^{n-j} & if\ n < m, \\ \sum_{j=0}^{m} 2\overline{a_j}(n-j+1)\hat{\phi}(2n-j+2)z^{n-j} & if\ n \geq m. \end{cases}$$

Proof.

$T_{f(e^{i\theta})\phi}(z^n) = T_{\sum_{j=0}^{m} a_j e^{ij\theta}}(z^n) = \sum_{j=0}^{m} T_{a_j e^{ij\theta}\phi}(z^n) = \sum_{j=0}^{m} 2a_p(n+j+1)\hat{\phi}(2n+j+2)z^{n+j}$.

Also,

$$\begin{aligned} T^*_{f(e^{i\theta})\phi}(z^n) &= T_{\sum_{j=0}^{m} \overline{a_j} e^{-ij\theta}\phi}(z^n) \\ &= \sum_{j=0}^{m} T_{\overline{a_j} e^{-ij\theta}\phi}(z^n) \quad by\ Lemma\ ??. \end{aligned}$$

$$= \begin{cases} \sum_{j=0}^{n} 2\overline{a_j}(n-j+1)\hat{\phi}(2n-j+2)z^{n-j} & if\ n < m, \\ \sum_{j=0}^{m} 2\overline{a_j}(n-j+1)\hat{\phi}(2n-j+2)z^{n-j} & if\ n \geq m. \end{cases}$$

The following theorem gives us an important results about the product of two poly-quasihomogenous Toeplitz operators.

Theorem 3.3 *Let Ψ_1 and Ψ_2 be two non-zero bounded poly-quasihomogenous functions of degrees m and s respectively. If there exists a function Ψ_3 such that $T_{\Psi_1}T_{\Psi_2} = T_{\Psi_3}$, then Ψ_3 is a sum of at most $(m+s)$ quasihomogenous functions, and the highest degree of them is $(m+s)$.*

Proof. Assume

$$\Psi_1 = f(e^{i\theta})\phi_1 \quad and \quad \Psi_2 = g(e^{i\theta})\phi_2,$$

whereϕ_1 and ϕ_2 are radial functions, and $f(e^{i\theta}) = \sum_{p=0}^{m} a_p e^{ip\theta}$ and $g(e^{i\theta}) = \sum_{j=0}^{s} b_j e^{ij\theta}$. Now, by Lemma 3, we have $\forall n \geq 0$

$T_{\Psi_1}T_{\Psi_2}(z^n) = T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1} T_{\sum_{j=0}^{s} b_j e^{ij\theta}\phi_2}(z^n) \qquad = T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1}(\sum_{j=0}^{s} 2\, b_j(n+j+1)\widehat{\phi_2}(2n+j+2)\, z^{n+j}) \qquad = \sum_{j=0}^{s} 2\, b_j(n+j+1)\widehat{\phi_2}(2n+j+2)T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1}(z^{n+j}) \qquad = \sum_{j=0}^{s} 2\, b_j(n+j+1)\widehat{\phi_2}(2n+j+2). \sum_{p=0}^{m} 2a_p(n+$

$j+p+1)\widehat{\phi_1}(2n+2j+p+2)z^{n+j+p} \qquad = \sum_{j=0}^{s}\sum_{p=0}^{m} 4a_p(n+j+p+1)b_j(n+j+1)\widehat{\phi_2}(2n+j+2)\widehat{\phi_1}(2n+2j+p+2)z^{n+j+p}$.

Now, if we denote $\lambda_{n,p,j} = 4a_p(n+j+p+1)b_j(n+j+1)\widehat{\phi_2}(2n+j+2)\widehat{\phi_1}(2n+2j+p+2)$. Then we get

$$T_{\Psi_1}T_{\Psi_2}(z^n) = \sum_{j=0}^{s}\sum_{p=0}^{m} \lambda_{n,p,j}z^{n+j+p}.$$

By Theorem 2.2, for all j,p, $\lambda_{n,p,j}z^{n+j+p} = T_{h_{p,j}}$, where $h_{p,j}$ is a quasihomogenous functions of degree $p+j$. Therefore,

$$T_{\Psi_1}T_{\Psi_2} = \sum_{j,p} T_{h_{p,j}} = T_{\sum_{p,j} h_{p,j}} = T_{\Psi_3}, where\ 0 \le j \le s, and\ 0 \le p \le m.$$

Hence Ψ_3 is a sum of $(m+s)$ quasihomogenous Toeplitz operators. Moreover, the highest degree of them is $(m+s)$.

