

Using Graphs to Depict Relationships among Elements in Various Topological Spaces

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In this article, we present a novel definition of discrete topological space, where a relationship was found between the elements of these spaces and their representation in graphs. Additionally, we compute fundamental graph parameters such as the chromatic number, clique number, and planar graph. Furthermore, we prove that a topological graph is a simple undirected connected graph with a cut vertex.

Keywords: graph theory, Topological graph, girth number, clique number, chromatic number, planer graph.

1. Introduction

A graph $\Delta = (V, E)$ is an ordered pair of disjoint sets (V, E) , where $V \neq \emptyset$ and E are a subset of unordered pairs of V . The elements $V = V(\Delta)$, and $E = E(\Delta)$ respectively vertices and edges of a graph (Δ) [8]. A topological graph theory is one of the important types in mathematics which is interesting for both graph theory and specialists in the topological space, Recently some topological graphs have been presented by some researchers such as “some results of domination on the discrete topological graph with it is inverse [4]. “Out topological Digraph Space and Some Related Properties”[7]. “The neighborhood topology converted from the undirected graphs” [5]. Also, some graphs that depend on multiple topological properties have been known in [2, 3, 6]. In this paper, we defined a new graph associated with discrete topology denoted by Δ_T which has the set vertex $V(\Delta_T) = \{M; M \in \tau \text{ and } M \neq \emptyset, X\}$ and two vertices M and N are adjacent where $M \cap N = \emptyset$, also we calculated Girth, Chromatic number, Clique number of this graph and we discussed when this graph is planar or non-planar. The vertices and edges of a graph (Δ) play a vital role in network science and have applications

in diverse domains such as spam detection, graph analysis, graph modeling, and community detection.

2. Fundamental Concepts

This part applies a new approach to the work of the topological spaces under investigation. Many graph properties have been studied and proven in discrete topological spaces.

Definition 2.1: let (X, τ) be a discrete topology on non-empty set X , then we can define a graph on τ which is defined by $\Delta\tau$ as follows the vertex set $V(\Delta\tau) = \{M; M \in \tau \text{ and } M \neq \emptyset, X\}$ and every two vertices M, N adjacent were $M \cap N = \emptyset$

Definition 2.2: The girth of a graph Δ refers to the length of the shortest cycle in Δ . [8].

Definition 2.3: A complete graph is a graph in which every vertex is adjacent to all vertices and denoted by K_n [8].

Definition 2.4: A clique in graph Δ is a fully connected subgraph of Δ , and the size of the largest fully connected subgraph in Δ is referred to as the clique number and denoted by $\omega(\Delta)$ [8].

Definition 2.5: Coloring the vertices of a graph entails assigning distinct colors to each vertex so that no two adjacent vertices share the same color. The minimum number of colors required for such a coloring is known as the chromatic number, represented by the symbol $\chi(\Delta)$ [8].

Definition 2.6: A planer graph is a graph whose edges intersect at the final points only. In other words, there is no intersection between the edges of the plane [8].

Proposition 2.7: Consider a non-empty set X of size n and a topology τ defined on X . If n equals 2, then the topological graph $\Delta\tau$ represents a complete graph K_2 .

Note: We refer to the vertices in the shapes mentioned in the article as follows:

$$\forall a_i = i; i = 1, 2, 3, \dots, n$$

Proof: by Definition 2.1 we have $\{a_1\}$ is not a subset of $\{a_2\}$ then $\{a_1\} \cap \{a_2\} = \emptyset$ for all vertices of the single element, since $\Delta\tau$ is a complete graph.



Figure 1: The graph $\Delta\tau$ is complete.

Proposition 2.8: Consider a non-empty set X of size n and a topology τ defined on X . If n equals 2, then the independent number of $(\Delta\tau) = 1$

Proof: by definition 2.1. According to the definition of an independent set, its vertices are not adjacent, and the proof is obtained.

Example 2.9: Let $X = \{a_1, a_2, a_3\}$, then $\tau = \{\emptyset, X, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\},$

$\{a_2, a_3\}\}$, and $V(\Delta) = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$ Let $X = \{a_1\}$, and $Y =$

$\{a_2\}$ be vertices of single elements. Since $\{a_1\}$ is not a subset of $\{a_2\}$ for all vertices of the single element, by the definition of the topological space, $X \cap Y = \emptyset$, Then X is adjacent to Y . We will have all the individual vertices contiguous, regarding the adjacency of individual vertices with groups of two vertices, each vertex is adjacent to vertices that it is not a part of. According to the definition, vertex $\{a_1\}$ is adjacent to vertex $\{a_2, a_3\}$. For vertex $\{a_2\}$, it is adjacent to vertex $\{a_1, a_3\}$, and vertex $\{a_3\}$ is adjacent to vertex $\{a_1, a_2\}$. Thus, we have the graph. As depicted in Figure 2.

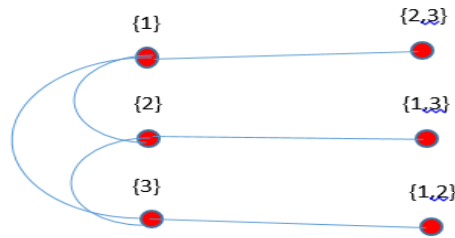


Figure 2: The graph $\Delta\tau$ when the cardinality of X is 3.

Example2.10:

If $|X|=4$, then $\tau = \{X, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\}$, so

$V = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\}$. We also notice that the individual vertices are adjacent to each other because their intersection is an empty set. Likewise, every single vertex is adjacent to a vertex of two elements that are not part of it. It is worth noting that the more the elements of the set increase, the more the intersection of the vertices becomes a non-empty set. This means there is no adjacency between them, so we note that every single-element vertex is shared with a single-element vertex containing three elements. In comparison, vertices that contain two elements are not adjacent to any three-element vertex because their intersection is a non-empty set. See Figure 3.

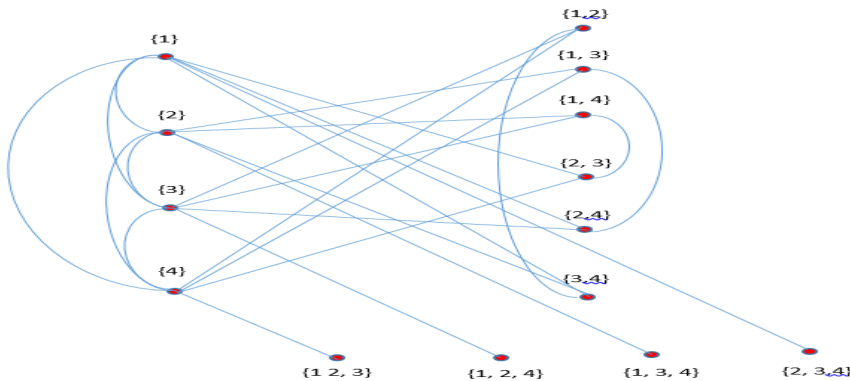


Figure 3: The graph $\Delta\tau$ when the cardinality of X is 4.

Example2.11: If $|X|=5$, and

$\tau = \{ \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}, \{a_4, a_5\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_2, a_5\}, \{a_1, a_3, a_4\}, \{a_1, a_3, a_5\}, \{a_1, a_4, a_5\}, \{a_2, a_3, a_4\}, \{a_2, a_4, a_5\}, \{a_2, a_3, a_5\}, \{a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}, \{a_1, a_3, a_4, a_5\}, \{a_1, a_2, a_4, a_5\}, \{a_2, a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4, a_5\} \}$, so $V = \{ \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_5\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_1, a_5\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_2, a_5\}, \{a_3, a_4\}, \{a_3, a_5\}, \{a_4, a_5\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_2, a_5\}, \{a_1, a_3, a_4\}, \{a_1, a_3, a_5\}, \{a_1, a_4, a_5\}, \{a_2, a_3, a_4\}, \{a_2, a_4, a_5\}, \{a_2, a_3, a_5\}, \{a_3, a_4, a_5\}, \{a_1, a_2, a_3, a_4\}, \{a_1, a_2, a_3, a_5\}, \{a_1, a_3, a_4, a_5\}, \{a_1, a_2, a_4, a_5\}, \{a_2, a_3, a_4, a_5\} \}$. As depicted in Figure 4.