Remark 3.4 *The above theorem is true if we replace Ψ_1 and Ψ_2 by there conjugates, we can prove them by using the adjoint operator.*

The following theorem states that the only idempotent poly-quasihomogenous different than the identity poly quasi (where the identity poly-quasihomogenous is T_1) is the zero poly-quasihomogenous.

Theorem 3.5 *Let $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ be a polynomial function of $e^{i\theta}$ of degree $m > 0$ and ϕ is a bounded radial function. If $T^2_{f(e^{i\theta})\phi} = T_{f(e^{i\theta})\phi}$, then $\phi = 0$.*

Proof. By Lemma , we have $\forall n \ge 0$

$$T_{f(e^{i\theta})\phi}(z^n) = \sum_{j=0}^{m} 2\, a_j(n+j+1)\hat{\phi}(2n+j+2)\, z^{n+j},$$

and

$$T^2_{f(e^{i\theta})\phi}(z^n) = \sum_{j=0}^{m} 4\, a_j^2\, (n+j+1)(n+2j+1)\hat{\phi}(2n+j+2)\hat{\phi}(2n+3j+2)\, z^{n+2j}.$$

Since $T^2_{f(e^{i\theta})\phi} = T_{f(e^{i\theta})\phi}$, then we have

$$\sum_{j=0}^{m} 2\, a_j(n+j+1)\hat{\phi}(2n+j+2)\, z^{n+j} = \sum_{j=0}^{m} 4\, a_j^2\, (n+j+1)(n+2j+1)\hat{\phi}(2n+j+2)\hat{\phi}(2n+3j+2)\, z^{n+2j} \qquad (3.1)$$

In the above equation, the highest degree of z on the right hand side is $n+2m$, however the coefficient of z^{n+2m} is zero on the left hand side. This implies that

$$\hat{\phi}(2n+m+2)\hat{\phi}(2n+3m+2) = 0.$$

Then, either $\hat{\phi}(2n+m+2) = 0$ or $\hat{\phi}(2n+3m+2) = 0, for\ all\ n \ge 0$, but $\sum_{n\in\mathbb{N}} \frac{1}{2n+1} = \infty$. Therefore, by Remark 2.4, $\phi = 0$.

The following states that a Toeplitz operator with bounded radial symbol commutes with a poly-quasihomogenous Toeplitz operator only in the trivial case.

Theorem 3.6 *Let ϕ_1 and ϕ_2 be two bounded radial functions, and $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ a polynomial in $e^{i\theta}$ of degree m. If T_{ϕ_1} commutes with $T_{f(e^{i\theta})\phi_2}$, then $\phi_2 = 0$ or ϕ_1 is constant.*

Proof. Since T_{ϕ_1} commutes with $T_{f(e^{i\theta})\phi_2}$, then $\forall n \geq 0$ we have

$$T_{\phi_1} T_{\sum_{j=0}^{m} a_j e^{ij\theta}\phi_2}(z^n) = T_{\sum_{j=0}^{m} a_j e^{ij\theta}\phi_2} T_{\phi_1}(z^n).$$

On the left side of the above equation, the monomial in z of the highest degree is z^{n+m}, and it comes only from $T_{\phi_1} T_{a_m e^{im\theta}\phi_2}(z^n)$. On the other hand, the monomial in z with highest degree on the right side of the above equation, also is z^{n+m} and it comes from $T_{a_m e^{im\theta}\phi_2} T_{\phi_1}(z^n)$ only. Therefore, $\forall n \geq 0$

$$T_{\phi_1} T_{a_m e^{im\theta}\phi_2}(z^n) = T_{a_m e^{im\theta}\phi_2} T_{\phi_1}(z^n), for\ every\ n \geq 0. \quad ($$

According to Theorem 2.5, equation (3.2) implies that $\phi_2 = 0$ or ϕ_1 is constant.

Remark 3.7 *In the above theorem, if T_{ϕ_1} commutes with $T_{\overline{f(e^{i\theta})}\phi_2}$, then we have the same conclusion i.e. $\phi_2 = 0$ or ϕ_1 is constant.*

The following theorem states that poly-quasihomogenous Toeplitz operator and adjoint of the poly-quasihomogenous Toeplitz operator commute only in the trivial case.