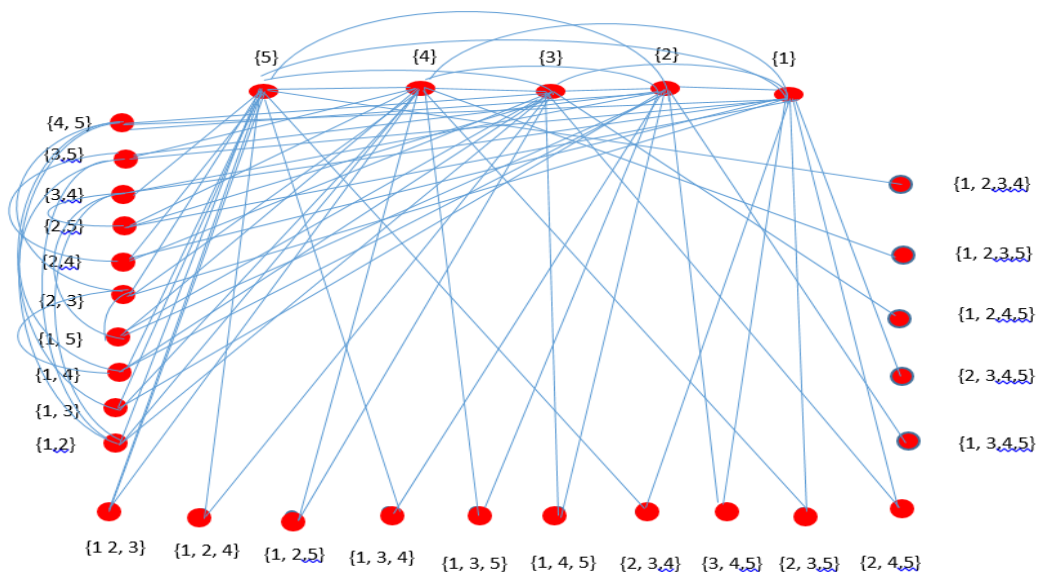


Figure 4: The graph $\Delta\tau$ when $|X| = 5$.

Proposition 2.12: Consider the topological graph G_τ derived from a non-empty set X . It can be asserted that $\Delta\tau$ is a simple graph.

Proof: Let $\{a\}$ and $\{b\}$ represent any two distinct vertices in $\Delta\tau$. Since $\{a\}$ is a subset of $\{a\}$ for all elements of τ , it follows that there are no self-loops in $\Delta\tau$. Moreover, if $\{a\} \cap \{b\} = \emptyset$, as per Definition 2.1, there exists a single edge between $\{a\}$ and $\{b\}$. Therefore, $\Delta\tau$ does not contain multiple edges between vertices.

Proposition 2.13: states that the topological graph $\Delta\tau$ is undirected.

Proof: By (Definition 2.1). The proof is obtained.

Proposition 2.14: Suppose the cardinality of set X is denoted by $|X| = n$. In this case, the order of the topological graph $\Delta\tau$ is determined to be $2^n - 2$.

Proof: Let τ represent a collection comprising all subsets of X with cardinality $2n$.

Consequently, the discrete topological graph $\Delta\tau$ includes all elements from τ , excluding only the empty set $\emptyset$ and the set X , as per (Definition 2.1).

Proposition 2.15: the topological graph $\Delta\tau$ is simple, as it does not contain self-loops or multiple edges between vertices.

Proof: BY (definition 2.1), the maximum vertices are $n - 1$ elements except for X thus, these vertices are only adjacent to vertices with single elements because they exclude only one vertex. On the other hand, these vertices cannot connect with vertices with two elements because the intersection does not give an empty set and therefore there is no adjacency between them. Therefore, when the single vertex is deleted, As a result, vertex $(n - 1)$ elements are cut vertex (see Example 2.11).

Proposition 2.16: Let $|X| = n$, ($n \geq 3$) and Consider the topological graph $\Delta\tau$ derived from a non-empty set X , then the girth number of $\Delta\tau$ equal 3.

Proof: by (definition 2.1) for all vertices of the single element its intersection is an empty set, this means that they are adjacent to each other so when $|X| = n$ ($n \geq 3$) then the least cycle is 3. (See Example 2.7).

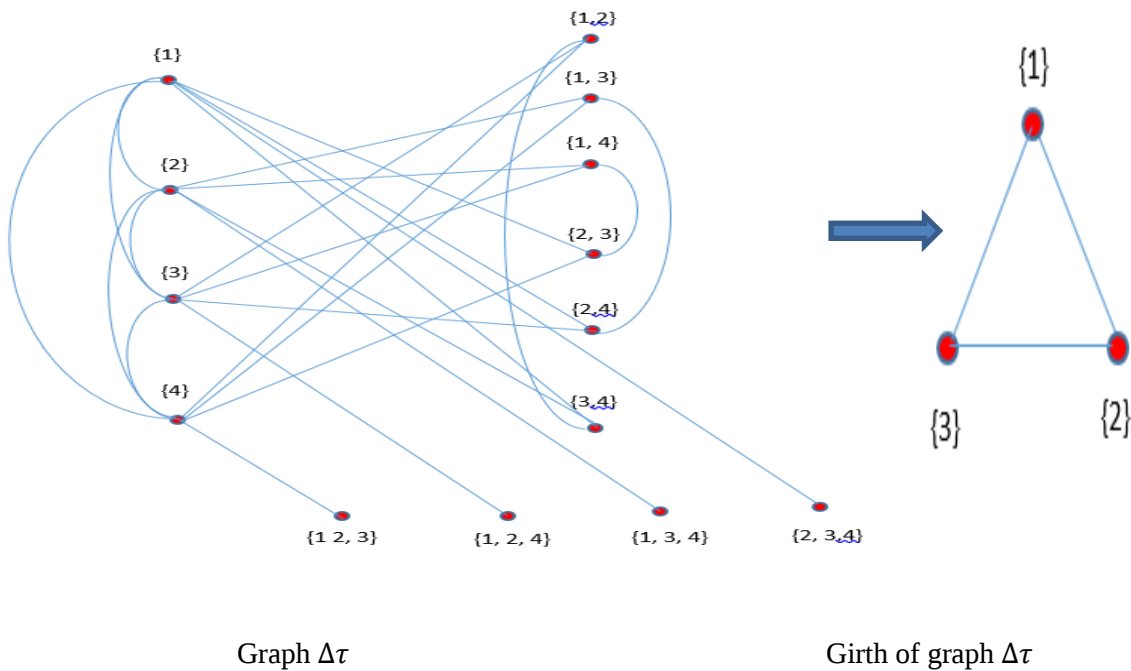


Figure 5: The Girth graph $\Delta\tau$ is 3.

Proposition 2.17: Let $|X| = n$, ($n \geq 3$) and $\Delta\tau$ be a topological graph on X , then the clique number of $\Delta\tau$ equal n .

Proof: by (definition 2.1) for all vertices of the single element its intersection is an empty set, this means that they are adjacent to each other so when $|X| = n$ ($n \geq 3$) then the largest complete subset $\omega(\Delta\tau)$ equal n .

Example 2.18: Let $X = \{a_1, a_2, a_3\}$, then $\tau = \{\emptyset, X, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$, and $V(\Delta) = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$. Then the largest complete subset $\omega(\Delta\tau)$ is 3.

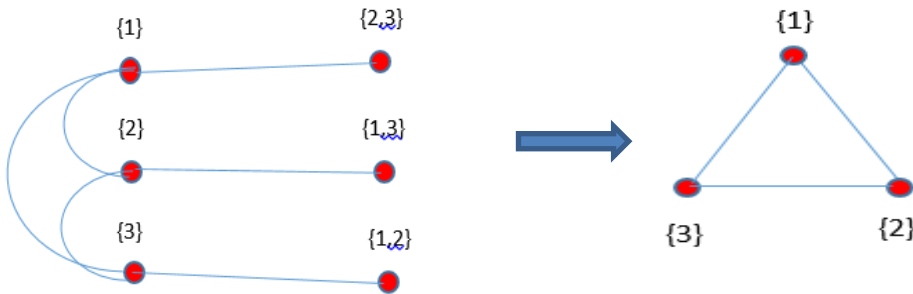


Figure 6: The Clique number of graph $\Delta\tau$ is 3.

Example 2.19:

If $|X|=4$, then $\tau = \{X, \emptyset, \{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\}$, so $V = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_4\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_1, a_4\}, \{a_2, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}, \{a_1, a_2, a_3\}, \{a_1, a_2, a_4\}, \{a_1, a_3, a_4\}, \{a_2, a_3, a_4\}\}$. Then the largest complete subset $\omega(\Delta\tau)$ equals 4.