Theorem 3.8 *Let ϕ_1 and ϕ_2 be two bounded radial functions, and $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ and $g(e^{i\theta}) = \sum_{j=0}^{s} b_j e^{ij\theta}$ be two polynomials in $e^{i\theta}$ of degrees m and s respectively, where $m \geq s \geq 0$. If*

$$T_{f(e^{i\theta})\phi_1} T^*_{g(e^{i\theta})\phi_2} = T^*_{g(e^{i\theta})\phi_2} T_{f(e^{i\theta})\phi_1},$$

then $\phi_1 = 0$ or $\phi_2 = 0$.

Proof. Note that $T^*_{g(e^{i\theta})\phi_2} = T_{\overline{g(e^{i\theta})}\phi_2}$

If $T_{f(e^{i\theta})\phi_1} = T_{\sum_{j=0}^{m} a_j e^{ij\theta}\phi_1}$ commutes with $T_{\overline{g(e^{i\theta})}\phi_2} = T_{\sum_{j=0}^{s} \overline{b_j} e^{-ij\theta}\phi_2}$, then $\forall n \geq 0$,

$$T_{\sum_{j=0}^{m} a_j e^{ij\theta}\phi_1} T_{\sum_{j=0}^{s} \overline{b_j} e^{-ij\theta}\phi_2}(z^n) = T_{\sum_{j=0}^{s} \overline{b_j} e^{-ij\theta}\phi_2} T_{\sum_{j=0}^{m} a_j e^{ij\theta}\phi_1}(z^n).$$

Now the terms in z of degree $(n - s + m)$, on both left and right sides of the above equation, come only from $T_{a_m e^{im\theta}\phi_1} T_{\overline{b_s} e^{-is\theta}\phi_2}$, and $T_{\overline{b_s} e^{-is\theta}\phi_2} T_{a_m e^{im\theta}\phi_1}$ respectively. This implies that

$$T_{a_m e^{im\theta}\phi_1} T_{\overline{b_s} e^{-is\theta}\phi_2} = T_{\overline{b_s} e^{-is\theta}\phi_2} T_{a_m e^{im\theta}\phi_1}.$$

By Theorem 2.6, the above equation implies $\phi_1 = 0$ or $\phi_2 = 0$.

Theorem 3.9 *Let ϕ_1,and ϕ_2 be bounded radial functions, and $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ and $g(e^{i\theta}) = \sum_{j=0}^{s} b_j e^{ij\theta}$ be polynomials in $e^{i\theta}$ of degrees m and s respectively. If $T_{e^{ip\theta}\phi_1}$*

commutes with $T_{(f(e^{i\theta})+\overline{g(e^{i\theta})})\phi_2}$, *then* $\phi_1 = 0$ *or* $\phi_2 = 0$.

Proof. If $p > 0$, then we have $\forall n \geq 0$

$$T_{e^{ip\theta}\phi_1} T_{(f(e^{i\theta})+\overline{g(e^{i\theta})})\phi_2}(z^n) = T_{(f(e^{i\theta})+\overline{g(e^{i\theta})})\phi_2} T_{e^{ip\theta}\phi_1}(z^n).$$

In the above equation the monomial in z of the smallest degree is z^{n+p-s} on both sides, Therefore,

$$T_{e^{ip\theta}\phi_1} T_{\overline{b_s}e^{-is\theta}\phi_2} = T_{\overline{b_s}e^{-is\theta}\phi_2} T_{e^{ip\theta}\phi_1}.$$

Then directly, by Theorem 2.6 , $\phi_1 = 0$ or $\phi_2 = 0$.

If $p < 0$, then we can prove the same result by using the term of the highest degree in z is z^{n+m+p}.

The following theorem shows the uniqueness of the commutant of poly-quasihomogenous Toeplitz operator.

Theorem 3.10 *Let* $f(e^{i\theta}) = \sum_{p=0}^{m} a_p e^{ip\theta}$ *and* $g(e^{i\theta}) = \sum_{j=0}^{s} b_j e^{ij\theta}$ *be two polynomials in* $e^{i\theta}$ *of degrees* $m > 0$ *and* $s > 0$ *respectively, and* $\phi_1 \neq 0$ *bounded radial function. If there exists a bounded radial function* ϕ_2 *not identically zero, such that*

$$T_{f(e^{i\theta})\phi_1} T_{g(e^{i\theta})\phi_2} = T_{g(e^{i\theta})\phi_2} T_{f(e^{i\theta})\phi_1},$$

then ϕ_2 is unique up to a constant factor.

Proof. If $T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1}$ commutes with $T_{\sum_{j=0}^{s} b_j e^{ij\theta}\phi_2}$, then $\forall n \geq 0$

$$T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1} T_{\sum_{j=0}^{s} b_j e^{ij\theta}\phi_2}(z^n) = T_{\sum_{j=0}^{s} b_j e^{ij\theta}\phi_2} T_{\sum_{p=0}^{m} a_p e^{ip\theta}\phi_1}(z^n).$$

On the left side of the above equation, the monomial in z of highest degree is z^{n+s+m}, and it comes only from $T_{a_m e^{im\theta}\phi_1} T_{b_s e^{is\theta}\phi_2}$, and on the right side the monomial of highest degree in z is also z^{n+s+m}, and it comes from the product $T_{b_s e^{is\theta}\phi_2} T_{a_m e^{im\theta}\phi_1}$. Therefore,

$$T_{a_m e^{im\theta}\phi_1} T_{b_s e^{is\theta}\phi_2} = T_{b_s e^{is\theta}\phi_2} T_{a_m e^{im\theta}\phi_1}.$$

By Theorem 2.7, the above equation implies that ϕ_2 is unique up to a constant factor.

Remark 3.11 *If* $T_{\overline{f(e^{i\theta})}\phi_1}$ *commutes with* $T_{\overline{g(e^{i\theta})}\phi_2}$, *then* ϕ_2 *is unique up to a constant factor, we can prove it by using the adjoint operator.*

Corollary 3.12 *Let* ϕ_1 *and* ϕ_2 *be two radial functions, and* $f(e^{i\theta}) = \sum_{p=0}^{m} a_p e^{ip\theta}$ *and* $g(e^{i\theta}) = \sum_{j=0}^{m} b_j e^{ij\theta}$ *be two polynomials in* $e^{i\theta}$ *of degree* $m > 0$. *If* $T_{f(e^{i\theta})\phi_1}$ *commutes with* $T_{g(e^{i\theta})\phi_2}$, *then* $\phi_2 = c\phi_1$.

Proof. We know each Toeplitz operator commutes with itself, therefore $T_{f(e^{i\theta})\phi_1}$ commutes with itself. And since

$$T_{f(e^{i\theta})\phi_1}T_{g(e^{i\theta})\phi_2} = T_{g(e^{i\theta})\phi_2}T_{f(e^{i\theta})\phi_1},$$

then, by theorem 3.10, there exists a constant c such that $\phi_2 = c\phi_1$.

Remark 3.13 *The above theorem is true if we replace $f(e^{i\theta})$ and $g(e^{i\theta})$ by there conjugates, we can prove it by the adjoint operator.*

Theorem 3.14 *Let ϕ_1 and ϕ_2 be two bounded radials, and $f(e^{i\theta}) = \sum_{j=0}^{m} a_j e^{ij\theta}$ polynomial in $e^{i\theta}$ of degree m. Then $T_{e^{ip\theta}\phi_1}$ commutes with $T_{f(e^{i\theta})\phi_2}$ if and only if $T_{e^{ip\theta}\phi_1}$ commutes with $T_{a_j e^{ij\theta}\phi_2}$ for all $0 \le j \le m$. Moreover if $p < m$ and $a_p \ne 0$, then there exist $c \in \mathbb{C}$ such that $\phi_2 = c\phi_1$*

Proof. Since $f(e^{i\theta})\phi_2 = \sum_{j=0}^{m} a_j e^{ij\theta}\phi_2$, then directly by Theorem 2.8 $T_{e^{ip\theta}\phi_1}$ commutes with $T_{f(e^{i\theta})\phi_2}$ if and only if $T_{e^{ip\theta}\phi_1}$ commutes with $T_{a_j e^{ij\theta}\phi_2}$ for all $0 \le j \le m$.

If $p < m$ and $a_p \ne 0$, then, by the above, $T_{e^{ip\theta}\phi_1}$ commutes with $T_{a_p e^{ip\theta}\phi_2}$.

But the commutant of $T_{a_p e^{ip\theta}\phi_2}$ is unique, so by Theorem 2.7, $\phi_1 = c\phi_2$.