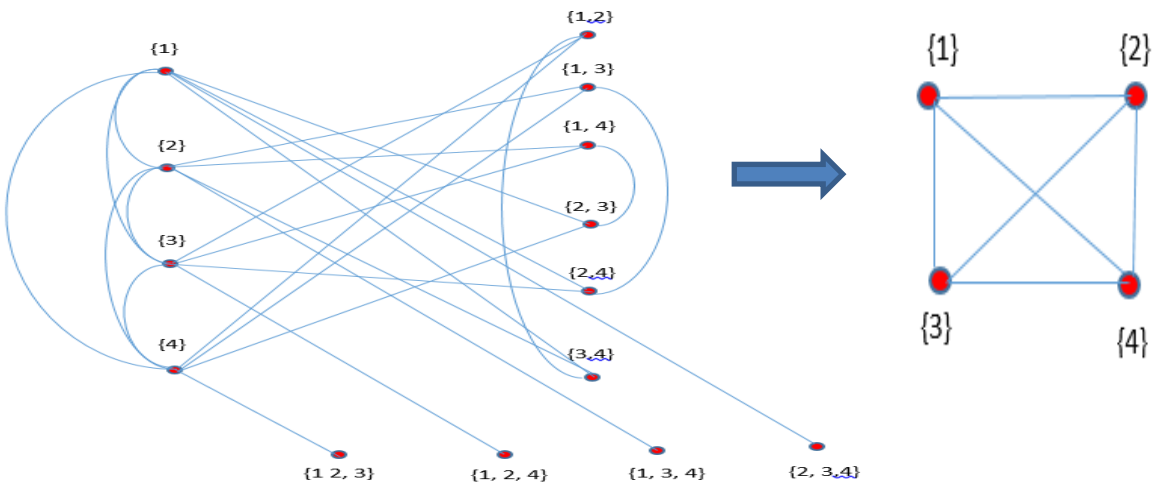


Figure 7: The Clique number of graph $\Delta\tau$ $\omega(\Delta\tau)$ is 4.

Example 2.20: If $|X|=5$, then The Clique number of graph $\Delta\tau$ is 5. (See Example 2.11).

Proposition 2.21: Suppose $|X| = n$, ($n \geq 2$), and $\Delta\tau$ denotes a topological graph on X . In this context, it is established that the chromatic number of $\Delta\tau$, denoted by $\chi(\Delta\tau)$, equals n

Proof: In this graph, the set of vertices can be divided into several families of subsets. Each family depends on the number of elements of its subsets, so the arrangement of the families is

as follows the first family has a singleton subset the second family whose subsets contain two numbers, and so on until a family whose subsets consist of $(n - 1)$ numbers. We can see that a subgraph associative of the first family is a complete subgraph and contains n vertices so we can color with n color shown in (Figures 8,9), from the definition of adjacent in this graph two vertices adjacent where the intersection of two subsets are equal empty set, so the rest of the vertices in the other families are not adjacent to at least one of the vertices in the first family, so the color of this vertex can be given to the non-adjacent vertex. Thus, $\chi(\Delta\tau) = n$.

Example 2.22: If $|X|=4$, then the number of colors of graph $\Delta\tau$ is 4.

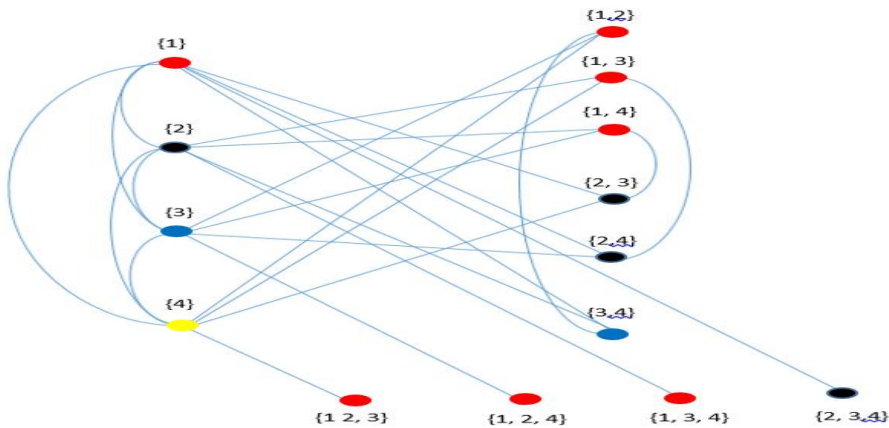


Figure 8: The number of colors in graph $\Delta\tau$ is 4.

Example 2.23: If $|X|=5$, then the number of colors in graph $\Delta\tau$ is 5.