Remark 3.15 *By theorem 2.9, we can see that $T_{a_p e^{ip\theta}\phi_2}$ commutes with $T_{a_l e^{il\theta}\phi_2}$, for all $0 < l, p \le m$.*

Corollary 3.16 *If $T_{e^{ip\theta}\phi_1}$ commutes with T_{ψ}, where $\psi(z) = \sum_{-\infty}^{\infty} e^{ik\theta}\psi_k$, then*

(a) $T_{e^{ik\theta}\psi_k}$ commutes with $T_{e^{ip\theta}\phi_1}$, for all $k \in \mathbb{Z}$.

(b) $T_{e^{ik\theta}\psi_k}$ commutes with $T_{e^{im\theta}\psi_m}$, for all $m, k \in \mathbb{Z}$.

(b) If $\psi_p \ne 0$, then there exist $c \in \mathbb{C}$ such that $\phi_1 = c\psi_p$.

Theorem 3.17 *Let ϕ_1 and ϕ_2 be two radial functions, and $f(e^{i\theta}) = \sum_{p=0}^{m} a_p e^{ip\theta}$ and $g(e^{i\theta}) = \sum_{j=0}^{s} b_j e^{ij\theta}$ be two polynomials in $e^{i\theta}$ of degrees m and s respectively. If $T_{f(e^{i\theta})\phi_1}T_{g(e^{i\theta})\phi_2} = 0$, then $\phi_1 = 0$ or $\phi_2 = 0$.*

Proof. Since $T_{f(e^{i\theta})\phi_1}T_{g(e^{i\theta})\phi_2} = 0$, then $\forall n \ge 0$ $T_{f(e^{i\theta})\phi_1}T_{g(e^{i\theta})\phi_2}(z^n) = 0$.

By Lemma 3

$$T_{f(e^{i\theta})\phi_1}T_{g(e^{i\theta})\phi_2}(z^n) = \sum_{j=0}^{s}\sum_{p=0}^{m} 4a_p(n+j+p+1)b_j(n+j+1)\widehat{\phi_2}(2n+j+2)\widehat{\phi_1}(2n+2j+p+2)z^{n+j+p}$$

this is polynomial in z of degree $(n+m+s)$ and equal zero for all $z \in \mathbb{D}$, this implies that the coefficients are all zeros. Therefore, $\forall\, n \ge 1$ the coefficient of the highest degree of z is

$$4a_m(n+s+m+1)b_s(n+s+1)\widehat{\phi_2}(2n+s+2)\widehat{\phi_1}(2n+2s+m+2)=0,$$

This implies that $\forall n \geq 0$

$$\widehat{\phi_2}(2n+s+2)=0 \quad or \quad \widehat{\phi_1}(2n+2s+m+2)=0.$$

Now, let $B_1=\{n: \widehat{\phi_1}(2n+2s+m+2)=0\}$ and $B_2=\{n: \widehat{\phi_2}(2n+s+2)=0\}$.

It is clear

$$\sum_{n\in \mathbb{N}} \frac{1}{2n+1}=\infty.$$

Since

$$\sum_{n\in \mathbb{N}} \frac{1}{2n+1} \leq \sum_{n\in B_1} \frac{1}{2n+1}+\sum_{n\in B_2} \frac{1}{2n+1}.$$

at least one of the series $\sum_{n\in B_1} \frac{1}{2n+1}$ or $\sum_{n\in B_2} \frac{1}{2n+1}$ diverges, and so by Remark 2.4 $\phi_1=0$ or $\phi_2=0$.

Theorem 3.18 *Let ϕ_1, and ϕ_2 be bounded radial functions, and $f_1(e^{i\theta})=\sum_{j=0}^{m} a_j e^{ij\theta}$, $f_2(e^{i\theta})=\sum_{j=0}^{m} b_j e^{ij\theta}$, $g_1(e^{i\theta})=\sum_{j=0}^{s} c_j e^{ij\theta}$ and $g_2(e^{i\theta})=\sum_{j=0}^{p} d_j e^{ij\theta}$ polynomials in $e^{i\theta}$ of degrees m, m, s and p respectively, where $m>s>0$ and $m>p>0$. If $(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})$ commutes with $(T_{f_2(e^{i\theta})\phi_2}+T_{g_2(e^{i\theta})\phi_1})$, then $\exists\, c \in \mathbb{C}$ such that $\phi_2=c\phi_1$.*

Proof. Since $(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})$ commutes with $(T_{f_2(e^{i\theta})\phi_2}+T_{g_2(e^{i\theta})\phi_1})$, then $\forall n \geq 0$

$$(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})(T_{f_2(e^{i\theta})\phi_2}+T_{g_2(e^{i\theta})\phi_1})(z^n)=(T_{f_2(e^{i\theta})\phi_2}+T_{g_2(e^{i\theta})\phi_1})(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})(z^n)$$

In the above equation the monomial of z of the highest degree is z^{n+2m} on both sides, Therefore,

$$T_{a_m e^{im\theta}\phi_1}T_{b_m e^{im\theta}\phi_2}=T_{b_m e^{im\theta}\phi_2}T_{a_m e^{im\theta}\phi_1}.$$

Then directly, by Theorem 2.7,then $\exists\, c \in \mathbb{C}$ such that $\phi_2=c\phi_1$.

Remark 3.19 *The above theorem is true, if we replace $f_1(e^{i\theta})$ and $f_2(e^{i\theta})$ by there conjugates.*

Theorem 3.20 *Let ϕ_1,ϕ_2,ψ_1, and ψ_2 be bounded radial functions, and $f_1(e^{i\theta})=\sum_{j=0}^{m} a_j e^{ij\theta}$, $f_2(e^{i\theta})=\sum_{j=0}^{s} b_j e^{ij\theta}$ polynomials in $e^{i\theta}$ of degree m and $g_1(e^{i\theta})=\sum_{j=0}^{s} c_j e^{ij\theta}$ $g_2(e^{i\theta})=\sum_{j=0}^{p} d_j e^{ij\theta}$ polynomials in $e^{i\theta}$ of degree s, where $s<m$. If $(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})$ commutes with $(T_{f_2(e^{i\theta})\psi_1}+T_{g_2(e^{i\theta})\psi_2})$, then $\psi_1=c_1\phi_1$ and $\psi_2=c_2\phi_2$.*

Proof. Since $(T_{f_1(e^{i\theta})\phi_1}+T_{g_1(e^{i\theta})\phi_2})$ commutes with $(T_{f_2(e^{i\theta})\phi_2}+T_{g_2(e^{i\theta})\phi_1})$, then $\forall n \geq 0$

$$(T_{f_1(e^{i\theta})\phi_1} + T_{g_1(e^{i\theta})\phi_2})(T_{f_2(e^{i\theta})\psi_1} + T_{g_2(e^{i\theta})\psi_2})(z^n) = (T_{f_2(e^{i\theta})\psi_1} + T_{g_2(e^{i\theta})\psi_2})(T_{f_1(e^{i\theta})\phi_1} + T_{g_1(e^{i\theta})\phi_2})(z^n) \qquad (3.3)$$

In the above equation the monomial in z of the highest degree is z^{n+2m} on both sides, Therefore,

$$T_{a_m e^{im\theta}\phi_1} T_{b_m e^{im\theta}\psi_1} = T_{b_m e^{im\theta}\phi_1} T_{a_m e^{im\theta}\psi_1}.$$

Then directly, by Theorem 2.7, $\exists\, c_1 \in \mathbb{C}$ such that $\psi_1 = c_1\phi_1$.

On the other hand, the terms in z of degree $n + 2s$, on both left and right sides of (3.3), come from only $T_{a_s e^{is\theta}\phi_1} T_{b_s e^{is\theta}\psi_1}(z^n) + T_{c_s e^{is\theta}\phi_2} T_{d_s e^{is\theta}\psi_2}(z^n)$ and $T_{b_s e^{is\theta}\psi_1} T_{a_s e^{is\theta}\phi_1}(z^n) + T_{d_s e^{is\theta}\psi_2} T_{c_s e^{is\theta}\phi_2}(z^n)$ respectively.

But $\psi_1 = c_1\phi_1$, so

$$T_{c_s e^{is\theta}\phi_2} T_{d_s e^{is\theta}\psi_2}(z^n) = T_{d_s e^{is\theta}\psi_2} T_{c_s e^{is\theta}\phi_2}(z^n).$$

Then directly, by Theorem 2.7, $\exists\, c_2 \in \mathbb{C}$ such that $\psi_2 = c_2\phi_2$.

Remark 3.21 *The above theorem is true, if we replace $f_1(e^{i\theta})$, $f_2(e^{i\theta})$, $g_1(e^{i\theta})$ and $g_2(e^{i\theta})$ by there conjugates.*

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