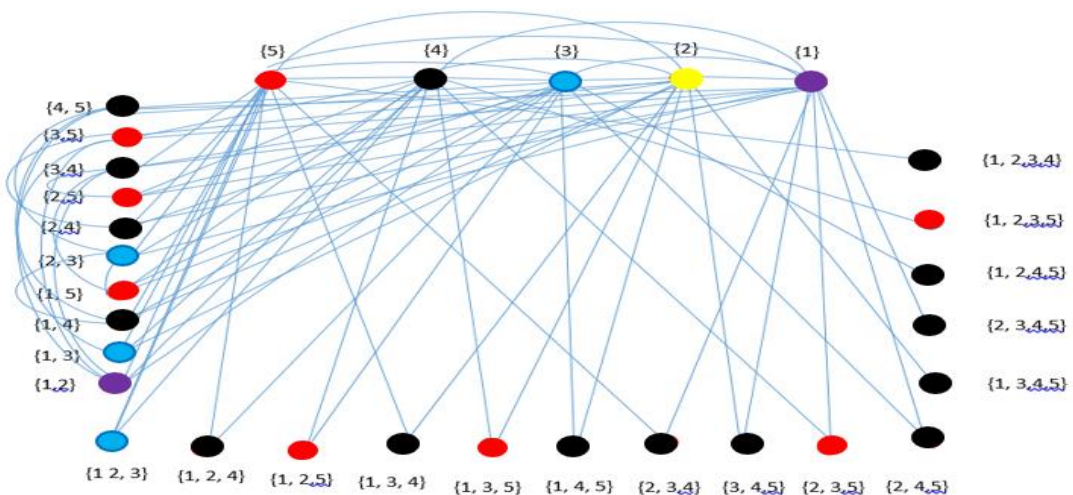


Figure 9: The chromatic number of graph $\Delta\tau$ is 5.

Proposition 2.24: Let $|X| = n$, $(n \leq 3)$ and Consider the topological graph $\Delta\tau$ derived from a non-empty set X then the $(\Delta\tau)$ is a planer graph.

Proof: is clear by (definition 2.5), so there are no vertices intersecting since the graph($\Delta\tau$) is planer.

Example 2.25: Let $X = \{a_1, a_2, a_3\}$, then $\tau = \{\emptyset, X, \{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\},$

$\{a_2, a_3\}\}$, and $V(\Delta) = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_3\}\}$. Then the topological graph ($\Delta\tau$) is the planer graph. (See Figure 2).

Proposition 2.26: Let $|X| = n$, ($n > 3$) and let $\Delta\tau$ be a topological graph on X then the ($\Delta\tau$) is not a planer graph.

Proof: We can prove that by two cases as follows

Case 1: if $|X|=4$, we can see that the number of singletons set equal 4 and the subgraph consisting of this vertex as well $\{a_1, a_2\}, \{a_1, a_3\}, \{a_2, a_4\}, \{a_3, a_4\}$ is a subgraph has the crossing number 2 is shown in figure [10] so, this subgraph is non-planar and thus deduce that the graph is non-planar.

Case 2: if $|X| > 5$, in this case, we have a graph that includes a subgraph isomorphic to the complete graph K_5 which consists of the singleton sets, thus the graph is non-planar (See Figure 3).

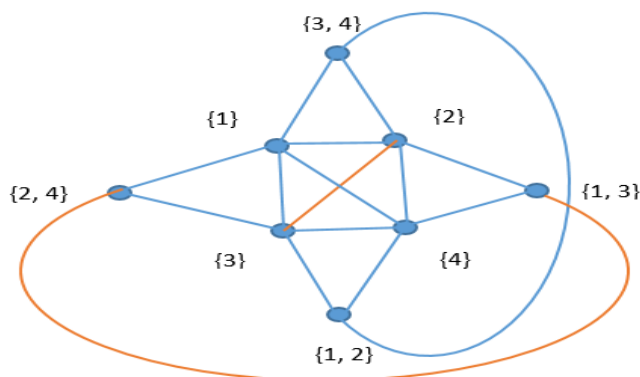


Figure 10: where $|X| = 4$, the graph $\Delta\tau$ is not a planer graph.

3. Conclusions:

This study presented a new construction of a discrete Topological graphic. Also, many properties of the graph have been studied in this space. Our chart is a simple, connected graph. Moreover, the Girth number, Clique number, chromatic number, and planer graph, were calculated by studying many other properties.

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